ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 4, Fall 2007

SOME EXTENSIONS OF THE MARKOV INEQUALITY FOR POLYNOMIALS

D. DRYANOV, R. FOURNIER AND S. RUSCHEWEYH

ABSTRACT. Let $\mathbf{D}$ denote the unit disc of the complex plane and $\mathcal{P}_n$ the class of polynomials of degree at most n with complex coefficients. We prove that

$$\max_{z\in\partial\mathbf{D}}\left|\frac{p_k(z)-p_k(\bar z)}{z-\bar z}\right| \le n^{1+k}\max_{0\le j\le n}\left|\frac{p(e^{ij\pi/n})+p(e^{-ij\pi/n})}{2}\right|,$$

where $p_0 := p$ belongs to $\mathcal{P}_n$ and for $k \ge 0$, $p_{k+1}(z) := zp_k'(z)$. We also obtain a new proof of a well-known inequality of Duffin and Schaeffer and sharpenings of some other classical inequalities.

Introduction. Let $\mathcal{P}_n$ be the class of polynomials

$$p(z) = \sum_{k=0}^{n} a_k(p) z^k$$

of degree at most n with complex coefficients. We define, together with $\mathbf{D} := \{z \mid |z| < 1\}$,

$$\|p\|_{\mathbf{D}} := \max_{z\in\partial\mathbf{D}} |p(z)| \quad \text{and} \quad \|p\|_{[-1,1]} := \max_{-1\le x\le 1} |p(x)|.$$

The famous inequalities of, respectively, Bernstein and Markov state that for any $p \in \mathcal{P}_n$,

$$\|p'\|_{\mathbf{D}} \le n\|p\|_{\mathbf{D}} \tag{1}$$

and

$$\|p'\|_{[-1,1]} \le n^2\|p\|_{[-1,1]}, \tag{2}$$

2000 AMS *Mathematics Subject Classification.* Primary 41A17.
The second author acknowledges support of NATEQ (Québec). The third author acknowledges partial support from the German-Israeli Foundation (grant G-809-234.6/2003), from FONDECYT (grants 1040366 and 7040069) and from DGIP-UTFSM (grant 240104).
Received by the editors on Dec. 6, 2004, and in revised form on April 6, 2005.

while

$$\|p'\|_{[-1,1]} \le n^2 \max_{0\le j\le n} \big|p\big(\cos(j\pi/n)\big)\big| \tag{3}$$

is a far reaching extension of (2) obtained by Duffin and Schaeffer [**3**] in 1941. We refer the reader to the recent book by Rahman and Schmeisser [**4**] or to the survey paper by Bojanov [**1**] for historical remarks and generalizations of these inequalities.

Let us consider a polynomial $p(z) := \sum_{k=0}^n a_k(p)z^k$ in $\mathcal{P}_n$ and an associated polynomial $P(z) := \sum_{k=0}^n a_k(p)T_k(z)$ where T_k denotes, for each integer $k \ge 0$, the kth Chebyshev polynomial, i.e., $T_k(\cos\theta) = \cos(k\theta)$ for any real number θ. We have

$$P(\cos\theta) = \frac{p(e^{i\theta}) + p(e^{-i\theta})}{2}$$

and, applying (3) to P, we obtain the inequality

$$\left|\frac{e^{i\theta}p'(e^{i\theta}) - e^{-i\theta}p'(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}}\right| \le n^2 \max_{0\le k\le n}\left|\frac{p(e^{ik\pi/n}) + p(e^{-ik\pi/n})}{2}\right| \tag{4}$$

valid for any real θ and equivalent to the Duffin and Schaeffer inequality.

