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A SECTOR ISOPERIMETRIC PROBLEM

WILLIAM CALBECK

ABSTRACT. We find the sector of largest area that can be formed by taking curves of fixed length that begin at some point on the positive x axis and connecting their endpoints to the origin with radial lines. We also discuss a more general question.

1. Statement of problem. The standard isoperimetric result in two dimensions is that of all possible closed curves of fixed length, the circle is the one that encloses the greatest area. See, for instance, the article [**2**] and its references. In this paper we are interested in looking not at a closed curve but at an open curve of fixed length s which we will assume starts at some point $(r, 0)$ on the x axis. We want to maximize the area of the sector created by connecting its endpoint to the origin. So the idea is to find which curve maximizes the area shown in Figure 1.

We will begin by assuming that the curve has length $s = 1$ and starts at the point $(1, 0)$ and show later the result for arbitrary s and r values which turns out to be no more difficult.

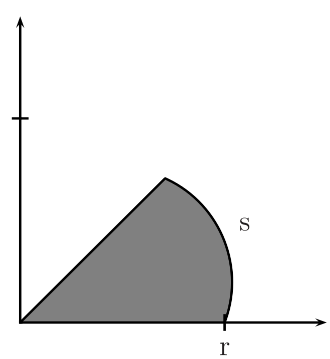

FIGURE 1. Maximize the shaded region.

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1.1. Examples with area $= 0.5$. There are two particularly interesting curves we would like to look at first. One curve is part of the unit circle and the other is a vertical line segment. Both curves are shown in Figure 2.

What is interesting about both of these curves is that they each produce the same area of $1/2$. In fact, these two curves have a property which has been called the equal ratio property [**4**] meaning that the ratio of the area of the sector to the length of the curve remains constant regardless of how long the curve is. In this case it will always be $1/2$. These are the only types of curves that have this property. Besides providing examples in which the area of the sector is easy to compute, these examples also show us that

a) Circular arcs are not necessarily the shapes that produce the maximum area.

b) There is reason to believe there may be a curve that produces a greater area whose path lies "between" these two curves.

As it turns out circular arcs are the shapes that produce the maximum area if we are not required to stay "between" the two curves, it is just that the circles are not centered on the x axis. But before we find the formulas for these circular arcs we want to give some of the basics of working with arclength and area for curves given both parametrically and in polar coordinates.

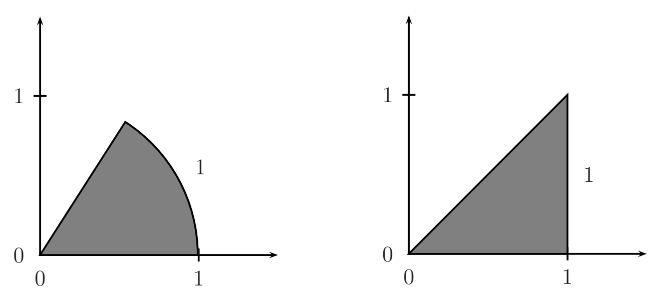

FIGURE 2. Two "equal ratio" curves that both produce an area of 0.5.

1.2. Polar coordinates. In polar coordinates we can state the problem this way. Among all piecewise differentiable functions $f(\theta)$ defined on some interval $[0, \beta]$ such that $f(0) = 1$ and

$$\int_0^\beta \sqrt{f(\theta)^2 + (f'(\theta))^2}\, d\theta = 1 \tag{1}$$

find the function that produces the maximum sector area given by

$$\frac{1}{2}\int_0^\beta f(\theta)^2\, d\theta. \tag{2}$$

The condition $f(0) = 1$ starts the curve at the point $(1, 0)$ on the x axis and condition (1) says the curve $r = f(\theta)$ has length 1. The two example curves can be written in polar coordinates as a) circle $f(\theta) = 1$ and b) vertical line $f(\theta) = \sec\theta$. Note that when the problem is stated in polar form the ending angle β will in general vary from curve to curve. In our examples the ending angle for the circle is 1 radian (approximately 57.3°), but for the vertical line the ending angle is 45°. Finding the arclength for curves by antidifferentiating the formula (1) is usually not possible except for simple functions $f(\theta)$; however, some results can be obtained numerically. There is one other function we will work with in a moment for which we can calculate both the arclength and the area integrals and that is the exponential function $f(\theta) = a^\theta$.

