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COMMON INVARIANT SUBSPACES FOR FINITELY QUASINILPOTENT COLLECTIONS OF POSITIVE OPERATORS ON A BANACH SPACE WITH A SCHAUDER BASIS

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ABSTRACT. We prove that if $\mathcal{C} \neq \{0\}$ is a collection of continuous positive operators on a Banach space with a Schauder basis that is finitely quasinilpotent at a nonzero positive vector, then $\mathcal{C}$ and its positive commutant $\mathcal{C}'_+$ have a common nontrivial invariant closed subspace.

In 1995, Y.A. Abramovich, C.D. Aliprantis and O. Burkinshaw [**4**] showed that every continuous positive operator S on a Banach space X with a Schauder basis which commutes with a nonzero continuous positive operator T on X that is quasinilpotent at a nonzero positive vector has a nontrivial invariant closed subspace. In this paper, using the Abramovich-Aliprantis-Burkinshaw technique based on the idea from [**2, 4**], we extend the result to a collection $\mathcal{C}$ of operators on X and obtain the result that $\mathcal{C}$ and its positive commutant $\mathcal{C}'_+$ have a common nontrivial invariant closed subspace. In particular, all continuous positive operators on a Banach space X with a Schauder basis which commute with a nonzero continuous positive operator T on X that is quasinilpotent at a nonzero positive vector have a common nontrivial invariant closed subspace.

In order to do this, we first recall some of the basic terminologies and facts from [**4, 5**] and others. For the notions and facts not stated in the text we refer to [**1–15**] and so on.

In this note, the word *operator* will be synonymous with *linear transformation*. Let X be a Banach space and $B(X)$ the Banach

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algebra of all continuous operators on X. Let W and $\mathcal{C}$ be subsets of X and $B(X)$ respectively. Define

$$\|W\| = \sup\{\|x\|;\ x \in W\} \quad \text{and} \quad \mathcal{C}W = \{Tx;\ T \in \mathcal{C},\ x \in W\}.$$

If W is a singleton $\{x\}$, we shall write $\mathcal{C}x$ instead of $\mathcal{C}\{x\}$. If $\mathcal{D}$ is another subset of $B(X)$, we will write $\mathcal{C}\mathcal{D} = \{TS;\ T \in \mathcal{C},\ S \in \mathcal{D}\}$. The powers $\mathcal{C}^n$ are defined inductively by $\mathcal{C}^1 = \mathcal{C}$, $\mathcal{C}^n = \mathcal{C}\mathcal{C}^{n-1}$ for all $n = 2, 3, \dots$.

A collection $\mathcal{F}$ of operators in $B(X)$ is said to be quasinilpotent at a vector $x_0 \in X$ if $\lim_{n\to\infty} \|\mathcal{F}^n x_0\|^{1/n} = 0$. A collection $\mathcal{C}$ of operators in $B(X)$ is said to be finitely quasinilpotent at a vector $x_0 \in X$ if every finite subset $\mathcal{F}$ of $\mathcal{C}$ is quasinilpotent at a vector x_0.

Let E be an ordered vector space. An operator T on E is said to be positive, in symbols $T \geq 0$, if $Tx \geq 0$ holds for each $x \geq 0$.

From now on, we consider only a Banach space X with a Schauder basis and fix a Schauder basis $\{e_n\}$ of X. It follows from [**4**] that X can be regarded as an ordered vector space equipped with the positive cone

$$C = \left\{x = \sum_{n=1}^{\infty} \alpha_n e_n;\ \alpha_n \geq 0 \text{ for each } n = 1, 2, \dots\right\},$$

that the functional $\{f_n\}$ defined by

$$f_n(x) = \alpha_n \quad \text{for each} \quad x = \sum_{n=1}^{\infty} \alpha_n e_n$$

is a continuous linear functional on X, and that f_n is also automatically positive with respect to the positive cone C. Moreover, the sequence of the continuous linear functional $\{f_n\}$ satisfies $f_n(e_m) = \delta_{nm}$.