Given a nonnegative number t and a polynomial $p(z) := \sum_{k=0}^n a_k(p)z^k \in \mathcal{P}_n$ we define

$$p_t(z) := \sum_{k=0}^n k^t a_k(p) z^k.$$

Clearly, $p_t \in \mathcal{P}_n$, $p_0 = p$ and $p_{t+1}(z) = zp_t'(z)$ for $t \ge 0$. Our main result is the following

Theorem 1. *For any integer $j \ge 0$ and polynomial $p \in \mathcal{P}_n$,*

$$\left|\frac{p_j(e^{i\theta}) - p_j(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}}\right| \le n^{1+j} \max_{0\le k\le n}\left|\frac{p(e^{ik\pi/n}) + p(e^{-ik\pi/n})}{2}\right| \tag{5}$$

for all real θ.

Our proof of Theorem 1 is completely independent of the known proofs of (3). This Theorem 1 therefore contains (3) as a special case ($j=1$, compare with (4)). It also follows easily from (5) that

$$(6)\quad |p'_{j-1}(e^{i\theta})| \le n^j \max_{0\le k\le n} \left| \frac{p(e^{i(\theta+k\pi/n)}) + p(e^{i(\theta-k\pi/n)})}{2} \right|, \quad \theta \quad \text{real},$$

for all $p \in \mathcal{P}_n$ and integer $j \ge 1$. It is therefore also clear that our Theorem 1 contains an improvement of Bernstein's inequality (1).

Some lemmas. Using the notation

$$\sum_{j=0}^{n}{}'' a_j := \frac{a_0}{2} + \sum_{j=1}^{n-1} a_j + \frac{a_n}{2},$$

our auxiliary results are as follows:

Lemma 1. *For any real* φ, $n \ge 2$ *and* $z \in \mathbf{D}$

$$\frac{z}{(1-ze^{i\varphi})(1-ze^{-i\varphi})} = \sum_{j=0}^{n}{}'' \frac{c_n(j,\varphi)}{2} \left(\frac{1}{1-ze^{ij\pi/n}} + \frac{1}{1-ze^{-ij\pi/n}} \right) - \frac{2(z^n - \cos(n\varphi))z^{n+1}}{(1-ze^{i\varphi})(1-ze^{-i\varphi})(1-z^{2n})},$$

where

$$c_n(j,\varphi) = \frac{(-1)^j}{n} \frac{\cos(j\pi) - \cos(n\varphi)}{\cos(j\pi/n) - \cos(\varphi)}$$

and

$$\sum_{j=0}^{n}{}'' |c_n(j,\varphi)| \le n.$$

Lemma 2. *For any real* φ, $n \ge 2$ *and* $z \in \mathbf{D}$,

$$\frac{2(1-z\cos\varphi)}{(1-ze^{i\varphi})(1-ze^{-i\varphi})} = \sum_{j=0}^{n}{}'' \frac{d_n(j,\varphi)}{2} \left(\frac{1}{1-ze^{ij\pi/n}} + \frac{1}{1-ze^{-ij\pi/n}} \right) + \frac{2z^{n+1}(\cos(n+1)\varphi - \cos(n-1)\varphi)}{(1-ze^{i\varphi})(1-ze^{-i\varphi})(1-z^{2n})},$$

where

$$d_n(j,\varphi) = \frac{(-1)^{j-1}}{n}\,\frac{\cos(n+1)\varphi - \cos(n-1)\varphi}{\cos(j\pi/n) - \cos(\varphi)}$$

and

$$\sum_{j=0}^{n}{}'' |d_n(j,\varphi)| \le 2\left|\frac{n\sin\varphi}{\sin(n\varphi)}\right|.$$

We only prove Lemma 2 in details. Let us fix $\varphi \in \mathbf{R}$ and consider

$$L_\varphi(z) := \frac{(1 - z\cos\varphi)(1 - z^{2n}) - z^{n+1}(\cos(n+1)\varphi - \cos(n-1)\varphi)}{(1 - ze^{i\varphi})(1 - ze^{-i\varphi})}$$

together with

$$R_\varphi(z) := \frac{1}{4n}\sum_{j=-n+1}^{n}(-1)^{j-1}\,\frac{\cos(n+1)\varphi - \cos(n-1)\varphi}{\cos(j\pi/n) - \cos(\varphi)}\,\frac{1 - z^{2n}}{1 - ze^{ij\pi/n}}.$$

