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THE MORI PROPERTY IN RINGS WITH ZERO DIVISORS, II

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ABSTRACT. A commutative ring R is said to be a Mori ring if it satisfies the ascending chain condition on regular divisorial ideals. Contrary to what happens in Mori domains, examples exist which show if P is a prime ideal of a Mori ring R, then R_P need not be a Mori ring. However, if the total quotient ring of R is von Neumann regular, then it is the case that R_P is Mori whenever R is Mori. In fact, when the total quotient ring is von Neumann regular, then R is a Mori ring if and only if R_P is a Mori ring for each maximal t-ideal P and each regular nonunit of R is contained in at most finitely many maximal t-ideals.

1. Introduction. An integral domain D is said to be a Mori domain if it satisfies the Ascending Chain Condition (ACC) on divisorial ideals [**23**]. Following the terminology introduced in [**19**], we say that a ring R is a *Mori ring* if it satisfies ACC on regular divisorial ideals, *regular* meaning each ideal contains an element that is not a zero divisor. There are several conditions which are known to be equivalent to a domain D being Mori including having D satisfy the Descending Chain Condition on those descending chains of divisorial ideals whose intersection is nonzero [**26**, Théorème I.1] and having D_M a Mori domain for each maximal t-ideal M with each nonzero nonunit a unit in all but finitely many such localizations [**26**, Théorème I.2], [**24**, Théorème 2.1]. Theorem 2.22 of [**19**] shows that R is a Mori ring if and only if it satisfies DCC on those descending chains of regular divisorial ideals whose intersections are regular ideals. One of our main purposes here is to extend the local characterization of Mori domains to reduced rings whose total quotient rings are von Neumann regular. We shall also consider the problem of determining when the polynomial ring $R[\mathrm{x}]$ is a Mori ring.

In this paper R will always denote a commutative ring with identity. Also, we use $T(R)$ to denote the total quotient ring of R and $Q_0(R)$ to

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denote the ring of finite fractions over R. The ring of finite fractions can be described in two different ways, yielding naturally isomorphic but different rings. One is to build it as a subring of the complete ring of quotients using only those homomorphisms defined on *semi-regular ideals*—the ideals which contain finitely generated ideals with no nonzero annihilators. In this form, R and $T(R)$ naturally embed in $Q_0(R)$ by viewing multiplication by $a/b \in T(R)$ as an R-module homomorphism from the regular ideal bR into R (where $a, b \in R$ with b regular). The other is to simply say that it consists of those elements $a(\mathrm{x})/b(\mathrm{x}) \in T(R[\mathrm{x}])$, the total quotient ring of the polynomial ring $R[\mathrm{x}]$, with $a(\mathrm{x}) = \sum a_i \mathrm{x}^i$, $b(\mathrm{x}) = \sum b_i \mathrm{x}^i \in R[\mathrm{x}]$ such that $a_i b_j = a_j b_i$ for each i and j. We shall employ both ways of viewing $Q_0(R)$. Note that via the embedding of R and $T(R)$ into $Q_0(R)$ based on the former view, $T(R[\mathrm{x}]) = T(Q_0(R)[\mathrm{x}])$ with each element of $Q_0(R)$ equal to the appropriate ratio of polynomials employed in the latter form of the construction. See either [**17, 18**] for more details on each construction. Also see [**16**] for the construction of the complete ring of quotients.

We need several more definitions from [**19**]. An R-module $B \subseteq Q_0(R)$ is a *fractional ideal* of R if there is a semi-regular ideal $I \subseteq R$ such that $IB \subseteq R$. If, in addition, B contains an element that is not a zero divisor and there is a regular element $r \in R$ such that $rB \subseteq R$, then it is a *regular fractional ideal* of R. In this case B is an R-submodule of $T(R)$. An R-module $J \subseteq Q_0(R)$ is a *semi-regular fractional ideal* of R if J contains a finite dense subset and there is a finitely generated semi-regular ideal I of R such that $IJ \subseteq R$. Note that each regular fractional ideal of R is also a semi-regular fractional ideal of R.

For a semi-regular fractional ideal J, we break with the conventional notation of colons and let $(R : J) = \{t \in Q_0(R) \mid tJ \subseteq R\}$. By default, $(R : J)$ is a semi-regular fractional ideal of R. Moreover, if J is a regular fractional ideal of R, then $(R : J)$ contains a regular element of R, and it is easy to show that $(R : J)$ is a regular fractional ideal as well, see [**19**, Lemma 2.1]. We say that $(R : (R : J))$ is the *divisorial closure* of J and that J is *divisorial* if $(R : (R : J)) = J$. We employ the standard condensed form J_v to denote $(R : (R : J))$. Clearly, if $J \subseteq A$, then $(R : A) \subseteq (R : J)$ and hence $J_v \subseteq A_v$. It follows that $(R : J) = (R : J)_v$ for each semi-regular fractional ideal J. We also may form the "t" of J in the same way as done for integral domains. Specifically, for each semi-regular fractional ideal J, set $J_t = \cup A_v$ with the union taken over the finitely generated semi-regular fractional ideals contained in J. If $J = J_t$, then J is a *t-ideal*. As with ideals in an integral domain, if J is a semi-regular ideal with $J_t \neq R$, then it is contained in a maximal t-ideal. Also, each maximal t-ideal is prime [**19**, Lemma 2.3].

For a prime ideal P of R we let $R_{(P)} = \{t \in T(R) \mid rt \in R$ for some $r \in R \backslash P\}$, and we let $R_{\langle P\rangle} = \{q \in Q_0(R) \mid rq \in R$ for some $r \in R\backslash P\}$. For an ideal I, we let $(I)_{(P)} = \{t \in T(R) \mid rt \in I$ for some $r \in R\backslash P\}$, and we let $\langle I\rangle_{\langle P\rangle} = \{q \in Q_0(R) \mid rq \in I$ for some $r \in R\backslash P\}$. In both cases, these extensions may be larger than the "simple" extensions of $IR_{(P)}$ and $IR_{\langle P\rangle}$ obtained by simply using the elements of I to generate ideals. Moreover, ideals which are not contained in P may extend to proper ideals in one or both rings.

Examples 3.3, 3.4 and 3.5 of [**19**] show that when we deal with semi-regular divisorial ideals, it is possible for a ring to satisfy ACC on semi-regular divisorial ideals yet have an infinite descending chain of semi-regular divisorial ideals whose intersection is semi-regular, and it is possible for a ring to satisfy DCC on chains of semi-regular divisorial ideals with semi-regular intersections without satisfying ACC on semi-regular divisorial ideals. Since both Mori domains and Mori rings satisfy both ACC and some form of DCC, a *Q_0-Mori ring* is required to satisfy both ACC on semi-regular divisorial ideals and DCC on those descending chains of semi-regular divisorial ideals with semi-regular intersection. In [**19**] the latter is referred to as the "restricted DCC on semi-regular divisorial ideals".

A particularly nice type of ring to deal with is one where each finitely generated ideal that contains only zero divisors has a nonzero annihilator and each ideal that contains a regular element can be generated by its regular elements. Rings that satisfy the former are referred to as *McCoy rings* [**4**] and rings that satisfy the latter are referred to as *Marot rings* [**12**]. For a ring R, R[x] is both McCoy ring ([**22**, Proposition 6] and [**13**, Theorem 1]) and a Marot ring ([**12**, Theorems 7.2 and 7.5]).

2. Preliminary results. We will use several results from [**19**]. For convenience we collect many of these in this section.

A useful fact when dealing with the v and t operations in integral domains is that an ideal of the form $(a :_D b)\,(:= \{d \in D \mid db \in aD\})$

is always a divisorial ideal of D. A simple manipulation shows that $(a :_D b) = (D : (1, b/a))$. We will use a similar result in our work.

Lemma 2.1 [**19**, Lemma 2.1 (a)]. *Let A be a semi-regular fractional ideal of R. Then $(R : A)$ is a semi-regular divisorial fractional ideal of R. In particular, for each nonzero $t \in Q_0(R)$, the ideal $(R : (1, t))$ is a semi-regular divisorial ideal of R. Moreover, $t \in R$ if and only if $(R : (1, t)) = R$.*

Lemma 2.2 [**19**, Lemma 2.1 (b)]. *Let J be a regular fractional ideal of R. Then $(R : J) \subseteq T(R)$ and $(R : J)$ is a regular fractional ideal of R.*

For rings which satisfy ACC on semi-regular divisorial ideals, the t and v operations coincide (but not necessarily on semi-regular fractional ideals). So $A_v = A_t$ for each semi-regular ideal of a Q_0-Mori ring. The operations coincide for regular ideals (and regular fractional ideals) in Mori rings.

Lemma 2.3 [**19**, Lemma 2.4]. *Let $\{J_n\}$ be a nonempty collection of semi-regular divisorial fractional ideals of R. If $\cap J_n$ contains a semi-regular fractional ideal of R, then $\cap J_n$ is a semi-regular divisorial fractional ideal.*

Theorem 2.4 [**19**, Theorem 2.19]. *The following are equivalent for a ring R.*

(1) *R is a Q_0-Mori ring.*

(2) *Each semi-regular ideal is contained in at most finitely many maximal t-ideals and for each maximal t-ideal M, $R_{\langle M\rangle}$ is a Q_0-Mori ring.*

(3) *Each semi-regular divisorial ideal is contained in at most finitely many maximal t-ideals and for each maximal t-ideal M, $R_{\langle M\rangle}$ is a Q_0-Mori ring.*

(4) *Each semi-regular fractional ideal that is comparable with R contains a finitely generated semi-regular ideal with the same divisorial closure.*

Theorem 2.5 [**19**, Theorem 2.22]. *The following are equivalent for a ring R.*

(1) *R is a Mori ring.*

(2) *R satisfies the restricted* DCC *on regular divisorial ideals, i.e., on those descending chains of regular divisorial ideals with regular intersection.*

(3) *For each regular fractional ideal J there is a finitely generated regular fractional ideal $B \subseteq J$ such that $B_v = J_v$.*

(4) *For each regular ideal I, there is a finitely generated regular ideal $A \subseteq I$ such that $A_v = I_v$.*

(5) *Each regular ideal is contained in at most finitely many maximal t-ideals and for each regular maximal t-ideal M, $R_{(M)}$ satisfies* ACC *on regular divisorial ideals.*

(6) *Each regular ideal is contained in at most finitely many maximal t-ideals and for each regular maximal t-ideal M, $R_{(M)}$ satisfies the restricted* DCC *on regular divisorial ideals.*

Theorem 2.6 [**19**, Theorem 2.5]. *The following are equivalent for a ring R.*

(1) *R satisfies* ACC *on semi-regular divisorial ideals.*

(2) *For each ascending chain of finitely generated semi-regular ideals $\{I_m\}$, there is an ideal I_n in the chain such that $(I_n)_v = (I_k)_v$ for each $k \geq n$.*

(3) *For each semi-regular ideal I there is a finitely generated semi-regular ideal $J \subseteq I$ such that $J_v = I_v$.*

Corollary 2.7 [**19**, Corollary 2.6]. *Let R be a ring which satisfies* ACC *on semi-regular divisorial ideals. Then $I_t = I_v$ for each semi-regular ideal. In particular, each semi-regular maximal t-ideal is divisorial.*

Theorem 2.8 [**19**, Theorem 2.7]. *The following are equivalent for a ring R.*

(1) *R satisfies the restricted* DCC *on semi-regular divisorial ideals.*

(2) *If J is a semi-regular fractional ideal that contains R, then there is a finitely generated semi-regular fractional ideal B contained in J such that $B_v = J_v$.*

Lemma 2.9 [**19**, Lemma 2.3 (c), (d)]. *Let I and J be fractional semi-regular ideals of R. Then $(IJ)_v = (IJ_v)_v = (I_vJ_v)_v$ and $(IJ)_t = (IJ_t)_t = (I_tJ_t)_t$.*

For a semi-regular prime P, if $s \in Q_0(R)$ is such that there is a semi-regular ideal J for which $sJ \subseteq R$ and J is not contained in P, then we have $st = r \in R$ for some $t \in J \backslash P$. It follows that "s" can be represented as the fraction r/t in R_P. There is no guarantee that this image of s is a nonzero element of R_P. But if this image is zero, then so is every other image of s. To see this, assume $r/t = 0$ as an element of R_P. Then there is an element $b \in R \backslash P$ such that $br = 0$. Now let $a \in (R : (1, s))$ and let $c = sa$. Then we have $0 = 0a = (br)a = bsta = (bt)(sa) = (bt)c$ with $bt \in R \backslash P$, so $sa = c = 0/bt = 0$ as an element of R_P. On the other hand, there is no guarantee that an element of $Q_0(R_P)$, or even of R_P, is the image of an element of $Q_0(R)$. But we can extend each element of $Q_0(R)$ to an element of $Q_0(R_P)$, although some nonzero elements may extend to the zero element of R_P.

