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THE STEREOGRAPHIC PROJECTION IN BANACH SPACES

FRANCISCO J. GARCÍA-PACHECO AND JUAN B. SEOANE-SEPÚLVEDA

ABSTRACT. We give a new and direct proof of the fact that, in any infinite dimensional Banach space, the unit sphere minus any one point is homeomorphic to a closed hyperplane. The proof involves L-structures and geometric concepts as, for instance, rotund, smooth and exposed points.

1. Preliminaries and background. It is well known [**1**] that the Euclidean unit sphere $\mathcal{S}^n$ minus any one point is homeomorphic to $\mathbf{R}^n$; this homeomorphism is known as the *stereographic projection*. This stereographic projection can be generalized to infinite dimensional spaces or, more particularly, to infinite dimensional real Banach spaces. This is the aim of this paper, to give a new and direct proof of this result, i.e., that the unit sphere minus any one point is homeomorphic to a closed hyperplane in any real Banach space.

On the other hand, to establish homeomorphisms between unit balls and/or unit spheres in a Banach space, it suffices to consider isomorphisms of Banach spaces. In other words, if X and Y are isomorphic Banach spaces, and $T : X \to Y$ is an isomorphism, then the mapping $T_B : \mathcal{B}_X \to \mathcal{B}_Y$, given by

$$\begin{cases} T_B : \mathcal{B}_X \longrightarrow \mathcal{B}_Y \\ x \longmapsto T_B x, \end{cases}$$

where

$$T_B x = \begin{cases} Tx/\|Tx\| \cdot \|x\| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

and where $\mathcal{B}_X$ is the unit ball of X, is an homeomorphism whose restriction to $\mathcal{S}_X$ (the unit sphere of X) induces an homeomorphism between $\mathcal{S}_X$ and $\mathcal{S}_Y$. This fact will be used later on to establish the main result. Next, let us recall the definition of the L^2-*summand vector*, see [**3**].

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Definition 1.1. Let X be any real Banach space and $x \in X$. We say that x is an L^2-summand vector of X if span $\{x\}$ has a topological complement M such that $\|m+\delta x\|^2 = \|m\|^2 + \|\delta x\|^2$, for every $m \in M$ and any $\delta \in \mathbf{R}$.

We also have that, if $x \neq 0$ is an L^2-summand vector of X, there exists $x^* \in X^*$ so that:

(1) $x^* x = 1$,

(2) $\|x^*\| = \|x\|^{-1}$, and

(3) $\operatorname{Ker}(x^*) = M$.

It can be proved that, in this case, x^* is an L^2-summand vector of X^*, and it will be called the *L^2-summand functional of x*.

Let us remember some definitions that we will use in this paper, see [**2, 4**] for further details:

Definition 1.2. Let X be a Banach space and $x \in \mathcal{S}_X$. Then:

(1) x is called an *exposed point* of $\mathcal{B}_X$ if there exists $f \in \mathcal{S}_{X^*}$ so that $\{x\} = f^{-1}(\{1\}) \cap \mathcal{B}_X$.

(2) We say that x is a rotund point of $\mathcal{B}_X$ if every $y \in \mathcal{S}_X \setminus \{x\}$ verifies $\|(x+y)/2\| < 1$.

(3) x is a smooth point of $\mathcal{B}_X$ if there exists a unique $f \in \mathcal{S}_{X^*}$ so that $f(x) = 1$.

It is well known that any exposed and smooth point is also a rotund point [**2**].

Now we can state the main result:

Theorem. *Let X be a real Banach space and $x \in \mathcal{S}_X$. Then $\mathcal{S}_X \setminus \{x\}$ is homeomorphic to a closed hyperplane.*

2. The new proof. In order to give the proof of the main theorem we will need some previous results, that we show now. A first result is related to L^2-summand vectors and says:

Lemma 2.1. *Let X be a real Banach space, $x \in \mathcal{S}_X$ an L^2-summand vector and x^* the L^2-summand functional of x. Let $(x_n)_n \subset \mathcal{S}_X \setminus \{-x\}$ be a sequence convergent to $-x$. Then, we have*

$$\left\| \frac{x_n + x}{x^* x_n + 1} \right\| \longrightarrow \infty \quad \text{as} \quad n \to \infty.$$