It is readily seen that L_φ and R_φ are polynomials in $\mathcal{P}_{2n-1}$. A simple computation gives

$$R_\varphi(e^{-ij\pi/n}) = L_\varphi(e^{-ij\pi/n}) = (-1)^j\,\frac{\sin(n\varphi)\sin(\varphi)}{\cos(j\pi/n) - \cos(\varphi)}$$

at the $2n$ distinct points $e^{-ij\pi/n}$, $j = -n+1, \dots, n$. Clearly then the polynomials L_φ and R_φ must coincide on the whole complex plane and the identity of Lemma 2 follows. We further have with $Z = e^{i\varphi}$

$$\begin{aligned}
&\sum_{j=0}^{n}{}'' \frac{1}{|\cos j\pi/n - \cos\varphi|} \\
&\quad = \sum_{j=0}^{n}{}'' \frac{2}{|1 - Ze^{ij\pi/n}||1 - Ze^{-ij\pi/n}|} \\
&\quad = \frac{1}{|1 - Z|^2} + \sum_{j=1}^{n-1}\frac{2}{|1 - Ze^{ij\pi/n}||1 - Ze^{-ij\pi/n}|} + \frac{1}{|1 + Z|^2} \\
&\quad \le \frac{1}{|1 - Z|^2} + \sum_{j=1}^{n-1}\frac{1}{|1 - Ze^{ij\pi/n}|^2} + \frac{1}{|1 - Ze^{-ij\pi/n}|^2} + \frac{1}{|1 + Z|^2}
\end{aligned}$$

$$= \sum_{j=0}^{2n-1} \frac{1}{|1 - w_j Z|^2} = \sum_{j=0}^{2n-1} \frac{-w_j Z}{(1 - w_j Z)^2} = \frac{n^2}{\sin^2(n\varphi)}$$

where $\{w_j\}_{j=0}^{2n-1}$ is the set of distinct $2n$th roots of unity. It follows that

$$\sum_{j=0}^{n}{}'' |d_n(j,\varphi)| = \sum_{j=0}^{n}{}'' \frac{2|\sin(n\varphi)||\sin(\varphi)|}{n|\cos j\pi/n - \cos\varphi|} \leq \frac{2n|\sin\varphi|}{|\sin n\varphi|}.$$

This completes the proof of Lemma 2. □

A similar proof holds for Lemma 1. It is based on the fact that the polynomials (in $\mathcal{P}_{2n-1}$)

$$\ell_\varphi(z) := z\,\frac{1 - e^{in\varphi}z^n}{1 - e^{i\varphi}z}\,\frac{1 - e^{-in\varphi}z^n}{1 - e^{-i\varphi}z}$$

and

$$r_\varphi(z) := \frac{1}{2n} \sum_{j=-n+1}^{n} (-1)^j\, \frac{\cos(j\pi) - \cos(n\varphi)}{\cos(j\pi/n) - \cos(\varphi)}\, \frac{1 - z^{2n}}{1 - e^{ij\pi/n}z}$$

also satisfy $r_\varphi(e^{-ij\pi/n}) = \ell_\varphi(e^{-ij\pi/n})$, $j = -n+1, \dots, n$. A proof that

$$\sum_{j=0}^{n}{}'' |c_n(j,\varphi)| \leq n$$

can be found in [**2**]. □

We end this section by an application of Lemma 2.