The next few results concern extending the local properties of Mori domains to Mori rings and Q_0-Mori rings. We will use $\overline{u}$ to denote an element of $Q_0(R_P)$, no matter whether it is the image of an element of $Q_0(R)$ or not. As with the results quoted above from [**19**], the next few lemmas extend well-known results for nonzero ideals of integral domains to semi-regular ideals in rings with nonzero zero divisors. In particular, the "local" characterizations of v-ideals and t-ideals in Lemmas 2.11 and 2.13 are related to results in [**8**], see also [**14**, Proposition 2.8].

Lemma 2.10. *Let P be a semi-regular prime ideal of R, and let A be a finitely generated semi-regular ideal. Then AR_P is a finitely generated semi-regular ideal of R_P and each element of $(R : A)$ extends to an element of $(R_P : AR_P)$.*

Proof. Let $A = (a_1, a_2, \dots, a_n)$, and let $\overline{b} \in \operatorname{Ann}(AR_P)$ and write $\overline{b} = \overline{r}/\overline{t}$ for some $r \in R$ and $t \in R\backslash P$. Then we have $\overline{ra_i}/\overline{t} = 0$ in R_P for each i, i.e., there is an element $q_i \in R\backslash P$ such that $q_i r a_i = t\,0 = 0$. But then $q = q_1 q_2 \cdots q_n \in R\backslash P$ is such that $qra_i = 0$. Since A is semi-regular, we must have $qr = 0$, hence $\overline{b} = 0$ and AR_P is semi-regular.

For $u \in (R : A)$ we simply use the image of $s_i = ua_i \in R$ in R_P as the value of the R_P homomorphism $\overline{u}$ from AR_P into R_P. As an element of $T(R_P[\mathrm{x}])$, $\overline{u}$ can be represented as the quotient $\overline{s}(\mathrm{x})/\overline{a}(\mathrm{x})$ with $\overline{s}(\mathrm{x}) = \sum \overline{s}_i \mathrm{x}^i$ and $\overline{a}(\mathrm{x}) = \sum \overline{a}_i \mathrm{x}^i$, just as u can be represented as the quotient $s(\mathrm{x})/a(\mathrm{x})$ with $s(\mathrm{x}) = \sum s_i \mathrm{x}^i$ and $a(\mathrm{x}) = \sum a_i \mathrm{x}^i$. Thus, each element of $(R : A)$ extends to an element of $(R_P : AR_P)$. $\square$

Note that $u \in (R : A)$ will extend to the zero map if and only if P does not contain the annihilator of the ideal $(s_1, s_2, \dots, s_n)$, where $s_i = ua_i$ as in the last proof.

The next lemma deals with characterizing ideals locally. Of particular importance are the characterizations of divisorial ideals.

Lemma 2.11. *Let R be a ring. Then the following hold.*

(a) $R = \{t \in Q_0(R) \mid \overline{t} \in R_M$ *for each* $M \in t\operatorname{Max}(R)\}$.

(b) *If J is a semi-regular divisorial ideal of R, then* $J = \{t \in Q_0(R) \mid \overline{t} \in (JR_M)_v$ *for each* $M \in t\operatorname{Max}(R)\}$.

(c) *If I is a finitely generated semi-regular ideal of R, then* $I_v = \{t \in Q_0(R) \mid \overline{t} \in (IR_M)_v$ *for each* $M \in t\operatorname{Max}(R)\}$.

Proof. We start by showing that $R = \{t \in Q_0(R) \mid \overline{t} \in R_M$ for each $M \in t\operatorname{Max}(R)\}$. Fix a maximal t-ideal M. Then having $\overline{t} \in R_M$ implies that there is an element $s \in R\backslash M$ and an element $b \in R$ such that $\overline{t} = b/s$ in R_M. Let $A = (a_1, a_2, \dots, a_n)$ be any finitely generated semi-regular ideal that multiplies t into R, and let $b_i = ta_i$ for each i. So we have $b_i = \overline{t}a_i = ba_i/s$ as elements of R_M. As A is finitely generated, there is an element $q \in R\backslash M$ such that $0 = q(sb_i - ba_i) = q(sta_i - ba_i) = (qst - qb)a_i$ for each i. Since A is semi-regular, we must have $qst = qb$. Thus, $qs \in (R : (1, t))\backslash M$. As this must happen for each maximal t-ideal, no maximal t-ideal can contain the semi-regular divisorial ideal $(R:(1,t))$. Hence, $(R:(1,t)) = R$ and we have $t \in R$ as desired.

For a semi-regular ideal B of R, let $B^{(v)} = \{t \in Q_0(R) \mid \overline{t} \in (BR_M)_v$ for each $M \in t\,\mathrm{Max}\,(R)\}$. By the above, it is clear that $B^{(v)}$ is contained in R. It is a straightforward exercise to show that it is an ideal that contains B.

For (b), assume J is a semi-regular divisorial ideal, and let $u \in (R : J)$. Since J is divisorial, we need only show that $uJ^{(v)}$ is contained in R. For each maximal t-ideal M, u extends to an element $\overline{u}$ in $(R_M : JR_M)$. Thus $\overline{(ut)} = \overline{u}\,\overline{t} \in R_M$ for each $t \in J^{(v)}$. As $R = \{t \in Q_0(R) \mid \overline{t} \in R_M$ for each $M \in t\,\mathrm{Max}\,(R)\}$, we have $ut \in R$ and therefore $J^{(v)} = J$.

Now consider the case that $I = (a_1, a_2, \ldots, a_n)$ is a finitely generated semi-regular ideal of R. By part (b), we have $I^{(v)} \subseteq I_v$ since $(IR_M)_v \subseteq (I_vR_M)_v$ for each $M \in t\,\mathrm{Max}\,(R)$. So all we need do is show $(IR_M)_v = (I_vR_M)_v$. For this it suffices to show $I_vR_M(R : IR_M) \subseteq R_M$. To this end, let $\overline{q} \in (R : IR_M)$. We have $\overline{q}\,\overline{a}_i \in R_M$ for each i. Since I is finitely generated, there is an element $t \in R\backslash M$ and elements $b_1, b_2, \ldots, b_n \in R$ such that $\overline{q}\,\overline{a}_i = b_i/t \in R_M$. We also have $a_j(b_i/t) = a_i(b_j/t)$ as elements of R_M. So there is an element $s \in R\backslash M$ such that $st(a_ib_j - a_jb_i) = 0$ for each i and j. Thus, $a_i(stb_j) = a_j(stb_i)$. With this we may define an R-module homomorphism $h \in (R : I)$ by setting $ha_i = stb_i$. So $\overline{h}\,a_i = stb_i = (st)\overline{q}\,a_i$ in R_M.

For $a \in I_v$, set $ha = b$. In R_M we have $\overline{h}\,a = b$. But as st is a unit of R_M and IR_M is semi-regular, $(st)^{-1}\overline{h} = \overline{q}$. Thus, we simply have $\overline{q}\,a = b/st \in R_M$. The result follows. □

Remark 2.12. If some element $s \in Q_0(R)$ is integral over R, then its image in $Q_0(R_M)$ is integral over R_M. Thus, a consequence of part (a) of the previous lemma is that if each R_M is integrally closed in its ring of finite fractions for each maximal t-ideal M, then R is integrally closed in $Q_0(R)$.

Another simple consequence of Lemma 2.11 is that if some semi-regular ideal I is such that $(R_M : IR_M) = R_M$ for each maximal t-ideal M, then $(R : I) = R$ and $I_v = R$.

For t-ideals, we can prove a result similar to Lemma 2.11 under the additional assumption that each semi-regular divisorial ideal is contained in at most finitely many maximal t-ideals. Note that even for an ideal I of an integral domain D, I_v need not equal $\cap(ID_M)_v$ with the intersection taken over the maximal t-ideals of D. For example, if D is an almost Dedekind domain that is not Dedekind, then each maximal ideal is a t-ideal and at least one maximal ideal is locally principal but not invertible. For such a maximal ideal N, $N_v = D$ while $(ND_N)_v = ND_N$. Thus, N_v properly contains the intersection $N = N_t = \bigcap(ND_M)_v$. This example also illustrates the fact that knowing MD_M is divisorial does not guarantee that M is divisorial. The related converse for maximal t-ideals also fails, knowing M is a maximal t-ideal of an integral domain D, does not guarantee that MD_M will be a t-ideal of D_M (see for example [**28**]).

Lemma 2.13. *Let R be a ring for which each semi-regular divisorial ideal is contained in only finitely many maximal t-ideals. Then the following hold.*

(a) *For each maximal t-ideal M, MR_M is the maximal t-ideal of R_M.*

(b) *If I is a semi-regular t-ideal, then $I = \{t \in Q_0(R) \mid \bar{t} \in (IR_M)_t$ for each $M \in tMax(R)\}$.*

Proof. Let M be a maximal t-ideal of R, and let C be a finitely generated semi-regular ideal of R_M. To show that MR_M is a t-ideal and thus the maximal t-ideal of R_M, it suffices to show $C_v \subseteq MR_M$. Now since C is finitely generated, there is a finitely generated ideal A of R such that $AR_M = C$, but there is no guarantee that A is semi-regular. However, M is semi-regular, so it contains a finitely generated semi-regular ideal B, and it does no harm to assume B contains A. Thus BR_M contains C and we have $C_v \subseteq (BR_M)_v$. Since B_v is contained in only finitely many maximal t-ideals, there is a finitely generated semi-regular ideal $I \supseteq B$ such that M is the only maximal t-ideal that contains I. By Lemma 2.11 we know that $I_v = \{t \in Q_0(R) \mid \bar{t} \in (IR_N)_v$ for each $N \in t\,\mathrm{Max}\,(R)\}$. As $IR_N = R_N$ for each maximal t-ideal $N \neq M$, we must have $(IR_M)_v \neq R_M$. Thus $C_v \subseteq MR_M$, and we have that MR_M is a t-ideal.

For (b), let I be a t-ideal of R, and let A be a finitely generated semi-regular ideal contained in I. By our assumption, only finitely many maximal t-ideals contain A_v, so only finitely many contain I.

Let $M_1, M_2, \dots, M_n$ be the maximal t-ideals that contain I. Consider the set $I^{(t)} = \{t \in Q_0(R) \mid t \in (IR_M)_t \text{ for each } M \in t\operatorname{Max}(R)\}$. As above, there is a finitely generated semi-regular ideal B which is contained in I and in no maximal t-ideal other than $M_1, M_2, \dots, M_n$. Thus, we may consider I as the union $\cup B_v$ where each ideal B is finitely generated, semi-regular and contained in no maximal t-ideal other than $M_1, M_2, \dots, M_n$.