Proof. For each $n \in \mathbf{N}$ we decompose x_n as $x_n = m_n + \delta_n x$, where $m_n \in \operatorname{Ker}(x^*)$ and $\delta_n \in \mathbf{R}$. Since $(\delta_n)_n \subset (-1, 1]$ and converges to -1, and $\|m_n\|^2 + \delta_n^2 = 1$ for every $n \in \mathbf{R}$, we obtain

$$\begin{aligned} \left\| \frac{x_n + x}{x^* x_n + 1} \right\| &= \frac{\|x_n + x\|}{|x^* x_n + 1|} = \frac{\sqrt{\|m_n\|^2 + (\delta_n + 1)^2}}{\delta_n + 1} \\ &= \frac{\sqrt{\|m_n\|^2 + \delta_n^2 + 1 + 2\delta_n}}{\delta_n + 1} = \frac{\sqrt{1 + 1 + 2\delta_n}}{\delta_n + 1} \\ &= \sqrt{2} \cdot \frac{\sqrt{1 + \delta_n}}{1 + \delta_n} = \frac{\sqrt{2}}{\sqrt{1 + \delta_n}} \longrightarrow \infty \quad \text{as} \quad n \to \infty. \qquad \square \end{aligned}$$

Now, we will need one more result. The next result is a weaker case of the main theorem. But we will need it in order to complete the final proof.

Theorem 2.2. *Let X be a real Banach space. Let $x \in \mathcal{S}_X$ be an exposed point of $\mathcal{B}_X$, and denote by $f \in \mathcal{S}_{X^*}$ the functional that characterizes x as an exposed point of $\mathcal{B}_X$. Then*:

(1) *There exists a continuous function $\phi : \mathcal{S}_X \setminus \{-x\} \to f^{-1}(\{1\})$.*

(2) *If x is a smooth point of $\mathcal{B}_X$ as well as an exposed point, then ϕ is a bijection.*

(3) *If x is an L^2-summand vector, then ϕ is a homeomorphism.*

Proof. (1) Define, for every $y \in \mathcal{S}_X \setminus \{-x\}$, the stereographic projection

$$\phi(y) = -x + \frac{2}{f(y) + 1} \cdot (y + x).$$

Notice that ϕ is well defined, since $\{x\} = f^{-1}(\{1\}) \cap \mathcal{B}_X$, and that ϕ is continuous.

(2) Let us first see that ϕ is one to one. Suppose there are $y \neq z \in \mathcal{S}_X \setminus \{-x\}$ with $\phi(y) = \phi(z)$. Then

$$\frac{1}{f(y)+1} \cdot (y+x) = \frac{1}{f(z)+1} \cdot (z+x),$$

and we obtain

$$z = -x + \frac{f(z)+1}{f(y)+1} \cdot (y+x);$$

in other words, the above equation means that $-x$, y and z are collinear, but this is in contradiction with the fact that $-x$ is a rotund point of $\mathcal{B}_X$ (remember that any exposed and smooth point of the unit ball is a rotund point of the unit ball). So ϕ must be one to one.

To see that ϕ is onto, let $v \in f^{-1}(\{1\})$. Since $\phi(x) = x$, we can assume that $v \neq x$. Consider the segment $[-x, v]$; we show that $(-x, v)$ intersects $\mathcal{B}_X$. Suppose not; then for every $t \in (0,1)$,

$$\|tv + (1-t)(-x)\| > 1.$$

For every $t < 0$ we obtain

$$(-f)(tv + (1-t)(-x)) = -t + (1-t) = 1 - 2t > 1,$$

so we have that $\|tv + (1-t)(-x)\| > 1$. If $t > 1$ we have

$$f(tv + (1-t)(-x)) = t + (t-1) = 2t - 1 > 1,$$

and, again, $\|tv + (1-t)(-x)\| > 1$. Notice also that $\|v\| > 1$, since $v \in f^{-1}(\{1\})$ and $v \neq x$. To summarize, if $t \neq 0$, then $\|tv + (1-t)(-x)\| > 1$. Consider now $V = \operatorname{span}\{-x, v\}$, and let $g \in V^*$ be the unique element so that $g(-x) = g(v) = 1$. Since $\|tv + (1-t)(-x)\| > 1$ for $t \neq 0$, $g \in \mathcal{S}V^*$. Let $G \in \mathcal{S}_{X^*}$ be the Hahn-Banach extension of g. The smoothness of $-x$ allows us to infer that $-f = G$, which yields to