Corollary 1. *Let $p \in \mathcal{P}_n$ and $\varphi \in \mathbf{R}$. Then*

$$|p(e^{i\varphi}) + p(e^{-i\varphi})| \leq \begin{cases} n\left|\dfrac{\sin(\varphi)}{\sin(n\varphi)}\right| \max_{0\leq j\leq n} |p(e^{ij\pi/n}) + p(e^{-ij\pi/n})| & \text{if } e^{2in\varphi} \neq 1, \\ \max_{0\leq j\leq n} |p(e^{ij\pi/n}) + p(e^{-ij\pi/n})| & \text{if } e^{2in\varphi} = 1. \end{cases}$$

We first define the Hadamard product of two analytic functions $f(z) := \sum_{n=0}^{\infty} a_n(f)z^n$ and $g(z) := \sum_{n=0}^{\infty} a_n(g)z^n$ by

$$f \star g(z) := \sum_{n=0}^{\infty} a_n(f)a_n(g)z^n.$$

Then for any $p \in \mathcal{P}_n$ and $\varphi \in \mathbf{R}$, we obtain from Lemma 2,

$$\begin{aligned} p(ze^{i\varphi}) + p(ze^{-i\varphi}) &= \left(\frac{1}{1-ze^{i\varphi}} + \frac{1}{1-ze^{-i\varphi}}\right) \star p(z) \\ &= \frac{2(1-z\cos(\varphi))}{(1-ze^{i\varphi})(1-ze^{-i\varphi})} \star p(z) \\ &= {\sum_{j=0}^{n}}'' d_n(j,\varphi)\,\frac{p(ze^{ij\pi/n}) + p(ze^{-ij\pi/n})}{2}. \end{aligned} \tag{7}$$

Therefore,

$$\begin{aligned} |p(e^{i\varphi}) + p(e^{-i\varphi})| &\le \frac{1}{2} {\sum_{j=0}^{n}}'' |d_n(j,\varphi)|\,|p(e^{ij\pi/n}) + p(e^{-ij\pi/n})| \\ &\le \left|\frac{n\sin(\varphi)}{\sin(n\varphi)}\right| \max_{0\le j\le n} |p(e^{ij\pi/n}) + p(e^{-ij\pi/n})|, \end{aligned}$$

and the result follows. It can also be checked that the above inequality is strict when $p \in \mathcal{P}_n$, $n \ge 2$, $p \not\equiv 0$, $e^{i\varphi} \notin \{w_j\}_{j=0}^{2n-1}$.

Proof of Theorem 1. Let $q \in \mathcal{P}_n$ and $\varphi \in [0,\pi]$. By Lemma 1, we have

$$\begin{aligned} \frac{q(ze^{i\varphi}) - q(ze^{-i\varphi})}{e^{i\varphi} - e^{-i\varphi}} &= \frac{z}{(1-ze^{i\varphi})(1-ze^{-i\varphi})} \star q(z) \\ &= {\sum_{j=0}^{n}}'' \frac{c_n(j,\varphi)}{2}\left(q(ze^{ij\pi/n}) + q(ze^{-ij\pi/n})\right) \end{aligned}$$

and in particular for $z = 1$,

$$\frac{q(e^{i\varphi}) - q(e^{-i\varphi})}{e^{i\varphi} - e^{-i\varphi}} = {\sum_{j=0}^{n}}'' c_n(j,\varphi)\,\frac{q(e^{ij\pi/n}) + q(e^{-ij\pi/n})}{2}. \tag{8}$$

Letting now $\varphi = 0$ in (8) we obtain

$$q'(1) = {\sum_{j=0}^{n}}'' c_n(j,0) \, \frac{q(e^{ij\pi/n}) + q(e^{-ij\pi/n})}{2}.$$

and more generally

$$e^{i\varphi} q'(e^{i\varphi}) = {\sum_{j=0}^{n}}'' c_n(j,0) \, \frac{q(e^{i(\varphi+j\pi/n)}) + q(e^{i(\varphi-j\pi/n)})}{2}. \tag{9}$$