Let $t \in I^{(t)}$. Then, for each M_i there is a finitely generated semi-regular ideal $A_i \subseteq I$ such that $t \in (A_i R_{M_i})_v$. We may further assume that A_i is among those finitely generated semi-regular ideals which are contained in I and no maximal t-ideal other than $M_1, M_2, \dots, M_n$. But then the sum $A = A_1 + A_2 + \cdots + A_n$ is a finitely generated semi-regular ideal which is contained in I and no maximal t-ideal other than the M_i's. We have $t \in (AR_{M_i})_v$ for each M_i. For all other maximal t-ideals N, we have $t \in (AR_N)_v = (IR_N)_v = R_N$. Thus, we have $t \in A_v \subseteq I$. $\square$

3. Localization and polynomial rings. Our first result of this section is a local version of [**19**, Theorem 2.17].

Theorem 3.1. *If R is a ring for which each semi-regular divisorial ideal is contained in only finitely many maximal t-ideals and R_M satisfies* ACC *on semi-regular divisorial ideals for each maximal t-ideal M, then R satisfies* ACC *on semi-regular divisorial ideals.*

Proof. Assume each semi-regular divisorial ideal is contained in only finitely many maximal t-ideals and that R_M satisfies ACC on semi-regular divisorial ideals for each maximal t-ideal M.

Let M be a maximal t-ideal of R. By Lemma 2.13, MR_M is the maximal t-ideal of R_M. We first show that M is divisorial and the v of a finitely generated semi-regular ideal. Then we show that each semi-regular divisorial ideal of R is the v of some finitely generated semi-regular ideal.

As R_M satisfies ACC on semi-regular divisorial ideals, MR_M is divisorial and the v of some finitely generated semi-regular ideal. Since each semi-regular divisorial ideal of R is contained in only finitely many maximal t-ideals, there is a finitely generated semi-regular ideal C contained in M and no other maximal t-ideal such that $(CR_M)_v = MR_M$. By Lemma 2.11, $C_v = \{t \in Q_0(R) \mid \bar{t} \in (CR_N)_v$ for each $N \in t\operatorname{Max}(R)\}$. As in the proof of Lemma 2.13, we have $(CR_N)_v = CR_N = R_N$ for each maximal t-ideal $N \neq M$. But we also have $MR_N = R_N$ for $N \neq M$. Hence, $C_v = M$.

To complete the proof, let J be a semi-regular ideal of R. If $J_t = R$, then there is a finitely generated semi-regular ideal $A \subseteq J$ such that $A_v = J_v = R$. So we may assume $J_t \neq R$. Since each maximal t-ideal is divisorial, we also have $J_v \neq R$. Furthermore, each maximal t-ideal that contains J contains J_v as well. Let $M_1, M_2, \ldots, M_n$ be the maximal t-ideals that contain J_v (and J). Since R_{M_i} satisfies ACC on semi-regular divisorial ideals and $M_iR_{M_i}$ is its unique maximal t-ideal, $(JR_{M_i})_v \neq R_{M_i}$ and there is a finitely generated semi-regular ideal $A_i \subseteq J$ such that $(A_iR_{M_i})_v = (JR_{M_i})_v$. It follows that the ideal $A' = A_1 + A_2 + \cdots + A_n$ is finitely generated, semi-regular and contained in J with $(A'R_{M_i})_v = (JR_{M_i})_v$ for each M_i. As only finitely many maximal t-ideals of R contain A', there is a finitely generated semi-regular ideal $A \subseteq J$ such that the M_i's are the only maximal t-ideals that contain A. For $N \in t\operatorname{Max}(R)\backslash\{M_1, M_2, \ldots, M_n\}$, $AR_N = JR_N = R_N$ and for each M_i, $(AR_{M_i})_v = (JR_{M_i})_v$. By Lemma 2.11, $A_v = \{t \in Q_0(R) \mid \bar{t} \in (AR_M)_v$ for each $M \in t\operatorname{Max}(R)\}$. As $(AR_M)_v = (JR_M)_v$ for each $M \in t\operatorname{Max}(R)$, A_v contains J which implies $A_v = J_v$ and therefore R satisfies ACC on semi-regular divisorial ideals. $\square$

We have the following partial converse for Theorem 3.1. A key ingredient to the proof is that if $T(R)$ is von Neumann regular, then so is $T(R_P)$ for each prime ideal P, and each regular ideal of R_P is the localization of a regular ideal of R. To see this, suppose $r \in P$ is such that its image in R_P is regular. If r is not regular, then there is an idempotent $e \in T(R)$ and a regular element $s \in R$ such that $er = r$, $r(1-e) = 0$ and $s(1-e) \in R$. Since $rs(1-e) = 0$ and the image of r in R_P is regular, the image of $s(1-e)$ in R_P must be 0. This puts $s(1-e)$ in P and we have that $r + s(1-e)$ is a regular element of R with the same image in R_P as that of r.

Theorem 3.2. *Let R be a reduced ring whose total quotient ring is von Neumann regular. If R is a Mori ring, then R_P is a Mori ring for each prime ideal P.*

Proof. Since R is reduced, R_Q is a field for each minimal prime Q. As $T(R)$ is von Neumann regular, the primes that are not minimal ones are regular ideals of R. Moreover, for each prime $P \in \operatorname{Spec}(R)\backslash\operatorname{Min}(R)$, each semi-regular ideal of R_P is regular and is the localization of a regular ideal of R that is contained in P.

Assume R is a Mori ring, and let P be a regular prime of R. By Theorem 2.6 [**19**, Theorem 2.5] it suffices to show that for each regular ideal JR_P of R_P, there is a finitely generated regular ideal $AR_P \subseteq JR_P$ such that $(AR_P)_v = (JR_P)_v$, or equivalently, that $(R_P : AR_P) = (R_P : JR_P)$. We may assume that J is a regular ideal of R that is contained in P. Thus, by Theorem 2.6 [**19**, Theorem 2.5], there is a finitely generated regular ideal $A \subseteq J$ such that $A_v = J_v$. As the process of taking inverses reverses containment relations, it suffices to show $(R_P : AR_P) \subseteq (R_P : JR_P)$. To this end, let $u \in (R_P : AR_P)$. As A is a regular ideal of R, there is a regular element $b \in A$ and an element $w \in R_P$ such that $u = w/b$. Now select $c \in R$ and $s \in R\backslash P$ such that $w = c/s$ (as an element of R_P). Since A is finitely generated, there is an element $t \in R\backslash P$ such that $ts(c/b)$ is in $(R : A) = (R : J)$. Since both t and s are units of R_P, we have $uJR_P \subseteq R_P$ as desired. Hence, R_P is a Mori ring. $\square$

Corollary 3.3. *Let R be a reduced ring such that $T(R)$ is von Neumann regular. Then the following are equivalent.*

(1) *R is a Mori ring.*

(2) *R is a Q_0-Mori ring.*

(3) *Each regular divisorial ideal is contained in finitely many maximal t-ideals and R_M is Mori for each maximal ideal M.*

(4) *Each regular divisorial ideal is contained in finitely many maximal t-ideals and R_M is Mori for each maximal t-ideal M.*

Proof. Since $T(R)$ is von Neumann regular, each semi-regular ideal of R is regular. This makes the equivalence of (1) and (2) trivial.

Combining the fact that each semi-regular ideal of R is regular with Theorem 3.1 gives us that (4) implies (1). By Theorem 2.5 [**19**, Theorem 2.22], if R is a Mori ring, then each regular element is contained in at most finitely many maximal t-ideals. Thus (1) implies both (3) and (4) by Theorem 3.2. Theorem 3.2 also is enough to show that (3) implies (4) since R_M is von Neumann regular for each maximal ideal of M of R and R_P is naturally isomorphic to the ring $(R_M)_{PR_M}$. $\square$

The ring R in Example 3.3 of [**19**] is a reduced Q_0-Mori ring with a unique maximal t-ideal M which is also a maximal ideal of R. Localizing at M yields an integral domain which is not a Mori domain. Also, there are elements in the quotient field of R_M which are not images of elements of $Q_0(R)$.

It is known that if D is an integrally closed Mori domain, then the polynomial ring $D[\mathrm{x}]$ is also a Mori domain. In [**27**], Roitman gives a general scheme for starting with an arbitrary countable field K and constructing a Mori domain D that contains K for which the associated polynomial ring $D[\mathrm{x}]$ is not a Mori domain. On the other hand, he shows that if a Mori domain contains an uncountable field, then the associated polynomial ring is Mori [**27**], see also [**1**]. We are interested in when $R[\mathrm{x}]$ is Mori. Based on what is known for integral domains, we will consider this question only for integrally closed rings. As we will see, for reduced integrally closed rings, the associated polynomial ring is Mori if and only if the original ring is a finite direct sum of integrally closed Mori domains. So the total quotient ring is a finite direct sum of fields and therefore is a very simple type of von Neumann regular ring. A related problem is determining when $R[\mathrm{x}]$ satisfies ACC on annihilator ideals. A ring R is said to be a *Kerr ring* if the polynomial ring $R[\mathrm{x}]$ satisfies ACC on annihilator ideals [**5**]. If $R[\mathrm{x}]$ satisfies ACC on annihilators, then so does R since $\mathrm{Ann}\,(IR[\mathrm{x}]) = \mathrm{Ann}\,(I)R[\mathrm{x}]$ for each ideal I of R. If R is reduced, each annihilator of $R[\mathrm{x}]$ is extended from an annihilator of R. Thus, when R is reduced, R satisfies ACC on annihilators if and only if $R[\mathrm{x}]$ does. In [**15**], Kerr constructs a nonreduced ring R such that R satisfies ACC on annihilators but $R[\mathrm{x}]$ does not. Moreover, a general construction scheme given in [**2**] shows that for each positive integer n, there is a ring R such that $R[\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_n]$ satisfies ACC on annihilator ideals while $R[\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_n, \mathrm{x}_{n+1}]$ does not. In our final example we show how to construct an integrally closed nonreduced Mori ring $R \neq T(R)$ where $R[\mathrm{x}]$ is Mori but the total quotient ring has arbitrary dimension.

Recall that the content of a polynomial $f(\mathrm{x}) \in R[\mathrm{x}]$ is simply the ideal of R generated by the coefficients. A polynomial whose coefficients generate R as an ideal is said to have unit content. We will use $\mathcal{U}(R)$ to denote the set of polynomials with unit content in R. Localizing $R[\mathrm{x}]$ at the set $\mathcal{U}(R)$ gives the Nagata ring $R(\mathrm{x})$. Also we denote the content of $f(\mathrm{x})$ by $C(f)$.

One difference between integral domains and rings with zero divisors has to do with the form of maximal t-ideals in polynomial rings. For an integral domain D, the maximal t-ideals of $D[\mathrm{x}]$ come in two forms. First we have those of the form $PD[\mathrm{x}]$ where P is a maximal t-ideal of D. All of the others, if any, are of the form $Q = f(\mathrm{x})K[\mathrm{x}] \cap D[\mathrm{x}]$ for some irreducible polynomial $f(\mathrm{x})$ with the additional property that there is a polynomial $g(\mathrm{x}) \in Q$ for which $C(g)_v = D$ [**11**, Proposition 1.1]. The ideal Q is referred to as an *upper to zero* since its contraction to D is the zero ideal. If R is a ring with zero divisors, the zero ideal is no longer prime so there are no uppers to zero. Let M be a maximal t-ideal of $R[\mathrm{x}]$, and let $P = M \cap R$. There are three possibilities for P. The first is that P is a maximal t-ideal of R in which case $M = PR[\mathrm{x}]$. We will say that M is of *type* I if this occurs. The second possibility is similar to the upper to zero types for domains, namely P is a minimal prime ideal of R. These are referred to as being of *type* II. The third possibility is that P is neither a maximal t-ideal of R nor a minimal prime. In this case P cannot be semi-regular, but it need not have a nonzero annihilator. Maximal t-ideals of this sort are said to be of *type* III.