Let $\mathcal{C}$ be a collection of continuous positive operators on X. We denote by $\mathcal{C}'_+$ the set of all continuous positive operators S on X such that $TS = ST$ for all $T \in \mathcal{C}$ and say that $\mathcal{C}'_+$ is the positive commutant of $\mathcal{C}$.

Moreover, we say that the continuous positive operator T on X has a nontrivial order hyperinvariant closed subspace if there exists a nontrivial closed subspace M of X such that M is invariant under all operators in the positive commutant of T.

Now we are in a position to give the main result.

Theorem 1. *Let $\mathcal{C} \neq \{0\}$ be a collection of continuous positive operators on a Banach space X with a Schauder basis $\{e_n\}$ that is finitely quasinilpotent at a nonzero positive vector $x_0 \in X$. Then $\mathcal{C}$ and its positive commutant $\mathcal{C}'_+$ have a common nontrivial invariant closed subspace.*

Proof. Since $x_0 > 0$, it is easy to see that there are an appropriate scalar $\lambda > 0$ and a positive integer n_0 such that $\lambda x_0 \geq e_{n_0} > 0$. It is clear that $\mathcal{C}$ is finitely quasinilpotent at λx_0. Let $\mathcal{G}$ be the multiplicative semigroup generated by $\mathcal{C}$, and let $\mathcal{A}$ be the subalgebra of $B(X)$ generated by $\mathcal{C}\mathcal{C}'_+$. Then $\mathcal{G} = \cup_{n=1}^{\infty} \mathcal{C}^n$, and $\mathcal{A}$ is the set of all operators of the form $\sum_{j=1}^{n} \lambda_j S_j G_j$ with $S_j \in \mathcal{C}'_+$, $G_j \in \mathcal{G}$ and scalars λ_j.

We consider two cases separately:

Case 1. If there is an operator $A_0 \in \mathcal{A}$ such that $A_0 e_{n_0} \neq 0$, then $\mathcal{A}e_{n_0} = \{Ae_{n_0};\ A \in \mathcal{A}\}$ is a nonzero linear manifold in X.

First we prove that $\mathcal{A}e_{n_0}$ is invariant under $\mathcal{C}$ and $\mathcal{C}'_+$. To this end, take $y \in \mathcal{A}e_{n_0}$, $T \in \mathcal{C}$ and $S \in \mathcal{C}'_+$. Then there is an operator $A \in \mathcal{A}$ such that $y = Ae_{n_0}$. It follows from the definition of $\mathcal{A}$ that there exist operators $S_1, S_2, \dots, S_n \in \mathcal{C}'_+$, $G_1, G_2, \dots, G_n \in \mathcal{G}$ and scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $A = \sum_{j=1}^{n} \lambda_j S_j G_j$. Thus, we have $TA = \sum_{j=1}^{n} \lambda_j S_j T G_j$. Since $TG_j \in \mathcal{G}$, it follows that $TA \in \mathcal{A}$. Consequently, $Ty = TAe_{n_0} \in \mathcal{A}e_{n_0}$. On the other hand, we have $SA = \sum_{j=1}^{n} \lambda_j S S_j G_j$. Observing $SS_j \in \mathcal{C}'_+$, we obtain $SA \in \mathcal{A}$. Consequently $Sy = SAe_{n_0} \in \mathcal{A}e_{n_0}$.

We now show that $\overline{\mathcal{A}e_{n_0}} \neq X$. Let P denote the natural projection from X onto the linear manifold generated by e_{n_0}. It is clear that $0 \leq Px \leq x$ holds whenever $0 \leq x \in X$. We claim that

$$PSGe_{n_0} = 0 \tag{1}$$

for all $S \in \mathcal{C}'_+$, $G \in \mathcal{G}$. To this end, we write $PSGe_{n_0} = ae_{n_0}$ for some $a \geq 0$. Since P is a positive operator and the composition of positive operators is also a positive operator, it follows that the estimate

$$0 \leq a^k e_{n_0} = (PSG)^k e_{n_0} \leq (SG)^k e_{n_0} \leq (SG)^k (\lambda x_0) \tag{2}$$