$$-1 = -f(v) = G(v) = g(v) = 1,$$

so $(-x,v)\cap\mathcal{B}_X\neq\varnothing$. Let $t\in(0,1)$ with $tv+(1-t)(-x)\in\mathcal{B}_X$. If $tv+(1-t)(-x)\in\mathcal{S}_X$, then we are done since $\phi(tv+(1-t)(-x))=v$. If, on the other hand, $tv+(1-t)(-x)\notin\mathcal{S}_X$, then by Bolzano's theorem there must be $s\in(t,1)$ with $\|sv+(1-s)(-x)\|=1$ and, therefore, $\phi(sv+(1-s)(-x))=v$. So ϕ is onto.

(3) For this last part, notice that f is the L^2-summand functional of x, so we will write $f=x^*$. Take any fixed $y\in\mathcal{S}_X\setminus\{-x\}$. To see that ϕ^{-1} is continuous at $\phi(y)$ it suffices to show that, if we have a sequence $(y_n)_n\subset\mathcal{S}_X\setminus\{-x\}$ so that $(\phi(y_n))_n$ converges to $\phi(y)$, then $(y_n)_n$ has a subsequence which is convergent to some element in $\mathcal{S}_X\setminus\{-x\}$. Let us take a subsequence $(y_{n_k})_k$ so that $x^*y_{n_k}$ converges to some $\delta\in[-1,1]$. Since $\phi(y_{n_k})$ converges to $\phi(y)$, then

$$y_{n_k}\longrightarrow -x+\frac{\delta+1}{x^*y+1}\cdot(y+x)\quad\text{as}\quad k\to\infty.$$

Let us see that $-x+(\delta+1)/(x^*y+1)\cdot(y+x)\neq -x$. If $-x+(\delta+1)/(x^*y+1)\cdot(y+x)=-x$, then by the previous lemma we have that

$$\left\|\frac{y_{n_k}+x}{x^*y_{n_k}+1}\right\|\longrightarrow\infty\quad\text{as}\quad k\to\infty,$$

and thus $\|\phi(y_{n_k})\|\to\infty$ as $k\to\infty$, which is impossible. □

Now we are ready to state and give the new proof of the main result.

Theorem 2.3. *Let X be a real Banach space. For every $x\in\mathcal{S}_X$, $\mathcal{S}_X\setminus\{x\}$ is homeomorphic to a closed hyperplane.*

Proof. Take any fixed topological complement M for $\operatorname{span}\{x\}$. Consider the equivalent norm on X given by

$$\lfloor y\rfloor=\sqrt{\|m\|^2+\|x\|^2}$$

for every $y\in X$, where $y=m+\delta x$, with $m\in M$ and $\delta\in\mathbf{R}$. Denote by Y the space X endowed with the norm $\lfloor\cdot\rfloor$; then $x\in\mathcal{S}_Y$ and x is an L^2-summand vector of Y. Therefore, by the previous theorem, $\mathcal{S}_Y\setminus\{x\}$

is homeomorphic to a closed hyperplane. Now, the mapping

$$\begin{aligned} \mathcal{S}_X &\longrightarrow \mathcal{S}_Y \\ z &\longmapsto \frac{z}{\lfloor z \rfloor}, \end{aligned}$$

is a homeomorphism that maps x to itself, so $\mathcal{S}_X \setminus \{x\}$ is homeomorphic to a closed hyperplane. □

REFERENCES

1. A. Aizpuru, *Apuntes y notas de topología*, Servicio de publicaciones de la Universidad de Cádiz, Cádiz, Spain, 1996.

2. P. Bandyopadhyay and B. Lin, *Some properties related to nested sequence of balls in Banach spaces*, Taiwanese J. Math. **5** (2001), 19–34.

3. J.W. Carlson and T.L. Hicks, *A characterization of inner product spaces*, Math. Japon. **23** (1978/79), 371–373.

4. R.E. Megginson, *An introduction to Banach space theory*, Grad. Texts in Math., Springer-Verlag, New York, 1998.

Department of Mathematics, Kent State University, Kent, Ohio, 44242
E-mail address: fgarcia@math.kent.edu

Department of Mathematics, Kent State University, Kent, Ohio, 44242
E-mail address: jseoane@math.kent.edu