The following result appears in [**20**]. It is a generalization of [**21**, Proposition 7].

Lemma 3.4 [**20**, Lemma 5.1]. *Let A be a semi-regular ideal of a ring R.*

(a) $(AR[\mathrm{x}])^{-1} = A^{-1}R[\mathrm{x}]$ *and* $(AR(\mathrm{x}))^{-1} = A^{-1}R(\mathrm{x})$.

(b) $(AR[\mathrm{x}])_v = A_vR[\mathrm{x}]$ *and* $(AR(\mathrm{x}))_v = A_vR(\mathrm{x})$.

Lemma 3.5. *Let I be an ideal of the ring R.*

(a) *If I has no nonzero annihilators, then $(I+\mathrm{x}R[\mathrm{x}])_v = R[\mathrm{x}]$.*

(b) *If I has a nonzero annihilator, then $(I+\mathrm{x}R[\mathrm{x}])^{-1} = \mathrm{x}^{-1}\mathrm{Ann}\,(I)+R[\mathrm{x}]$ and $(I+\mathrm{x}R[\mathrm{x}])_v = J+\mathrm{x}R[\mathrm{x}]$ where $J = \mathrm{Ann}\,(\mathrm{Ann}\,(I))$.*

(c) *The divisorial ideals of $R[\mathrm{x}]$ that contain* x *are all of the form $B+\mathrm{x}R[\mathrm{x}]$ where $B = \mathrm{Ann}\,(C)$ for some ideal C of R.*

Proof. For (a) and the first equality in (b), it suffices to prove that for each ideal I, $(I+\mathrm{x}R[\mathrm{x}])^{-1} = \mathrm{x}^{-1}\mathrm{Ann}\,(I)+R[\mathrm{x}]$. Let $A = I+\mathrm{x}R[\mathrm{x}]$. Since x is in A, A^{-1} is contained in $\mathrm{x}^{-1}R[\mathrm{x}] = \mathrm{x}^{-1}R+R[\mathrm{x}]$. Thus, to describe A^{-1} it suffices to know which elements of $(1/\mathrm{x})R$ multiply each element of I into $R[\mathrm{x}]$. Obviously, if s is a nonzero element of R, $s\mathrm{x}^{-1}$ is not in $R[\mathrm{x}]$. On the other hand, for each nonzero element $b \in \mathrm{Ann}\,(I)$ and each $a \in I$, $(b/\mathrm{x})a = 0$. It follows that $A^{-1} = \mathrm{x}^{-1}\mathrm{Ann}\,(I)+R[\mathrm{x}]$. In the case that I has a nonzero annihilator, it is also clear that $(I+\mathrm{x}R[\mathrm{x}])_v = J+\mathrm{x}R[\mathrm{x}]$.

The ideals of $R[\mathrm{x}]$ that properly contain $\mathrm{x}R[\mathrm{x}]$ are all of the form $B+\mathrm{x}R[\mathrm{x}]$ for some nonzero ideal B of R. By (b), such an ideal is divisorial if and only if $B = \mathrm{Ann}\,(\mathrm{Ann}\,(B))$. $\square$

For an ideal B of R that has no nonzero annihilator, $(R[\mathrm{x}] : \mathrm{x}R[\mathrm{x}] + BR[\mathrm{x}])$ must simply be $R[\mathrm{x}]$. As each regular t-ideal is divisorial in a Mori ring, each t-prime containing x must be of the form $\mathrm{x}R[\mathrm{x}]+PR[\mathrm{x}]$ for some prime P that is an annihilator in R.

Theorem 3.6. *If $R[\mathrm{x}]$ is a Mori ring, then*

(i) *R is a Q_0-Mori ring with ACC on annihilator ideals and only finitely many minimal primes,*

(ii) *$Z(R[\mathrm{x}])$ is a finite union of prime ideals, each extended from a prime of R,*

(iii) *$Z(R)$ is a finite union of prime ideals, and*

(iv) *$Q_0(R) = T(R)$ and R is a McCoy ring.*

Proof. Assume $R[\mathrm{x}]$ is a Mori ring. By the preceding lemma [**20**, Lemma 5.1], each semi-regular divisorial ideal of R extends to a regular divisorial ideal of $R[\mathrm{x}]$. Thus, R must be a Q_0-Mori ring. It is also the case that, for each regular ideal J of $R[\mathrm{x}]$, $J_t = J_v$ by Corollary 2.7 [**19**, Corollary 2.6].

Let P be a minimal prime of R, the ideal $\mathrm{x}R[\mathrm{x}] + PR[\mathrm{x}]$ is a height one regular prime of $R[\mathrm{x}]$ which is minimal over (x), so it is a t-prime of $R[\mathrm{x}]$ [**19**, Lemma 2.3]. As each regular t-prime is divisorial, Lemma 3.5 implies that P must have a nonzero annihilator and it must be the annihilator of some ideal. Having $R[\mathrm{x}]$ Mori also requires that x be contained in only finitely many regular t-primes (Theorem 2.5 [**19**, Theorem 2.22]). Thus, R has only finitely many minimal primes.

Now let Q be a prime of $R[\mathrm{x}]$ that is maximal among those primes which contain only zero divisors. Since $R[\mathrm{x}]$ is a McCoy ring, each finitely generated ideal contained in Q has a nonzero annihilator. It follows that Q must be extended from a prime P of R where each finitely generated ideal contained in P has a nonzero annihilator. This in turn implies that $(P + \mathrm{x}R[\mathrm{x}])_t \neq R[\mathrm{x}]$. As $(P + \mathrm{x}R[\mathrm{x}])_t = (P + \mathrm{x}R[\mathrm{x}])_v$, we must have $(P + \mathrm{x}R[\mathrm{x}])_t = \mathrm{Ann}\,(\mathrm{Ann}\,(P)) + \mathrm{x}R[\mathrm{x}]$ by Lemma 3.5. Since $(P + \mathrm{x}R[\mathrm{x}])_t \neq R[\mathrm{x}]$, P has a nonzero annihilator and $P = \mathrm{Ann}\,(\mathrm{Ann}\,(P))$. As with the minimal primes of R, the set $Z(R[\mathrm{x}]$ must be a finite union of prime ideals, each extended from a prime of R with a nonzero annihilator.

Let $P_1, P_2, \ldots, P_n$ be primes of R, such that $\cup P_i R[\mathrm{x}] = Z(R[\mathrm{x}])$. By the above, each P_i has a nonzero annihilator. For each zero divisor $b \in R$, the ideal $bR[\mathrm{x}]$ is contained in $Z(R[\mathrm{x}]$. Thus that $b \in P_i$ for some i and we have $\cup P_i = Z(R)$. Since we have a finite union, each finitely generated ideal of R that is contained in $Z(R)$ must be contained in at least one of the P_i's. It follows that each such finitely generated ideal has a nonzero annihilator. Thus, R is a McCoy ring and $Q_0(R) = T(R)$. $\square$

We record without proof the following elementary result.

Lemma 3.7. *Let $S = \sum_{i=1}^{n} D_i$ be a finite direct sum of integral domains. Then the following hold.*

(a) *S is reduced and $T(S)$ is von Neumann regular. Moreover, $T(S) = \sum_{i=1}^{n} K_i$ where K_i is the quotient field of D_i.*

(b) *Each ideal of S is of the form $I = \sum_{i=1}^{n} I_i$ where each I_i is an ideal of the corresponding D_i. Moreover, I is regular if and only if each I_i is nonzero.*

(c) *If $I = \sum_{i=1}^{n} I_i$ is a regular ideal, then $(S : I) = \sum_{i=1}^{n}(D_i : I_i)$. Hence $I_v = \sum_{i=1}^{n}(I_i)_v$.*

(d) *For each prime P_j of the domain D_j, the set $\widehat{P}_j = \{(a_1, a_2, \dots, a_n) \in S \mid a_j \in P_j\}$ is a prime of S. Moreover each prime of S has this form and $S_{\widehat{P}_j}$ is naturally isomorphic to $(D_j)_{P_j}$. Each minimal prime of S is simply the $\widehat{\ }$ extension of the zero ideal from a particular D_j.*

Lemma 3.8. *Let $S = \sum_{i=1}^{n} D_i$ be a finite direct sum of integral domains. Then S is a Mori ring if and only if each D_i is a Mori domain.*

Proof. Let $\{J_m\}$ be a chain of regular divisorial ideals of S, and let, for each m, $J_m = \sum_{i=1}^{n} J_{m,i}$. Then, by Lemma 3.7, each $J_{m,i}$ is a divisorial ideal of D_i. So, for each i we have a chain of divisorial ideals $\{J_{m,i}\}$. Clearly, a chain $\{J_m\}$ stabilizes if and only if each chain $\{J_{m,i}\}$ stabilizes. Hence, S is a Mori ring if and only if each D_i is a Mori domain. □

Theorem 3.9. *Let R be an integrally closed reduced ring. Then the following are equivalent.*

(1) *R[x] is a Mori ring.*

(2) *R is a Mori ring with only finitely many minimal primes.*

(3) *R is a Q_0-Mori ring with only finitely many minimal primes.*

(4) *R is a finite direct sum of integrally closed Mori domains.*

Proof. By Theorem 3.6, R[x] Mori implies R is a Q_0-Mori ring with only finitely many minimal primes. So (1) implies (3). As R is reduced, having finitely many minimal primes implies $T(R)$ is von Neumann regular. Thus, the equivalence of (2) and (3) follows from the fact that all semi-regular ideals are regular when the total quotient ring is von Neumann regular.

Next, we show that (4) implies (1). Assume that R is isomorphic to a finite direct sum of integrally closed Mori domains, say $S = D_1 \oplus D_2 \oplus \cdots \oplus D_m$. Then S is Mori by Lemma 3.8. We also have $S[\mathrm{x}] = \sum D_i[\mathrm{x}]$ Mori since a polynomial ring over an integrally closed Mori domain is also a Mori domain [**25**, Théorème 3.5]. Thus, $R[\mathrm{x}]$ is a Mori ring.

To complete the proof we show that (2) implies (4). Assume R is a Mori ring with finitely many minimal primes. Since R is reduced and integrally closed, it is isomorphic to the finite direct sum $\sum_{i=1}^{n} R/P_i$ where $P_1, P_2, \dots, P_n$ are the minimal primes of R and each R/P_i is an integrally closed integral domain [**12**, Lemma 8.14, Corollary 8.15]. Thus, by Lemma 3.8, each R/P_i is an integrally closed Mori domain. $\square$

In [**3**], Endo essentially proved that if R is a reduced ring whose total quotient ring is von Neumann regular, then R is integrally closed if and only if R_M is an integrally closed integral domain for each maximal ideal M. He does not make this statement explicitly, but it is a trivial consequence of combining his Propositions 2, 5 and 6. Of course, if R_M is an integrally closed domain for some maximal ideal M, then R_P is also an integrally closed domain for each prime P contained in M.

It seems likely that some of the statements in the next two lemmas may be known, but we do not know of a reference. For the most part, the assumption in Lemma 3.12 that R be integrally closed is to guarantee that R contains all of the idempotents of $T(R)$.

Lemma 3.10. *Let $r \in R$ be a regular element, and let $e \in R$ be an idempotent. Then, the following are equivalent.*

(1) $(r, e)_v \neq R$.

(2) $(r, e) \neq R$.

(3) $1 - e \notin rR$.

(4) $(1 - e)/r \notin R$.

Proof. To start, first note that if $(r,e)_v \neq R$, then we have $(r,e) \neq R$. This implies we cannot have $1-e$ in rR, which in turn implies $(1-e)/r \notin R$. But $(1-e)/r \in (r,e)^{-1}$, so we are back to $(r,e)_v \neq R$. This completes the cycle of implications. □

Lemma 3.11. *Let r be a regular element of R. Then for each idempotent $e \in R$, (r,e) is an invertible ideal of R. Moreover, if $(r,1-e)_v = R$, then $e \in rR$.*

Proof. We always have $(1-e)/r \in (r,e)^{-1}$. This puts $1-e \in (r,e)(r,e)^{-1}$. As e is also in this ideal, (r,e) is invertible.