We shall prove the following statement by induction on $k \geq 0$: there exist real numbers $\alpha_{j,k}(\theta)$, $j = 0, 1, \dots, n$, such that for any $p \in \mathcal{P}_n$ and $n \geq 1$,

$$\frac{p_k(e^{i\theta}) - p_k(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}} = \sum_{j=0}^{n} \alpha_{j,k}(\theta) \, \frac{p(e^{ij\pi/n}) + p(e^{-ij\pi/n})}{2}, \quad \theta \in \mathbf{R} \tag{10}$$

and $\sum_{j=0}^{n} |\alpha_{j,k}(\theta)| \leq n^{1+k}$, $\theta \in \mathbf{R}$. The truth of Theorem 1 is clearly a consequence of (10). A proof of (10) for $k = 0, 1$ has been given in [**2**]; clearly such a proof also follows from (8) and Lemma 1. Let us now assume that (10) is valid for a certain integer k and any polynomial $q \in \mathcal{P}_n$. By (9), we obtain

$$e^{\pm i\theta} p_k'(e^{\pm i\theta}) = {\sum_{j=0}^{n}}'' c_n(j,0) \, \frac{p_k(e^{i(\pm\theta+j\pi/n)}) + p_k(e^{i(\pm\theta-j\pi/n)})}{2}. \tag{11}$$

Now, since

$$\frac{p_{k+1}(e^{i\theta}) - p_{k+1}(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}} = \frac{e^{i\theta} p_k'(e^{i\theta}) - e^{-i\theta} p_k'(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}},$$

it follows from (11) that

$$\begin{aligned} &\frac{p_{k+1}(e^{i\theta}) - p_{k+1}(e^{-i\theta})}{e^{i\theta} - e^{-i\theta}} \\ &\qquad = {\sum_{j=0}^{n}}'' c_n(j,0) \, \frac{p_k(e^{ij\pi/n} e^{i\theta}) - p_k(e^{ij\pi/n} e^{-i\theta})}{2(e^{i\theta} - e^{-i\theta})} \\ &\qquad\quad + {\sum_{j=0}^{n}}'' c_n(j,0) \, \frac{p_k(e^{-ij\pi/n} e^{i\theta}) - p_k(e^{-ij\pi/n} e^{-i\theta})}{2(e^{i\theta} - e^{-i\theta})}. \end{aligned}$$

Applying the induction hypothesis, we get

$$\frac{p_k(e^{\pm ij\pi/n}e^{i\theta}) - p_k(e^{\pm ij\pi/n}e^{-i\theta})}{e^{i\theta}-e^{-i\theta}} = \sum_{l=0}^{n} \alpha_{\ell,k}(\theta)\,\frac{p(e^{i(\pm j\pi/n+\ell\pi/n)}) + p(e^{i(\pm j\pi/n-\ell\pi/n)})}{2}$$

with $\sum_{\ell=0}^{n} |\alpha_{\ell,k}(\theta)| \le n^{1+k}$. Finally, we have

$$\begin{aligned}\frac{p_{k+1}(e^{i\theta}) - p_{k+1}(e^{-i\theta})}{e^{i\theta}-e^{-i\theta}} &= \frac{1}{2}\sum_{j=0}^{n}{}'' c_n(j,0) \sum_{\ell=0}^{n} \alpha_{\ell,k}(\theta)\,\frac{p(e^{i(j+\ell)\pi/n}) + p(e^{-i(j+\ell)\pi/n})}{2} \\ &\quad + \frac{1}{2}\sum_{j=0}^{n}{}'' c_n(j,0) \sum_{\ell=0}^{n} \alpha_{\ell,k}(\theta)\,\frac{p(e^{i(j-\ell)\pi/n}) + p(e^{-i(j-\ell)\pi/n})}{2}.\end{aligned}$$