1.3. Parametric equations. When we deal with parametric equations, the greatest simplifying assumption we can make is that the curve is written in terms of the arclength parameter. In this case we will be dealing with a piecewise differentiable curve $r(t) = (x(t), y(t))$ that maps the unit interval $[0, 1]$ into the plane such that $x(0) = 1$, $y(0) = 0$ and $(x'(t))^2 + (y'(t))^2 = 1$ for each value of t in our interval. The problem then can be stated as follows. Among all such curves find the one that maximizes the following integral

$$\frac{1}{2}\int_0^1 (x(t)y'(t) - x'(t)y(t))\, dt. \tag{3}$$

It is not hard to derive this formula as a limit of Riemann sums if one sees that the integrand is $r(t) \times r'(t)$ and $1/2$ of this is the area of an

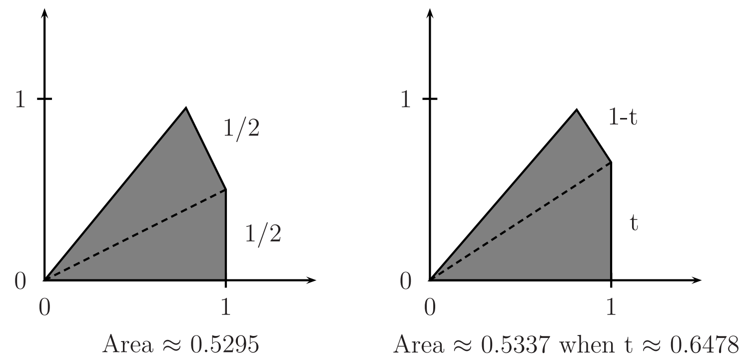

FIGURE 3. Two curves that produce an area greater than 0.5 and lie between the two curves in Figure 2.

infinitesimal triangle having vertices at $(0, 0)$, $r(t)$ and $r(t + \Delta t)$. For an interesting paper concerning this "radial area element," we refer the reader to [**1**]. One can also derive this formula from Green's theorem and the fact that the integrand is zero along the radial boundaries of our sector. We will return to parametric equations after giving a few more examples to show that we can easily improve on the area value of 0.5 of our first two examples.

1.4. Examples with area > 0.5 that lie between the circle and vertical line. We will start by constructing the first curve in Figure 3 that lies between our circle and the vertical line. For this purpose we will take a line that goes up a distance of 1/2 and then angles towards the y axis. The moment our curve turns towards the y axis the rate at which the area increases changes from what it was if we had continued to go straight up. Because of the equal ratio property if we had continued to go up we would be gaining area at the rate of 1/2 square unit per unit of length. But, by turning, we gain area at the rate of 1/2 square unit times the perpendicular distance from the line to the origin. This is basically just a statement of the area formula 1/2 base $\times$ height for a triangle where we think of the curve as being the base and the perpendicular distance from the curve (which may have to be extended) to the origin as being the height. The initial straight up curve had a height of 1 but, after making a turn, the height is whatever the closest distance is from the new line to the origin. If we want to maximize this new height, then the new curve should turn so that it is perpendicular to the radial line at the instant it makes its turn. A simple calculation using the Pythagorean theorem and adding the areas of the two triangles yields an area of $(1 + (\sqrt{5}/2))/4 \approx 0.5295$.

One might speculate that this could then be improved upon by taking three line segments of length $1/3$ and bending twice. However, this yields an area of approximately 0.5266 which is less than our two line segment example. But we can improve on the two line segment example if, instead of going $1/2$ and $1/2$, we go a distance t and $1 - t$ where $0 < t < 1$ is chosen to maximize the area. See the second curve in Figure 3. In terms of t the area would be

$$A(t) = \frac{1}{2}\left(t + (1-t)\sqrt{1+t^2}\right) \tag{4}$$

One can then take the derivative and solve for t algebraically (this results in a fourth degree polynomial equation in t) or find the maximum value numerically (any good graphing calculator will do). The maximum value for t is approximately $t = 0.6478$ which yields an area $A(t) \approx 0.5337$. Further improvement can be gained by bending again and again, etc. The best point to bend a segment of length s is somewhere between $(1/2)s$ and $(2/3)s$ and can be found by solving a fourth degree polynomial.

Some other curves that one might try that stay between the circle and the vertical line are circles of radius greater than 1 and centered on the negative x axis and ellipses with major axis on the y-axis. Neither of these yield any improvement over our $A(t)$.

1.5. Examples with area > 0.5 that go outside the circle and vertical line. If we are really looking to find curves that have the maximum area, then we have to consider the ones that go outside of our initial two curves. Figure 4 shows two such curves that happen again to yield the exact same value for the area although they seem to be completely unrelated.