By Lemma 3.10, $(r,1-e)_v = R$ implies $(r,1-e) = R$. So $1 = sr + t(1-e)$ for some $s,t \in R$. Simply multiply by e to obtain $e = esr + et(1-e) = esr \in rR$. □

Lemma 3.12. *Let R be an integrally closed ring whose total quotient ring is von Neumann regular. If R is a Mori ring, then for each regular element r there are at most finitely many minimal primes which are not comaximal with r. Moreover, the intersection of these minimal primes is contained in the principal ideal rR.*

Proof. Assume R is a Mori ring. Since R is integrally closed and $T(R)$ is von Neumann regular, R_P is an integrally closed integral domain for each prime P [**3**, Propositions 2, 5, 6]. Thus, each prime contains a unique minimal prime. The Mori assumption implies that each regular element is contained in at most finitely many maximal divisorial ideals (each of which is prime). So there must be at most finitely many minimal primes P for which $(rR+P)_v \neq R$. Let P be a minimal prime of R such that $rR + P \neq R$. Since R is a Mori ring, there is a finitely generated ideal $A = (r, a_1, a_2, \dots, a_n)$ with each $a_i \in P$ such that $A_v = (rR+P)_v$. As R is integrally closed with $T(R)$ von Neumann regular and P is a minimal prime, there is an idempotent $e \in P$ such that $A \subseteq (r,e)$. Thus, $(r,e)_v = (rR+P)_v$. By Lemma 3.10, $(r,e)_v \neq R$ if and only if $(r,e) \neq R$. Thus, $rR + P \neq R$ if and only if $(rR+P)_v \neq R$. The first statement is now clear.

For the second, let $P_1, P_2, \dots, P_n$ denote the minimal primes of R that are not comaximal with r, and let e be an idempotent in the intersection $\cap P_i$. Then $1-e$ is in none of these primes. Moreover, no divisorial prime that contains r can contain $1-e$. Hence, $(r, 1-e)_v = R$. But this means $e \in rR$ by Lemma 3.11. Since R is integrally closed, each element of $\cap P_i$ is a multiple of some idempotent in the intersection. Thus, each element of $\cap P_i$ is contained in rR. $\square$

Theorem 3.13. *Let R be a reduced ring whose total quotient ring is von Neumann regular. Then the following are equivalent.*

(1) *R is an integrally closed Mori ring.*

(2) *Each regular element is contained in at most finitely many maximal t-ideals and R_M is an integrally closed Mori domain for each maximal ideal M.*

(3) *Each regular element is contained in at most finitely many maximal t-ideals and R_M is an integrally closed Mori domain for each maximal t-ideal M.*

(4) $R(\mathrm{x})$ *is an integrally closed Mori ring.*

Proof. If $R(\mathrm{x})$ is Mori, then R is Mori since each regular divisorial ideal of R extends to a divisorial ideal of $R(\mathrm{x})$ [**20**, Lemma 5.1]. Thus, (4) implies (1).

The equivalence of (1) and (2) can be established by combining our Theorem 3.3 with Endo's result about R being integrally closed if and only if R_M is an integrally closed integral domain for each maximal ideal M. That (2) implies (3) is by Theorem 3.3 and the fact that a localization of an integrally closed integral domain is integrally closed (and a domain).

Theorem 3.3 combined with Remark 2.12 is enough to verify (3) implies (1).

Assume R is an integrally closed Mori ring. By Theorem 3.3 it suffices to show that $R(\mathrm{x})_M$ is Mori for each maximal t-ideal M of $R(\mathrm{x})$ and that each regular divisorial ideal of $R(\mathrm{x})$ is contained in only finitely many maximal t-ideals.

By Lemma 3.10, for each regular element r of R and each idempotent e, either $(r, e)_v \neq R$ or $(r, e) = R$. It follows that if P is a minimal prime, then either $rR + P = R$ or $(rR + P)_t \neq R$. As R is Mori, only finitely many maximal t-ideals contain r. As R is integrally closed and $T(R)$ is von Neumann regular, localizing at such an ideal yields an integrally closed domain. Thus, each maximal t-ideal contains a unique minimal prime. Hence, there are only finitely many minimal primes that are not comaximal with r. For a polynomial $s(\mathrm{x})$ with regular content, there must be at most finitely many minimal primes which are not comaximal with the content of $s(\mathrm{x})$. It follows that there are at most finitely many maximal t-ideals of $R(\mathrm{x})$ that contain $s(\mathrm{x})$. This in turn implies that each regular divisorial ideal of $R(\mathrm{x})$ is contained in only finitely many maximal t-ideals.

Suppose M is a maximal t-ideal of $R(\mathrm{x})$. Since each prime of positive height is regular, $R(\mathrm{x})$ has no maximal t-ideals of type III. Those of type II localize to one-dimensional discrete valuation domains, so to Mori domains. The remaining maximal t-ideals are those which are extensions of maximal t-ideals of R. Assume $M = PR(\mathrm{x})$ is of this type. Then R_P is an integrally closed Mori domain by (3). Thus, $R_P[\mathrm{x}]$ is a Mori domain by [**25**, Théorème 3.5]. Thus, $R_P(\mathrm{x}) = R(\mathrm{x})_M$ is a Mori domain by [**24**, Théorème 2.2]. Hence, $R(\mathrm{x})$ is a Mori ring by Theorem 3.3. □

4. Idealization.

The next two theorems and the example that follows are built using the technique of idealization of a module, see for example [**12**, Chapter VI]. The basic notion is to take a nonzero R-module B and create a new ring denoted $R(+)B$ from the direct sum $R \oplus B$ by defining products as $(r,a)(s,b) = (rs, rb + sa)$. For our purposes we will start with an integral domain D and take a particular type of D-module. In our first result we show how to start with an arbitrary integrally closed quasilocal integral domain and build a nonreduced Mori ring with the property that the corresponding polynomial ring is a Mori ring.

Theorem 4.1. *Let D be an integrally closed quasilocal domain with maximal ideal M, and let $R = D(+)F$ where $F = D/M$.*

(1) *Since $MF = \{0\}$, $M(+)F$ annihilates $(0)(+)F$. Also $Z(R) = M(+)F$ so R is a McCoy ring and $R = T(R) = Q_0(R)$.*

(2) *$Z(R[\mathrm{x}]) = M[\mathrm{x}](+)F[\mathrm{x}]$, with nonzero annihilator $(0)(+)F[\mathrm{x}]$. Thus, each regular element of $R[\mathrm{x}] = D[\mathrm{x}](+)F[\mathrm{x}]$ is of the form $(u(\mathrm{x}), b(\mathrm{x}))$ where $u(\mathrm{x}) \in \mathcal{U}(D)$ and $b(\mathrm{x}) \in F[\mathrm{x}]$.*

(3) *For an ideal J of $R[\mathrm{x}]$ generated by elements $(u(\mathrm{x}), a(\mathrm{x}))$ and $(m, b(\mathrm{x}))$ where $u(\mathrm{x}) \in \mathcal{U}(D)$, $m \in M\backslash\{0\}$ and $a(\mathrm{x}), b(\mathrm{x}) \in F[\mathrm{x}]$, $J_v = I(+)F[\mathrm{x}]$ where $I = u(\mathrm{x})D[\mathrm{x}] + M[\mathrm{x}]$. Note that $I = D[\mathrm{x}]$, and $J_v = R[\mathrm{x}]$, if and only if $u(\mathrm{x})$ is constant modulo M.*

(4) *The maximal t-ideals of $R[\mathrm{x}]$ are of two forms, the primes of the form $P(+)F[\mathrm{x}]$ where P is an upper to zero of $D[\mathrm{x}]$ that contains a polynomial which is a nonzero constant modulo $M[\mathrm{x}]$, and the primes of the form $N(+)F[\mathrm{x}]$ where $N = u(\mathrm{x})D[\mathrm{x}] + M[\mathrm{x}]$ is a maximal ideal of $D[\mathrm{x}]$ with $u(\mathrm{x}) \in \mathcal{U}(D)$ irreducible modulo M.*

(5) *If P is an upper to zero of $D[\mathrm{x}]$ that contains a polynomial which is a nonzero constant modulo $M[\mathrm{x}]$, then $R[\mathrm{x}]_Q$ is rank one discrete valuation domain for $Q = P(+)F[\mathrm{x}]$.*

(6) *If $u(\mathrm{x}) \in \mathcal{U}(D)$ is irreducible modulo M, then $N = u(\mathrm{x})D[\mathrm{x}] + M[\mathrm{x}]$ is a maximal ideal of $D[\mathrm{x}]$, $N' = N(+)F[\mathrm{x}]$ is a divisorial maximal ideal of $R[\mathrm{x}]$ and $R[\mathrm{x}]_{N'}$ is a Mori ring.*

(7) *$R[\mathrm{x}]$ is a Mori ring.*

Proof. First note that for each $m \in M$ and each $b \in F$, $(m, b)(0, 1) = (0, 0)$. On the other hand, if u is a unit of D, then $(u, c)(u^{-1}, -u^{-2}c) = (1, 0)$ for each $c \in F$. Thus, the only regular elements of R are its units and there are no proper semi-regular ideals. Moreover, $M(+)F$ annihilates $(0)(+)F$. Thus, $R = T(R) = Q_0(R)$ is a McCoy ring.

For the remainder of the proof, we let $\overline{f}(\mathrm{x})$ denote the image of $f(\mathrm{x})$ in $F[\mathrm{x}]$ for each element $f(\mathrm{x}) \in D[\mathrm{x}]$.

Since the maximal ideal $M(+)F$ has a nonzero annihilator, namely the ideal $(0)(+)F$, the only regular nonunits of $R[\mathrm{x}]$ are the polynomials of the form $(u(\mathrm{x}), b(\mathrm{x}))$ where $u(\mathrm{x})$ is a nonconstant polynomial with unit content. Thus, $T(R[\mathrm{x}]) = D(\mathrm{x})(+)F(\mathrm{x})$. Moreover, $M[\mathrm{x}](+)F[\mathrm{x}]$ annihilates $(0)(+)F[\mathrm{x}]$.

To prove statement (3), let $u(\mathrm{x}) \in \mathcal{U}(D)$, $m \in M\backslash\{0\}$ and $a(\mathrm{x}), b(\mathrm{x}) \in F[\mathrm{x}]$, and consider the ideal J generated by $(u(\mathrm{x}), a(\mathrm{x}))$ and $(m, b(\mathrm{x}))$. Since $m \neq 0$, $(D[\mathrm{x}] : (u(\mathrm{x}), m))$ is contained in $K[\mathrm{x}]$ where K is the quotient field of D. As $D(\mathrm{x}) \cap K[\mathrm{x}] = D[\mathrm{x}]$, $(R[\mathrm{x}] : J)$ must be contained in $D[\mathrm{x}](+)F(\mathrm{x})$. As m kills $F(\mathrm{x})$, the inverse of J consists of those elements of the form $(d(\mathrm{x}), c(\mathrm{x}))$ where $d(\mathrm{x}) \in D[\mathrm{x}]$ and $c(\mathrm{x}) \in F(\mathrm{x})$ are such that $d(\mathrm{x})a(\mathrm{x}) + u(\mathrm{x})c(\mathrm{x})$ is in $F[\mathrm{x}]$. Also for each $g(\mathrm{x}) \in F[\mathrm{x}]$, $(0, g(\mathrm{x}))(d(\mathrm{x}), c(\mathrm{x})) = (0, d(\mathrm{x})g(\mathrm{x})) \in R[\mathrm{x}]$. As $d(\mathrm{x})a(\mathrm{x})$ is always in $F[\mathrm{x}]$, the only restriction is that $u(\mathrm{x})c(\mathrm{x})$ must be in $F[\mathrm{x}]$. If $u(\mathrm{x})$ is constant modulo M, $c(\mathrm{x})$ must be in $F[\mathrm{x}]$ and thus in this case the inverse of J is trivial, i.e., $(R[\mathrm{x}] : J) = R[\mathrm{x}]$. On the other hand, if $u(\mathrm{x})$ is not constant modulo M, then the inverse of J is simply $D[\mathrm{x}](+)(1/\overline{u}(\mathrm{x}))F[\mathrm{x}]$. Note that each element of this fractional ideal will multiply all of $M[\mathrm{x}](+)F[\mathrm{x}]$ into $R[\mathrm{x}]$. Also note that if $v(\mathrm{x}) \in \mathcal{U}(D)$ is such that $\overline{u}(\mathrm{x})$ divides $\overline{v}(\mathrm{x})$, then $v(\mathrm{x})$ will be in the ideal $I = u(\mathrm{x})D[\mathrm{x}] + M[\mathrm{x}]$ as this ideal is the inverse image of $\overline{u}(\mathrm{x})F[\mathrm{x}]$. Thus, $J_v = I(+)F[\mathrm{x}]$.