Clearly, the right-hand side of the above is a sum of the type

$$\sum_{j=0}^{n} \alpha_{j,k+1}(\theta)\,\frac{p(e^{ij\pi/n}) + p(e^{-ij\pi/n})}{2}, \quad \alpha_{j,k+1}(\theta) \quad \text{real},$$

and obviously, since

$$\sum_{j=0}^{n} |a_{j,k+1}(\theta)| \le \sum_{j=0}^{n}{}'' |c_n(j,0)| \sum_{\ell=0}^{n} |\alpha_{\ell,k}(\theta)| \le n \cdot n^{k+1} = n^{k+2},$$

the final result follows. □

We shall end this section with some remarks concerning the sharpness of Theorem 1. Let us first point out that the inequality (5) becomes an equality for certain choices of polynomials p; indeed

$$\left|\frac{p_j(e^{i\theta}) - p_j(e^{-i\theta})}{e^{i\theta}-e^{-i\theta}}\right| = n^{1+j} \max_{0\le k\le n}\left|\frac{p(e^{ik\pi/n}) + p(e^{-ik\pi/n})}{2}\right|$$

for any $j = 0, 1, 2, \ldots$, $\theta = 0$ or $\theta = \pi$ and $p(z) \equiv Kz^n$ for some complex constant K. As shown in [**2**], there are no other cases of equality if $j = 1$ but there are many other cases of equality if $j = 0$. To discuss the cases of equality for $j > 1$ seems to be beyond the scope of our method. It is not, however, difficult to establish that (compare with (10)) for $k = 0, 1, 2, \dots$,

$$\sum_{j=0}^{n} |\alpha_{j,k}(\theta)| = n^{1+k} \iff \theta = 0 \quad \text{or} \quad \theta = \pi$$

i.e., the inequality (5) is always strict if $\theta \neq 0$, π and the polynomial p does not vanish identically. We also remark that the statement

$$\left|\frac{p_j(z) - p_j(\bar{z})}{z - \bar{z}}\right| \le n \left|\frac{p_{j-1}(z) - p_{j-1}(\bar{z})}{z - \bar{z}}\right|, \quad z \in \partial\mathbf{D}, \quad j \ge 1, \quad p \in \mathcal{P}_n,$$

can be seen numerically to be false and therefore cannot yield a simpler inductive proof of Theorem 1.

Let us notice that the definition of $p_j(z) := \sum_{k=0}^{\infty} k^j a_k(p) z^k$ extends to positive but not necessarily integer values of j and it is therefore a legitimate (but apparently hard) question to ask whether or not Theorem 1 holds for these values of j. In this context, let us mention that the slightly weaker inequality

$$|p_t|_{\mathbf{D}} \le n^t |p|_{\mathbf{D}}, \quad p \in \mathcal{P}_n$$

holds for all real $t \ge 1$ but does not hold in general for $0 < t < 1$. This unpublished result is due to Mohopatra, Qazi, and Rahman [**4**, Section 14.5].

Concluding remarks. It is possible to apply the identity of Lemma 1 to Hadamard products of polynomials of degree greater than n. For example, for $p \in \mathcal{P}_{n+1}$, we have

$$\frac{p(e^{i\varphi}) - p(e^{-i\varphi})}{e^{i\varphi} - e^{-i\varphi}} = {\sum_{j=0}^{n}}'' c_n(j,\varphi) \frac{p(e^{ij\pi/n}) + p(e^{-ij\pi/n})}{2} + 2a_{n+1}(p)\cos(n\varphi) \tag{12}$$

and (9) gets transformed into

$$e^{i\varphi} p'(e^{i\varphi}) = {\sum_{j=0}^{n}}'' c_n(j,0) \frac{p(e^{i(\varphi + j\pi/n)} + p(e^{i(\varphi - j\pi/n)})}{2} + 2a_{n+1}(p) e^{i(n+1)\varphi}. \tag{13}$$