The first curve in Figure 4 represents the largest area that can be obtained by taking a singly jointed $1/2$, $1/2$ segment. The initial segment leaves the x-axis at an angle of approximately $68.5°$. It turns out this can't be improved upon by taking a jointed t, $1-t$ segment as

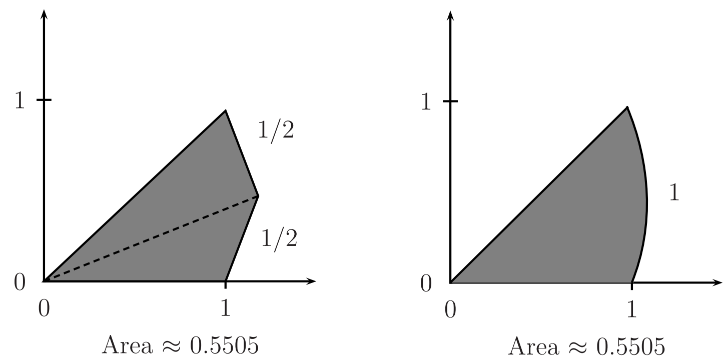

FIGURE 4. Two curves that produce the same area and lie outside the two curves in Figure 2.

was possible when the curve stayed between the circle and the vertical line. The area of this figure can easily be computed as the sum of two triangles. The second curve in Figure 4 is the graph of the function $f(\theta) = a^{\theta}$ where a has been chosen so as to maximize the area. In order to find a we must first require the length of the curve to be one.

$$\int_0^{\beta} \sqrt{(a^{\theta})^2 + (a^{\theta} \ln a)^2}\, d\theta = 1. \tag{5}$$

One can integrate this and solve for a^{β}

$$a^{\beta} = 1 + \frac{\ln a}{\sqrt{1 + (\ln a)^2}}. \tag{6}$$

We can then use this value of a^{β} (without actually solving for β) in the area integral

$$\frac{1}{2} \int_0^{\beta} (a^{\theta})^2\, d\theta. \tag{7}$$

Doing a little calculus, using the a^{β} value, and substituting $u = \ln a$, leads to a fourth degree polynomial with missing first and third degree terms which can be solved for u using the quadratic formula. This yields a maximum area of $A = 1/8 \left(\sqrt{2\sqrt{3} - 3} + 2\sqrt[4]{3}\sqrt{2} \right) \approx 0.5505$ when $a = \exp\left(\sqrt{-1 + 2\sqrt{3}/3} \right)$.

1.6. The maximum area $A \approx 0.566109354$. Now that we have seen a few examples of curves and the areas they produce we need to make a few observations and apply some known results. One observation is that the curves that produce maximum area are always convex. This makes sense because if a curve had a concave "dimple" then there would clearly be a shorter curve that would produce larger area simply by going directly across the dimple. So because the maximal curves are convex the sector area can be written as the sum of the area of a triangle plus the area of the convex hull of the curve, the vertices of the triangle being the origin and the two endpoints of the curve. The paper [**3**] deals with curves of fixed length whose convex hulls have maximal area. In that paper a result is given that in dimension 2 the convex hull area is maximal for circular arcs. Using that result we then know that our curve must be the arc of some circle. One could also argue directly from the isoperimetric inequality for circles because if the maximal curve was not the arc of a circle then you could use it to construct a counterexample to the circle having the maximal area among all curves of fixed length. Now that we know the curve is the arc of some circle we just have to find its center and radius.

We are going to set this problem up in parametric form with arclength parameter as described above. Because the parameter functions $x(t)$ and $y(t)$ satisfy $(x'(t))^2 + (y'(t))^2 = 1$ there exists a function $\delta(t)$ such that $x'(t) = \cos(\delta(t))$ and $y'(t) = \sin(\delta(t))$. The unit tangent to the curve is $T = \langle x', y' \rangle$ and the curvature is $K = \|T'\| = \delta'$. Since we know our curve is the arc of a circle the curvature is some constant β. So we can write $\delta(t) = \alpha + \beta t$ for some constants α and β. The geometric significance of α is that it is the initial direction our curve heads in as it leaves the x axis. Therefore, we can now write $x(t)$ and $y(t)$ as follows

$$x(t) = 1 + \int_0^t \cos(\alpha + \beta s)\, ds, \qquad y(t) = \int_0^t \sin(\alpha + \beta s)\, ds \tag{8}$$

and evaluate the integral formula (3) in order to find the values of α and β that maximize it. After integrating and using some trig identities we can write the area as

$$A = A(\alpha, \beta) = \frac{1}{2\beta^2}\left(\beta \cos\alpha - \beta\cos(\alpha + \beta) - \sin\beta + \beta\right). \tag{9}$$