For (4) and (5), we start with the maximal t-ideals of $R[\mathrm{x}]$ that contain $M[\mathrm{x}](+)F[\mathrm{x}]$. Since D is quasi-local, the maximal ideals of $D[\mathrm{x}]$ that contain $M[\mathrm{x}]$ are of the form $N = u(\mathrm{x})D[\mathrm{x}] + M[\mathrm{x}]$ where $u(\mathrm{x})$ is a polynomial with unit content whose image in $F[\mathrm{x}]$ is an irreducible polynomial. By the argument given in the previous paragraph, the corresponding ideal $N' = N(+)F[\mathrm{x}]$ is divisorial. Thus, N' is both divisorial and a maximal ideal of $R[\mathrm{x}]$.

Now consider the primes of the form $Q = P(+)F[\mathrm{x}]$ where P is an upper to zero of $D[\mathrm{x}]$ that contains a polynomial which is a nonzero constant modulo $M[\mathrm{x}]$. Let P be such an upper, and let $v(\mathrm{x}) \in P \cap \mathcal{U}(D)$ be (a nonzero) constant modulo M. By (3), for any $m \in M\{0\}$ and $a(\mathrm{x}), b(\mathrm{x}) \in F[\mathrm{x}]$, the ideal generated by $(v(\mathrm{x}), a(\mathrm{x}))$ and $(m, b(\mathrm{x}))$ has a trivial inverse. Since $(v(\mathrm{x}), 0) \in Q$ is regular and Q is clearly minimal over $(v(\mathrm{x}), 0)$, Q is a t-ideal of $R[\mathrm{x}]$ [**20**, Theorem 3.1]. It follows that Q is a maximal t-ideal of $R[\mathrm{x}]$. Also note that since no nonzero element of M is contained in P, $R[\mathrm{x}]_Q$ is isomorphic to $D[\mathrm{x}]_P$, a rank-one discrete valuation domain. This completes the proof of statements (4) and (5).

All that is left is to prove (7) and the last conclusion in statement (6) that $R[\mathrm{x}]_{N'}$ is a Mori ring for $N' = N(+)F[\mathrm{x}]$ a maximal (divisorial) ideal where $N = u(\mathrm{x})D[\mathrm{x}] + M[\mathrm{x}]$ with $u(\mathrm{x}) \in \mathcal{U}(D)$ an irreducible modulo M (and not constant modulo M).

A polynomial ring is always a McCoy ring, i.e., each finitely generated ideal that contains only zero divisors has a nonzero annihilator ([**22**, Proposition 6] and [**13**, Theorem 1]). Hence, each semi-regular ideal of

$R[\mathrm{x}]$ is regular. Thus, by Theorem 3.1, to prove that $R[\mathrm{x}]$ is Mori all we need show is that each regular element of $R[\mathrm{x}]$ is contained in at most finitely many maximal t-ideals and that localizing at a maximal t-ideal yields a Mori ring. This task is made somewhat simpler by the fact that each regular ideal of $R[\mathrm{x}]$ is generated by the regular elements it contains, i.e., $R[\mathrm{x}]$ is a Marot ring [**12**, Section 7]. We first show that each regular element is contained in at most finitely many maximal t-ideals.

Let $(v(\mathrm{x}), a(\mathrm{x}))$ be a regular element of $R[\mathrm{x}]$. Then $v(\mathrm{x})$ is a nonconstant polynomial with unit content (and $a(\mathrm{x})$ is an arbitrary element of $F[\mathrm{x}]$). Since each upper to zero of $D[\mathrm{x}]$ is the contraction of a prime ideal of $K[\mathrm{x}]$ to $D[\mathrm{x}]$, $(v(\mathrm{x}), a(\mathrm{x}))$ is contained in at most finitely many maximal t-ideals of the form $P(+)F[\mathrm{x}]$ where P is one of the special types of uppers discussed above. The only other maximal t-ideals are the ideals of form $N(+)F[\mathrm{x}]$ where $N = u(\mathrm{x})D[\mathrm{x}] + M[\mathrm{x}]$ with $\overline{u}(\mathrm{x})$ irreducible in $F[\mathrm{x}]$. As $\overline{v}(\mathrm{x})$ is contained in at most finitely many primes of $F[\mathrm{x}]$, $(v(\mathrm{x}), a(\mathrm{x}))$ is contained in at most finitely many of this latter type of maximal t-ideal.

As proved above, if $Q = P(+)F[\mathrm{x}]$ where P is an upper to zero, then $R[\mathrm{x}]_Q$ is isomorphic to $D[\mathrm{x}]_P$ since no nonzero element of M is contained in P. As $D[\mathrm{x}]_P$ is a rank-one discrete valuation domain, it is Mori.

For the remainder of the proof we suppress the "(x)" when denoting polynomials and quotients of polynomials.

Let $N = uD[\mathrm{x}] + M[\mathrm{x}]$ where $\overline{u}$ is irreducible in $F[\mathrm{x}]$ and let $N' = N(+)F[\mathrm{x}]$. All we have to do to complete the proof is show that $R[\mathrm{x}]_{N'}$ is a Mori ring. The proof of this is quite involved. What we will do is determine the form that the inverse of a regular ideal of $R[\mathrm{x}]$ can take.

First note that since $Z(R[\mathrm{x}]) = M[\mathrm{x}](+)F[\mathrm{x}]$ is contained in N', $R[\mathrm{x}]_{N'} = D[\mathrm{x}]_N(+)F[\mathrm{x}]_{(\overline{u})}$ is contained in $T(R[\mathrm{x}])$. Let J be a regular ideal of $R[\mathrm{x}]$ that is contained in N', and let $J' = JR[\mathrm{x}]_{N'}$, $\mathcal{J} = \{(h, b) \in J \mid h \in \mathcal{U}(D)\}$, $\mathcal{J}_d = \{h \mid (h, b) \in \mathcal{J} \text{ for some } b \in F[\mathrm{x}]\}$ and $\mathcal{J}_f = \{b \mid (h, b) \in \mathcal{J} \text{ for some } h\}\backslash\{0\}$. Since $R[\mathrm{x}]$ is a Marot ring, the set $\mathcal{J}$ will generate J as an ideal of $R[\mathrm{x}]$ and the localization of J at N'. Let $(h, b) \in \mathcal{J}$. Since $h \in N$, there is a positive integer n and polynomials $v \in \mathcal{U}(D)\backslash N$ and $p \in M[\mathrm{x}]$ such that $\overline{h} = \overline{v}\,\overline{u}^n$ and

$h = vu^n + p$. We will refer to these as the "standard forms" of h. Similarly, for each $b \in \mathcal{J}_f$ there is a unit $w \in F[\mathrm{x}]_{(\overline{u})}$ and nonnegative integer k such that $b = w\overline{u}^k$. We call this the "standard form" for b. The integers n and k will be called the "standard powers" of h and b, respectively. Among all members $h \in \mathcal{J}_d$, let β denote the minimum integer n such that the standard power of h is n. Similarly, for all members $b \in \mathcal{J}_f$, let γ denote the smallest integer m such that the standard power of b is m. Consider the sum $(h, b) + (k, c)$ where the standard power for h is β and the standard power for c is γ. If the standard power for k is greater than β, then the standard power of $h + k$ will be β. Similarly, if the standard power of b is greater than γ, then the standard power of $b + c$ will be γ. Hence, there is an element $(h, b) \in \mathcal{J}$ such that the standard power of h is β and the standard power of b is γ. The remainder of the proof is based on this particular choice. Note that, if $\gamma \neq \beta$, then we must have $\gamma < \beta$ since the second component of $(h, b)(1, 1) = (h, b + \overline{h}) \in \mathcal{J}$ is a unit multiple of $\overline{u}^{\,\beta}$ otherwise.

Let $t \in D[\mathrm{x}]$ be a gcd for the set $\mathcal{J}_d$ as polynomials in $K[\mathrm{x}]$. For each $h \in \mathcal{J}_d$, we have a polynomial $s \in K[\mathrm{x}]$ such that $st = h$. Since h has unit content, the Dedekind-Mertens content formula [**7**, Theorem 28.1] implies that there is a nonnegative integer m such that $C(s)C(t)^{m+1} = C(st)C(t)^m = C(t)^m$. Since D is integrally closed, we have $D = C(st) \subseteq C(s)C(t) \subseteq D$. Thus, the content of t is an invertible ideal of D. As D is quasilocal, $C(t)$ must be a principal ideal of D. It follows that we may assume t has unit content in D and from this that it divides each member of $\mathcal{J}_d$ in $D[\mathrm{x}]$.

While we may not assume that t is in the ideal generated by $\mathcal{J}_d$, there is a nonzero element $q \in M$ such that qt is in the ideal generated by $\mathcal{J}_d$.

Since each element of $M[\mathrm{x}]$ kills $F[\mathrm{x}]$ and t divides each member of $\mathcal{J}_d$, $(R[\mathrm{x}]_{N'} : J')$ contains $(1/t)M[\mathrm{x}]_N(+)(1/\overline{u}^{\beta})F[\mathrm{x}](\overline{u})$. Note that the inverse of J' also contains $D[\mathrm{x}]_N(+)(1/\overline{u}^{\beta})F[\mathrm{x}]_{(\overline{u})}$.

Let $(r, c) \in (R[\mathrm{x}]_{N'} : J')$. It is clear that r must multiply each element of $\mathcal{J}_d$ into $D[\mathrm{x}]_N$. While t need not be in the ideal generated by $\mathcal{J}_d$, the element qt is. Thus, r can be written in the form s'/qt for some $s' \in D[\mathrm{x}]$. Since q is a nonzero element of D and each element of $\mathcal{J}_d$ has unit content, s' must be in $qD[\mathrm{x}]$ and therefore $s' = qs$ for some

$s \in D[\mathrm{x}]$ and we have that $r \in (1/t)D[\mathrm{x}]_N$. Hence, $(R[\mathrm{x}]_{N'} : J')$ is contained in $(1/t)D[\mathrm{x}]_N(+)F(\mathrm{x})$.

Let $(m, d) \in N'$ be a regular element of $R[\mathrm{x}]$ with $m = zu^\sigma + q'$ in standard form. If $\sigma < \beta$, then $(m, d)[(0)(+)(1/\overline{u}^\beta)F[\mathrm{x}]_{(\overline{u})}]$ is not contained in $R[\mathrm{x}]_{N'}$. Also, if m is not a multiple of t, then $m(1/t)M[\mathrm{x}]_N$ is not contained in $D[\mathrm{x}]_N$. Thus, if $(m, d) \in J'_v \cap R[\mathrm{x}]$, then m is a multiple of t, and σ, the standard power of m, is greater than or equal to β. These are not the only restrictions that we must consider, but at least we have that the inverse of J' is dependent on the gcd of $\mathcal{J}_d$ and the minimal standard power, β.