Therefore,

$$\frac{e^{i\varphi}p'(e^{i\varphi})-e^{-i\varphi}p'(e^{-i\varphi})}{e^{i\varphi}-e^{-i\varphi}}=\frac{1}{2}\sum_{j=0}^{n}{}'' c_n(j,0)$$
$$\times\left(\frac{p(e^{ij\pi/n}e^{i\varphi})-p(e^{ij\pi/n}e^{-i\varphi})}{e^{i\varphi}-e^{-i\varphi}}+\frac{p(e^{-ij\pi/n}e^{i\varphi})-p(e^{-ij\pi/n}e^{-i\varphi})}{e^{i\varphi}-e^{-i\varphi}}\right)$$
$$+\,2a_{n+1}(p)\,\frac{\sin(n+1)\varphi}{\sin(\varphi)}$$

and applying (12) to the polynomials $p(e^{\pm ij\pi/n}z)$,

$$\frac{e^{i\varphi}p'(e^{i\varphi})-e^{-i\varphi}p'(e^{-i\varphi})}{e^{i\varphi}-e^{-i\varphi}}$$
$$=\frac{1}{2}\sum_{j,\ell=0}^{n}{}'' c_n(j,0)c_n(\ell,\varphi)\left(\frac{p(e^{i(j+\ell)\pi/n})+p(e^{-i(j+\ell)\pi/n})}{2}\right.$$
$$\left.+\frac{p(e^{i(j-\ell)\pi/n})+p(e^{-i(j-\ell)\pi/n})}{2}\right)$$
$$+\,2a_{n+1}(p)\cos(n\varphi)\sum_{j=0}^{n}c_n(j,0)(-1)^j\cos(j\pi/n)+2a_{n+1}(p)\,\frac{\sin(n+1)\varphi}{\sin(\varphi)}.$$

We now use (13) with $\varphi=0$ and $p(z)\equiv z^{n+1}$ and get

$$(n-1)=\sum_{j=0}^{n}c_n(j,0)(-1)^j\cos(j\pi/n)$$

and finally, as in the proof of Theorem 1,

$$\left|\frac{e^{i\varphi}p'(e^{i\varphi})-e^{-i\varphi}p'(e^{-i\varphi})}{e^{i\varphi}-e^{-i\varphi}}-2a_{n+1}(p)\left(\frac{\sin(n+1)\varphi}{\sin(\varphi)}+(n-1)\cos(n\varphi)\right)\right|$$
$$\le n^2\max_{0\le j\le n}\left|\frac{p(e^{ij\pi/n})+p(e^{-ij\pi/n})}{2}\right|.$$

Some more of our results can be similarly generalized. For example we have, given $p\in\mathcal{P}_{n+1}$, $\varphi\in[0,\pi]$,

$$\left|\frac{p(e^{i\varphi})-p(e^{-i\varphi})}{e^{i\varphi}-e^{-i\varphi}}-2a_{n+1}(p)\cos(n\varphi)\right|$$
$$\le n\max_{0\le j\le n}\left|\frac{p(e^{ij\pi/n})+p(e^{-ij\pi/n})}{2}\right|.$$

REFERENCES

1. B. Bojanov, *Markov-type inequalities for polynomials and splines*, Approx. Theory **10**, Innov. Appl. Math., Vanderbilt Univ. Press, Nashville, TN, 2002, pp. 31–90.

2. D. Dryanov and R. Fournier, *Bernstein and Markov type inequalities*, Preprint CRM-2929, Centre de Recherches Mathématiques, Université de Montréal, 2003.

3. R.J. Duffin and A.C. Schaeffer, *A refinement of an inequality of the brothers Markoff*, Trans. Amer. Math. Soc. **50** (1941), 517–528.

4. Q.I. Rahman and G. Schmeisser, *Analytic theory of polynomials*, Oxford Univ. Press, Oxford, 2002.

Department of Mathematics and Statistics, Concordia University, 7141 Sherbrooke Str., Montreal, Qc H4B 1R6, Canada
E-mail address: ddryanov@alcor.concordia.ca

Centre de Recherches Mathématiques, Université de Montréal, C.P. 6128, succ. Centre-ville, Montréal, Qc H3C 3J7, Canada
E-mail address: fournier@DMS.umontreal.ca

Mathematisches Institut, Universität Würzburg, D-97074 Würzburg, Germany
E-mail address: ruscheweyh@mathematik.uni-wuerzburg.de