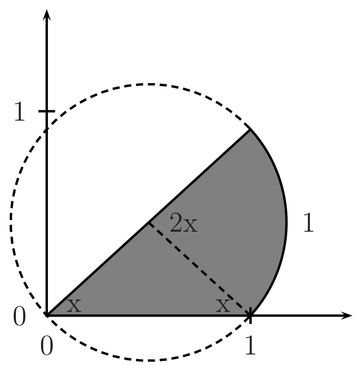

FIGURE 5. The curve with the largest area.

Setting the partial derivative $A_\alpha = 0$ yields $\alpha = \pi/2 - \beta/2$. Substituting this back into $A(\alpha, \beta)$ and then differentiating with respect to β, using a double angle trig identity, and factoring yields

$$A_\beta = \frac{1}{2\beta^3}\left(\beta\cos\left(\frac{\beta}{2}\right) - 2\sin\left(\frac{\beta}{2}\right)\right)\left(\beta - 2\cos\left(\frac{\beta}{2}\right)\right). \tag{10}$$

So either $\tan(\beta/2) = \beta/2$, which happens when $\beta = 0$ or $\cos(\beta/2) = \beta/2$, which happens when $\beta \approx 1.478$. The case $\beta = 0$ is our original vertical line and that doesn't interest us here. If we let $x = \beta/2$, then x is the solution to $x = \cos(x)$ which is ≈ 0.739 radians or $42.35°$. Figure 5 gives the picture of this maximal curve. The center of the circle is located on the line $x = 1/2$ and its radius is $1/\beta \approx 0.6765$.

1.7. The general case. The problem of finding the maximal curve for arbitrary r and s values is no more difficult than what we have just done for the case when $r = s = 1$. We can still write $\delta(t) = \alpha + \beta t$. The formulas for $x(t)$ and $y(t)$ would change only in that $x(0) = r$. Formula (9) would become

$$A = A(\alpha, \beta) = \frac{1}{2\beta^2}\left(r\beta\cos\alpha - r\beta\cos(\alpha + \beta s) - \sin(\beta s) + \beta s\right). \tag{11}$$

This time setting $A_\alpha = 0$ yields $\alpha = \pi/2 - (\beta s)/2$ and formula (10) becomes

$$A_\beta = \frac{1}{2\beta^3}\left(\beta s \cos\left(\frac{\beta s}{2}\right) - 2\sin\left(\frac{\beta s}{2}\right)\right)\left(r\beta - 2\cos\left(\frac{\beta s}{2}\right)\right). \tag{12}$$

So we again get two solutions to $A_\beta = 0$. Either $\tan(\beta s/2) = (\beta s)/2$, which happens when $\beta = 0$ or $\cos(\beta s/2) = (\beta r)/2$, which is the solution we are after. If we let $x = \beta s/2$, then Figure 5 is still the correct picture for our solution (with r and s replacing the two ones) only this time x is the solution not of $\cos(x) = x$ but rather $\cos(x) = (r/s)x$. Notice that when s is large compared to r the curve does not wrap around the origin but stays in the first quadrant.

2. Problems to think about. Since we spent some time looking at curves that stayed between the circle and the vertical line, one might ask what curve produces the largest area that lies between the two. This question has not been answered yet, but we conjecture the following.

2.1. Conjecture. *The curve of length* 1 *that produces that largest sector area that lies between the unit circle and the vertical line* $x = 1$ *can be obtained by traveling up the vertical line* $x = 1$ *until it is possible to begin tracing out the arc of a circle as was constructed in the maximal curve example such that the arc stays to the left of the line* $x = 1$. *This seems reasonable since at some point the maximal curve must leave the line* $x = 1$ *and when it does a circular arc would produce the largest area.*

2.2. Open question. Determine a procedure to find curves of fixed length that produce the maximum sector area with the extra constraint that the curve must lie between two given curves. Because the curve must now navigate between two given curves, trying to follow arcs of circles could lead to dead ends. A similar question could also be asked for closed curves that are forced to lie in an annular region between two given curves.

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LSU–Alexandria, Department of Mathematics and Physical Sciences, 8100 Highway 71 South, Alexandria, LA 71302
E-mail address: `billc@lsua.edu`