Let $\overline{t} = \overline{w}\,\overline{u}^\alpha$ and $t = wu^\alpha + m$ be the standard forms for t. Since t divides each member of $\mathcal{J}_d$, $0 \le \alpha \le \beta$. If $\alpha \le \gamma$ (the minimal standard power for elements of $\mathcal{J}_f$), then for each $(k, c) \in \mathcal{J}$ we have that $(1/\overline{t})$ will multiply c into $F[\mathrm{x}]_{(\overline{u})}$. Thus, in this case the inverse of J' will simply be $(1/t)D[\mathrm{x}]_N(+)(1/\overline{u}^\beta)F[\mathrm{x}]_{(\overline{u})}$. So, from here on, we assume $\gamma < \alpha$.

Let $\nu = \alpha - \gamma$. A simple calculation shows that $(u^\nu/t)D[\mathrm{x}]_N(+) \times (1/\overline{u}^\beta)F[\mathrm{x}]_{(\overline{u})}$ is contained in the inverse of J'. Note that if $(\ell, e) \in N'$ is such that ℓ is a multiple of t with standard form $vu^\beta + p'$ and e is a unit multiple of $\overline{u}^{\,\mu}$ where $\mu < \gamma$, then $(u^\nu/t, 0)(\ell, e)$ is not in $R[\mathrm{x}]_{N'}$ since the second component in the product is a unit multiple of $\overline{u}^{\nu+\mu-\alpha}$, an element which is not in $F[\mathrm{x}]_{(\overline{u})}$ since the exponent on $\overline{u}$ is negative.

Let r, v be elements of $D[\mathrm{x}]$ with v a unit in $D[\mathrm{x}]_N$, and let $d \in F(\mathrm{x})$. No matter what r is, r/vt will multiply each element of $\mathcal{J}_d$ into $D[\mathrm{x}]_N$ since t divides element in $\mathcal{J}_d$. The question is what restrictions apply to have $(r/vt, d)$ in the inverse of J'. Assume the standard form of r is $zu^\sigma + q'$. By the above, if $\sigma \ge \nu$, then we simply need to have d in $(1/\overline{u}^\beta)F[\mathrm{x}]_{(\overline{u})}$. Thus, we may assume $0 \le \sigma < \nu$.

We must have $(r/vt, d)(h, b) \in R[\mathrm{x}]_{N'}$. Thus $hd + (r/vt)b$ must be in $F[\mathrm{x}]_{(\overline{u})}$. Solving for d we have that d has the form $f/\overline{h} - \overline{r}b/\overline{v}\,\overline{t}\,\overline{h}$ for some $f \in F[\mathrm{x}]_{(\overline{u})}$. The element $(0, f/\overline{h})$ is in $(0)(+)(1/\overline{u}^\beta)F[\mathrm{x}]_{(\overline{u})}$ which we know is part of $(R[\mathrm{x}]_{N'} : J)$. Hence, we may assume $d = -\overline{r}b/\overline{v}\,\overline{t}\,\overline{h}$. Note that d is in $(\overline{u}^{\sigma+\gamma}/\overline{u}^{\alpha+\beta})F[\mathrm{x}]_{(\overline{u})}$. Now let (k, c) be an element of $\mathcal{J}$ and consider the product $(r/vt, d)(k, c)$. The second component of the product is the only one we need to look at; it is $(r/vt)c + kd = (r/vt)[(\overline{h}\,c - \overline{k}\,b)/\overline{h}]$. Since v is a unit in $D[\mathrm{x}]_N$, we may safely ignore it at this point. The element $(\overline{h}\,c - \overline{k}\,b)/\overline{h}$ is in $F[\mathrm{x}]_{(\overline{u})}$

and at least $\overline{u}^{\beta+\gamma}$ divides the numerator. Hence, at least $\overline{u}^{\,\gamma}$ divides the entire fraction in $F[\mathrm{x}]_{(\overline{u})}$. If that is the largest power of $\overline{u}$ that divides all such expressions obtained from elements of $\mathcal{J}$, then the inverse of J' is $((u^\nu/t)D[\mathrm{x}] + (1/t)M[\mathrm{x}])_N(+)(1/\overline{u}^\beta)F[\mathrm{x}]_{(\overline{u})}$. But it is possible that some larger power of $\overline{u}$ divides all of the expressions $(\overline{h}\,c - \overline{k}\,b)/\overline{h}$. A trivial case is when J is the principal ideal generated by (h, b). In that case each such expression is 0. Let $\overline{u}^\theta$ be the largest power of $\overline{u}$ that divides $(\overline{h}\,c-\overline{k}\,b)/\overline{h}$ for each $(k, c) \in \mathcal{J}$. Then, given the standard form of r and the "short" form of d above, $(r/vt, d)$ is in $(R[\mathrm{x}]_{N'} : J')$ if and only if $\sigma + \theta \geq \alpha$. Let $\pi \geq 0$ be the smallest nonnegative value for such a σ. It follows that the inverse of J' contains $(u^\pi/t, 0)(1, -b/\overline{h})R[\mathrm{x}]_{N'}$.

Here is the full description of the inverse of J' broken down into the cases based on the various values above.

$$
\begin{aligned}
&(R[\mathrm{x}]_{N'} : J')\\
&= \begin{cases}
(1/t)D[\mathrm{x}]_N(+)(1/\overline{u}^\beta)F[\mathrm{x}]_{(\overline{u})} & \text{if } \alpha \leq \gamma\\[2ex]
\big((u^\nu/t)D[\mathrm{x}] + (1/t)M[\mathrm{x}]\big)_N(+)(1/\overline{u}^\beta)F[\mathrm{x}]_{(\overline{u})} & \text{if } \theta = \gamma < \alpha\\[2ex]
\big((u^\nu/t)D[\mathrm{x}] + (1/t)M[\mathrm{x}]\big)_N(+)(1/\overline{u}^\beta)F[\mathrm{x}]_{(\overline{u})} & \text{if } \gamma < \alpha \text{ and}\\
\qquad\qquad + (u^\pi/t, 0)(1, -b/\overline{h})R[\mathrm{x}]_{N'} & \qquad \gamma < \theta.
\end{cases}
\end{aligned}
$$

Finally we may conclude that $R[\mathrm{x}]_{N'}$ satisfies ACC on regular divisorial ideals since, given a regular divisorial ideal J', to make a larger divisorial ideal requires at least one of the following to occur: (i) a lower degree for the gcd, which must be a factor of the gcd for $\mathcal{J}_d$; (ii) a smaller value of β and/or γ; or (iii) a smaller value of θ. For each we have at most finitely many "reductions." Hence $R[\mathrm{x}]_{N'}$ is a Mori ring and therefore by Theorem 3.1, $R[\mathrm{x}]$ is a Mori ring. $\square$

Theorem 4.2. *Let D be an integrally closed domain and $R = D(+)B$ where $B = \sum F_i$ with $F_i = D/M_i$ for each $M_i \in \operatorname{Max}(D)$. Then the following hold.*

(a) $R = T(R) = Q_0(R)$.

(b) $R[\mathrm{x}]$ *is a Mori ring if and only if* $\operatorname{Max}(D)$ *is finite.*

Proof. As in Theorem 4.1, the only regular elements of R are the elements of the form (u, b) where u is a unit of D and b is arbitrary. Also each proper ideal of R has a nonzero annihilator. Thus, $R = T(R) = Q_0(R)$ and the only regular elements of $R[\mathrm{x}]$ are those of the form (u, b) where $u \in D[\mathrm{x}]$ has unit content. Thus, $T(R[\mathrm{x}]) = D(\mathrm{x})(+)B(\mathrm{x})$ where $B(\mathrm{x}) = \sum F_i(\mathrm{x})$.

Since $R = T(R)$, each of its maximal ideals is contained in $Z(R)$. Thus, by Theorem 3.6, a necessary condition for $R[\mathrm{x}]$ to be Mori is for R to have only finitely many maximal ideals. As $M_i(+)B$ is a maximal ideal for each $M_i \in \mathrm{Max}\,(D)$, $\mathrm{Max}\,(D)$ is finite if $R[\mathrm{x}]$ is Mori.

For the converse, assume $\mathrm{Max}\,(D)$ is finite. As in the proof of Theorem 4.1, if M is a maximal ideal of D, then each maximal ideal of $D[\mathrm{x}]$ that contains $M[\mathrm{x}]$ is of the form $N = uD[\mathrm{x}] + M[\mathrm{x}]$ where u is a polynomial of unit content whose image in $(D/M)[\mathrm{x}]$ is irreducible. The proof of Theorem 4.1 shows that $N' = N(+)F[\mathrm{x}]$ is both maximal and divisorial with $R[\mathrm{x}]_{N'}$ a Mori ring.

Also we still have that if P is an upper to zero of $D[\mathrm{x}]$ such that $P(+)F[\mathrm{x}]$ is a maximal t-ideal, then $R[\mathrm{x}]_{P(+)F[\mathrm{x}]}$ is isomorphic to $D[\mathrm{x}]_P$, a rank-one discrete valuation domain.

Let $f \in D[\mathrm{x}]$ be a nonconstant polynomial with unit content, and let q be a nonzero element of D such that f and q are not comaximal in $D[\mathrm{x}]$. Since f has unit content, $(1/f)D[\mathrm{x}]$ is contained in $D(\mathrm{x})$. On the other hand, $(1/q)D[\mathrm{x}]$ is contained in $K[\mathrm{x}]$. It follows that $(D[\mathrm{x}] : (f, q)) = D[\mathrm{x}]$ since $K[\mathrm{x}] \cap D(\mathrm{x}) = D[\mathrm{x}]$. From this we have that for each $b, c \in B[\mathrm{x}]$, $(R[\mathrm{x}] : ((f, b), (q, c)))$ is contained in $D[\mathrm{x}](+)B(\mathrm{x})$. Let M_i be a maximal ideal of D. Then with regard to q we have two distinct possibilities. We could have $q \in M_i$, in which case $qF_i = (0)$. Or we could have $q \notin M_i$, in which case $qF_i = F_i$. Split B into the internal direct sum $C \oplus E$ where C consists of the sum of those F_i's where $qF_i = F_i$ and E consists of the sum of those F_j's where $qF_j = (0)$. Then $(R[\mathrm{x}] : ((f, b), (q, c))) = D[\mathrm{x}](+)(C[\mathrm{x}] \oplus (1/f)E[\mathrm{x}])$ and $((f, b), (q, c))_v = (f(\mathrm{x})D[\mathrm{x}] + J[\mathrm{x}])(+)B[\mathrm{x}]$ where J is the intersection of those M_j's that contain q and $\overline{f}$ is not a constant in $F_j[\mathrm{x}]$ (equivalently, f is not comaximal with $M_j[\mathrm{x}]$). Note that the only primes which contain such an ideal are the maximal ideals of the form $uD[\mathrm{x}] + M_j[\mathrm{x}]$ where u is a polynomial with unit content whose image in $F_j[\mathrm{x}]$ is an irreducible factor of $\overline{f}$. For a given M_j, there can be only finitely many such maximal ideals that contain f since $\overline{f}$ has only finitely many irreducible factors in $F_j[\mathrm{x}]$. Each of these ideals is divisorial. Since at most finitely many uppers to zero of $D[\mathrm{x}]$ contain f, at most finitely many maximal t-ideals based on uppers to zero contain (f, b). Hence, each regular nonunit of $R[\mathrm{x}]$ is contained in only finitely many maximal t-ideals and, therefore, $R[\mathrm{x}]$ is a Mori ring. $\square$

For a final example, we construct a nonreduced Mori ring R such that $R[\mathrm{x}]$ is Mori but R is not its own total quotient ring. Recall that a ring R is said to be a Prüfer ring if each finitely generated regular ideal is invertible [**9**].