holds for any positive integer k. Since $G \in \mathcal{G}$, G is an operator of the form $T_1 T_2 \cdots T_m$, where $T_1, T_2, \ldots, T_m \in \mathcal{C}$. It follows from (2) and the definition of $\mathcal{C}'_+$ that the estimate

$$(3) \qquad 0 \le a^k e_{n_0} \le (ST_1T_2\cdots T_m)^k(\lambda x_0) = S^k(T_1T_2\cdots T_m)^k(\lambda x_0)$$

holds for any positive integer k. Since f_{n_0} is a positive functional on X, it follows from (3) that $0 \le a^k = f_{n_0}(a^k e_{n_0}) \le f_{n_0}(S^k(T_1T_2\cdots T_m)^k(\lambda x_0))$. Set $\mathcal{F} = \{T_1, T_2, \ldots, T_m\}$. Noticing that $\mathcal{C}$ is finitely quasinilpotent at λx_0, one can obtain $\lim_{n\to\infty} \|\mathcal{F}^n(\lambda x_0)\|^{1/n} = 0$ so that

$$\begin{aligned} 0 \le a &\le \|f_{n_0}\|^{1/k}\|S^k(T_1T_2\cdots T_m)^k(\lambda x_0)\|^{1/k} \\ &\le \|f_{n_0}\|^{1/k}\|S\|\,\|(T_1T_2\cdots T_m)^k(\lambda x_0)\|^{1/k} \\ &\le \|f_{n_0}\|^{1/k}\|S\|\,\|(\mathcal{F}^m)^k(\lambda x_0)\|^{1/k} \\ &= \|f_{n_0}\|^{1/k}\|S\|\,(\|\mathcal{F}^{km}(\lambda x_0)\|^{1/(km)})^m \longrightarrow 0 \end{aligned}$$

as $k \to \infty$, from which it follows that $a = 0$.

For every $y \in \mathcal{A}e_{n_0}$, the definition of $\mathcal{A}e_{n_0}$ implies that there is an operator $A \in \mathcal{A}$ such that $y = Ae_{n_0}$. Thus, by the definition of $\mathcal{A}$ there are operators $S_1, S_2, \ldots, S_n \in \mathcal{C}'_+$, $G_1, G_2, \ldots, G_n \in \mathcal{G}$ and scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $Ae_{n_0} = \sum_{j=1}^n \lambda_j S_j G_j e_{n_0}$. Thus by (1) we obtain $P(y) = P(Ae_{n_0}) = \sum_{j=1}^n \lambda_j PS_jG_je_{n_0} = 0$. Hence it is easy to see that $f_{n_0}(y) = f_{n_0}(Py) = 0$ for every $y \in \mathcal{A}e_{n_0}$. Consequently $f_{n_0}(y) = 0$ for every $y \in \overline{\mathcal{A}e_{n_0}}$. Observing that $f_{n_0}(e_{n_0}) = 1$, we obtain $\overline{\mathcal{A}e_{n_0}} \ne X$.

From the above we conclude that $\overline{\mathcal{A}e_{n_0}}$ is a common nontrivial invariant closed subspace for $\mathcal{C}$ and $\mathcal{C}'_+$.

Case 2. If $Ae_{n_0} = 0$ for all $A \in \mathcal{A}$, then $\operatorname{Ker}\mathcal{A} = \{x; Ax = 0 \text{ for all } A \in \mathcal{A}\}$ is a nonzero closed subspace in X. It is easy to see by the identity operator $I \in \mathcal{C}'_+$ that $\{0\} \ne \mathcal{C} \subset \mathcal{G} \subset \mathcal{A}$, from which it follows that $\operatorname{Ker}\mathcal{A} \ne X$.