Example 4.3. Let V be a valuation domain with principal maximal ideal M and dimension greater than one, and let $P = \cap M^n$ be the prime ideal that sits just below M. Let $R = V(+)L$ where L is the quotient field of V/P. Then the following hold.

(a) $T(R) = V_P(+)L = Q_0(R)$ with $\dim(T(R)) = \dim(V) - 1$.

(b) Each regular ideal of R is principal, so R is a Marot ring and a Prüfer ring. Also R is a McCoy ring.

(c) R is an integrally closed Mori ring.

(d) $R[\mathrm{x}]$ is a Mori ring but $T(R)$ is not zero-dimensional.

Proof. Let $M = aV$. Obviously, the zero divisors on L as a V-module is the prime ideal P. In fact, each element of P annihilates L. Hence, $P(+)L = Z(R) = \mathrm{Ann}(L)$. Thus R is a McCoy ring. Moreover, $T(R) = V_P(+)L = Q_0(R)$. Also, R is integrally closed since V is integrally closed [**12**, Corollary 25.7]. The regular ideals of R are all of the form $IR = I(+)L$ where I is an ideal of V that is not contained in P. The only such ideals are the powers of M, each of which is a principal ideal. Hence, each regular ideal of R is principal and a power of $MR = M(+)L$. It follows that R is a Mori ring, a Q_0-Mori ring, a Marot ring and a Prüfer ring. Thus, $R(\mathrm{x}) = V(\mathrm{x})(+)L(\mathrm{x})$ is a Prüfer ring [**10**, Theorem 3.10].

Each prime $Q \subseteq P$ gives rise to a prime $QV_P(+)L$ of $T(R)$ and each prime of $T(R)$ has this form. Hence $\dim(T(R)) = \dim(V) - 1 > 0$.

Consider the polynomial ring $R[\mathrm{x}] = V[\mathrm{x}](+)L[\mathrm{x}]$. Since V is a valuation domain, the primes of $V[\mathrm{x}]$ are somewhat easy to describe. Of course there are the primes of the form $Q[\mathrm{x}]$ where Q is a prime of V. The other primes are all of the form $N = u(\mathrm{x})V[\mathrm{x}] + Q[\mathrm{x}]$ where $u(\mathrm{x})$ is a polynomial with unit content which is irreducible modulo Q. Thus, the primes of $R[\mathrm{x}]$ are of the form $Q[\mathrm{x}](+)L[\mathrm{x}]$ and $N(+)L[\mathrm{x}]$.

The ideal $MR[\mathrm{x}]$ is a regular ideal of $R[\mathrm{x}]$ which is also a maximal t-ideal [**20**, Theorem 5.5]. Since $P(+)L = \mathrm{Ann}\,(L)$, $MR[\mathrm{x}]$ is the only regular prime ideal of $R[\mathrm{x}]$ that does not contain a polynomial of unit content. Since M is the only maximal ideal of V, each polynomial outside $MV[\mathrm{x}]$ has unit content. Thus $V[\mathrm{x}]_{M[\mathrm{x}]} = V(\mathrm{x})$. Since V is a valuation domain, each polynomial in $V[\mathrm{x}]$ can be factored as a constant times a polynomial with unit content. It follows that each ideal of $V(\mathrm{x})$ is extended from an ideal of V. In particular, since $P(+)L = Z(R) = \mathrm{Ann}\,(L)$, the only regular ideals of $V(\mathrm{x})$ are those that are powers of M. Hence, $R[\mathrm{x}]_{MR[\mathrm{x}]}$ is a Mori ring.

The other maximal t-ideals of $R[\mathrm{x}]$ must be of types II and III, and each of these must contain an irreducible polynomial of unit content since V is a valuation domain.

From here on in the proof we will suppress the (x) and simply denote polynomials (and occasionally quotients of polynomials) with single letters.

Let $f \in V[\mathrm{x}]$ be an irreducible polynomial of unit content, and let $P_f = fV[\mathrm{x}]$ be the upper to zero generated by f. We have two cases to consider depending on whether or not f is constant modulo $P[\mathrm{x}]$ or not.

Case 1. f is constant modulo $P[\mathrm{x}]$.

In this case f modulo P is a (nonzero) constant. Thus, $(1/f)L[\mathrm{x}] = L[\mathrm{x}]$. Hence, $(1/f)R[\mathrm{x}] = (1/f)V[\mathrm{x}](+)L[\mathrm{x}]$. Since $C(f) = V$, $(V[\mathrm{x}] : (f, b)) = (1/f)V[\mathrm{x}] \cap (1/b)V[\mathrm{x}] = V[\mathrm{x}]$ for each nonzero $b \in V$. Thus, in this case $P_f(+)L[\mathrm{x}]$ is a maximal t-ideal of $R[\mathrm{x}]$. Since $(0)(+)L[\mathrm{x}]$ is a minimal prime of $R[\mathrm{x}]$, $P_f(+)L[\mathrm{x}]$ is of type II.

Case 2. f is not constant modulo $P[\mathrm{x}]$.

In this case $(1/f)L[\mathrm{x}]$ properly contains $L[\mathrm{x}]$. Thus, $(1/f)V[\mathrm{x}](+)L[\mathrm{x}]$

is properly contained in $(1/f)R[\mathrm{x}] = (1/f)V[\mathrm{x}](+)(1/f)L[\mathrm{x}]$. It is still the case that, for each nonzero $b \in V$, $(V[\mathrm{x}] : (f, b)) = V[\mathrm{x}]$. Let Q be a prime of $V[\mathrm{x}]$ minimal over $fV[\mathrm{x}] + PV[\mathrm{x}]$. We will show that $N = Q(+)L[\mathrm{x}]$ is a t-prime by showing that $J = (fV[\mathrm{x}]+PV[\mathrm{x}])(+)L[\mathrm{x}]$ is a regular divisorial ideal of $R[\mathrm{x}]$. Since $PL = (0)$ and $(V[\mathrm{x}] : fV[\mathrm{x}] + P[\mathrm{x}]) = V[\mathrm{x}]$, $(R[\mathrm{x}] : J) = V[\mathrm{x}](+)(1/f)L[\mathrm{x}]$. Let $(g, b) \in J_v$. As J contains $(0)(+)L[\mathrm{x}]$, we may assume $b = 0$. We have $(g/f)L[\mathrm{x}] \subseteq L[\mathrm{x}]$, so that we must have that f divides g as elements of $L[\mathrm{x}]$. Since f has unit content in V, its image in V/P also has unit content. Hence, there is a polynomial $h \in V[\mathrm{x}]$ such that $fh - g \in P[\mathrm{x}]$. But this is the same as saying that g is in the ideal of $V[\mathrm{x}]$ generated by f and P. Thus, $(g, 0)$ is in J and we have that J is divisorial and N is a t-prime. By the above, N must be of the form $(jV[\mathrm{x}] + P[\mathrm{x}])(+)L[\mathrm{x}]$ for some polynomial j with unit content that is irreducible (and not constant) modulo P. Note that j must divide f modulo P. Such a prime must be a maximal t-ideal of $R[\mathrm{x}]$ as the only prime of $V[\mathrm{x}]$ that might properly contain $jV[\mathrm{x}] + P[\mathrm{x}]$ is an ideal of the form $kV[\mathrm{x}] + M[\mathrm{x}]$ where k has unit content and, modulo M, is both irreducible and a divisor of j. Since $M(+)L$ is a maximal t-ideal of R, any ideal of the form $(kV[\mathrm{x}] + M[\mathrm{x}])(+)L[\mathrm{x}]$ is not a t-ideal of $R[\mathrm{x}]$. In particular, the ideal $((k, 0), (a, 0))$ has a trivial inverse. Thus, N is a maximal t-ideal of $R[\mathrm{x}]$. In some cases, N is also a maximal ideal. For example, suppose $j = a\mathrm{x} + 1$. Then $jV[\mathrm{x}] + M[\mathrm{x}] = V[\mathrm{x}]$, but $jV[\mathrm{x}] + P[\mathrm{x}]$ is prime.

The next task is to show that each regular element of $R[\mathrm{x}]$ is contained in only finitely many maximal t-ideals. To this end, let (g, b) be a regular element of $R[\mathrm{x}]$. Since V is a valuation domain and P is the prime just below the maximal ideal, it must be that $C(g)$ properly contains P. Moreover, there is a polynomial h with unit content in V and an integer $n \geq 0$ such that $g = a^n h$. Since a is not in P, b can be rewritten as $a^n c$ for some polynomial $c \in L[\mathrm{x}]$. Thus, our original element (g, b) can be factored as the product $(a^n, 0)(h, c)$. The only maximal t-ideal of $R[\mathrm{x}]$ that contains $(a^n, 0)$ is $MR[\mathrm{x}]$. For (h, c), first factor h into irreducible polynomials, each with unit content in V. Such a factorization is unique modulo rearrangements and multiplication by units from V. Now an irreducible factor of h can be constant modulo P, irreducible modulo P, or reducible modulo P. Let h_i be one of these irreducible factors of h. If h_i is constant modulo P, then $P_{h_i}(+)L[\mathrm{x}]$ is a maximal t-ideal of $R[\mathrm{x}]$ that contains (g, b). If h_i is irreducible modulo P, then $(h_iV[\mathrm{x}] + P[\mathrm{x}])(+)L[\mathrm{x}]$ is a maximal t-ideal of $R[\mathrm{x}]$ that contains (g, b). Finally if h_i is reducible modulo P, there are finitely many polynomials $h_{i,j}$ which have unit content and divide h modulo P and are irreducible both in $V[\mathrm{x}]$ and modulo P. Each of these polynomials together with $P[\mathrm{x}]$ generates a single maximal t-ideal. No other maximal t-ideal can contain (g, b). Hence, it is contained in only finitely many maximal t-ideals of $R[\mathrm{x}]$.

By Theorem 3.1, we may complete the proof by showing that localizing $R[\mathrm{x}]$ at each maximal t-ideal yields a Mori ring. We have already done this for $MR[\mathrm{x}]$.

Let N be a maximal t-ideal. If N is of type II, $N = P_f(+)L[\mathrm{x}]$ for some irreducible polynomial $f \in V[\mathrm{x}]$ that is constant modulo P. Since $P_f \cap V = (0)$, $N_{V\setminus P}$ is a type II maximal t-ideal of $T(R)[\mathrm{x}]$. Thus, by the proof of Theorem 4.1, $R[\mathrm{x}]_N$ is a Mori ring.

Next we consider the case that N is of type III. By the above, $N = (fV[\mathrm{x}]+P[\mathrm{x}])(+)L[\mathrm{x}]$ for some irreducible polynomial f of unit content that is not a constant modulo P. In this case, N contains the set of zero divisors of $R[\mathrm{x}]$. Thus, $R[\mathrm{x}]_N$ is contained in the total quotient ring of $R[\mathrm{x}]$. Moreover, no regular element of R is contained in N. Therefore, $R[\mathrm{x}]_N = T(R)[\mathrm{x}]_{N'}$ where $N' = (fV_P[\mathrm{x}]+PV_P[\mathrm{x}](+)L[\mathrm{x}] = NR[\mathrm{x}]_{V\setminus P}$. By either the proof of Theorem 3.2 or [**19**, Corollary 2.14], N' is a maximal t-ideal of $T(R)[\mathrm{x}]$. Since $T(R) = V_P(+)L$, $T(R)[\mathrm{x}]$ is a Mori ring by Theorem 4.1. Thus, by [**19**, Corollary 2.14], $R[\mathrm{x}]_N = T(R)[\mathrm{x}]_{N'}$ is a Mori ring. □

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