It only remains to show that $\operatorname{Ker}\mathcal{A}$ is invariant under $\mathcal{C}$ and $\mathcal{C}'_+$. To this end, take $x \in \operatorname{Ker}\mathcal{A}$, $T \in \mathcal{C}$ and $S \in \mathcal{C}'_+$. For any $A \in \mathcal{A}$, it follows from the definition of $\mathcal{A}$ that there are operators $S_1, S_2, \ldots, S_n \in \mathcal{C}'_+$, $G_1, G_2, \ldots, G_n \in \mathcal{G}$ and scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $A =$

$\sum_{j=1}^{n} \lambda_j S_j G_j$. Thus, we have $AT = \sum_{j=1}^{n} \lambda_j S_j G_j T$. Since $G_j T \in \mathcal{G}$, it follows that $AT \in \mathcal{A}$ and $ATx = 0$ for all $A \in \mathcal{A}$. Consequently, $Tx \in \operatorname{Ker} \mathcal{A}$. On the other hand, we have $AS = \sum_{j=1}^{n} \lambda_j S_j G_j S$. Since $G_j \in \mathcal{G}$, G_j is an operator of the form $T_{j_1} T_{j_2} \cdots T_{j_m}$, where $T_{j_1}, T_{j_2}, \ldots, T_{j_m} \in \mathcal{C}$. Thus, we have obtained

$$\begin{aligned} AS &= \sum_{j=1}^{n} \lambda_j S_j T_{j_1} T_{j_2} \cdots T_{j_m} S = \sum_{j=1}^{n} \lambda_j S_j S T_{j_1} T_{j_2} \cdots T_{j_m} \\ &= \sum_{j=1}^{n} \lambda_j S_j S G_j. \end{aligned}$$

Observing $S_j S \in \mathcal{C}'_+$, we obtain that $AS \in \mathcal{A}$ and $ASx = 0$ for all $A \in \mathcal{A}$. Consequently, $Sx \in \operatorname{Ker} \mathcal{A}$.

From the above we conclude that $\operatorname{Ker} \mathcal{A}$ is a common nontrivial invariant closed subspace for $\mathcal{C}$ and $\mathcal{C}'_+$, and this completes the proof of Theorem 1. $\square$

Theorem 2. *Let $\mathcal{C} \neq \{0\}$ be a commutative collection of continuous positive operators on a Banach space X with a Schauder basis $\{e_n\}$, and let every operator in $\mathcal{C}$ be quasinilpotent at the nonzero positive vector $x_0 \in X$. Then $\mathcal{C}$ and its positive commutant $\mathcal{C}'_+$ have a common nontrivial invariant closed subspace.*

Proof. The proof of the theorem is similar to that of Theorem 1 and is therefore omitted. $\square$

Corollary 1. *Every nonzero continuous positive operators on a Banach space with a Schauder basis which is quasinilpotent at a nonzero positive vector has a nontrivial order hyperinvariant closed subspace.*

Remark 1. It is worth mentioning that the positiveness hypothesis of operators in Theorem 1, Theorem 2 and Corollary 1 cannot be omitted. Indeed, Read [**14**] constructed a quasinilpotent continuous operator T on l_1 without nontrivial invariant closed subspace.

Finally, we provide an example of a noncommutative finitely quasinilpotent collection $\mathcal{C}$ of continuous positive operators for which the conditions of Theorem 1 are satisfied:

In [**2**], Abramovich, Aliprantis and Burkinshaw showed the operator $T : l_p \to l_p$ with matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1/2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1/3 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1/4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is quasinilpotent at e_2 but fails to be quasinilpotent, where the symbol e_n denotes the vector whose nth component is one and every other zero. Similarly, one can show that the operator $S : l_p \to l_p$ with matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1/2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1/3 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1/4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is quasinilpotent at e_2.

Set $\mathcal{C} = \{T, S\}$. Then the collection $\mathcal{C}$ of operators satisfies our demands. Indeed, the equalities

$$TSe_1 = \frac{1}{2}\, e_3, \qquad STe_1 = e_2 + \frac{1}{2}\, e_3,$$

and

$$T^n e_2 = S^n e_2 = \frac{1}{(n+1)!}\, e_{n+2}, \quad n = 1, 2, \ldots$$

are easily verified. Consequently, the equality

$$\|\mathcal{F}^n e_2\| = \|T^n e_2\| = \frac{1}{(n+1)!}$$

holds for $n = 1, 2, \ldots$ and every subset $\mathcal{F}$ of $\mathcal{C}$, from which it follows that

$$\lim_{n\to\infty} \|\mathcal{F}^n e_2\|^{1/n} = 0.$$

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