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HOMOLOGY OF ZERO-DIVISORS

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ABSTRACT. Let R be a commutative ring with unity. We define a semi-simplicial abelian group based on the structure of the semigroup of ideals of R and investigate various properties of the homology groups of the associated chain complex.

1. Introduction. Let R be a commutative ring with unity. The set $Z(R)$ of zero-divisors in a ring does not possess any obvious algebraic structure; consequently, the study of this set has often involved techniques and ideas from outside algebra. Several recent attempts, among them [**2, 3**] have focused on studying the so-called *zero-divisor graph* Γ_R, whose vertices are the zero-divisors of R, with xy being an edge if and only if $xy = 0$. This object Γ_R is somewhat unwieldy in that it has many symmetries; for example, if $u \in R^*$ is any unit, then $x \mapsto ux$ induces a (graph) automorphism of Γ_R. One way of treating this issue, following an idea of Lauve [**5**], is to work with the *ideal zero-divisor graph* $\mathcal{I}_R$. In effect, one replaces zero-divisors of R by proper ideals with nonzero annihilator; this is the approach adopted by the authors in [**1**]. Such a perspective also has its shortcomings; for instance, it does not adequately detect the phenomenon of there being three distinct proper ideals I, J, K in R with $IJK = 0$, but $IJ \neq 0$, $IK \neq 0$, $JK \neq 0$.

In this paper we adopt a different philosophy, using a new type of homology to study $Z(R)$ and capture the situation described above. Roughly speaking, if we denote by $\mathbf{Z}_n(R)$ the free abelian group generated by the set of $(n+1)$-tuples $(I_0, \dots, I_n)$ of distinct ideals of R such that $I_0 \cdot \dots \cdot I_n \neq 0$, there are obvious maps $\mathbf{Z}_n(R) \to \mathbf{Z}_{n-1}(R)$ obtained by forgetting one of the factors. This gives $\mathbf{Z}_{.}(R)$ the structure of a semi-simplicial abelian group; hence, we may speak of its associated chain complex $\mathbf{C}_{.}(R)$. Our homology groups $H_*(R)$ are then defined as the homology groups of a certain quotient of $\mathbf{C}_{.}(R)$. The idea behind this construction was sketched by Lauve in [**5**], although the precise definition is due to the authors.

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After giving a precise definition of these homology groups $H_*(R)$, we study the group $H_0(R)$ in depth and compute $H_1(\mathbb{Z}/p^r\mathbb{Z})$ when p is a prime and $r \geq 1$ is an integer. We then give some conditions on R sufficient to ensure that $H_n(R) = 0$ for $n > 0$. In the last section we consider the *Euler characteristic* $\chi(R) = \sum_{n=0}^{\infty}(-1)^n \operatorname{rk} H_n(R)$. Using some ideas from partition theory, we prove the surprising result that $\chi(\mathbb{Z}/p^r\mathbb{Z})$ is always either 0, 1, or 2, depending on the value of r relative to the "pentagonal" numbers $m(3m-1)/2$ and the related numbers $m(3m+1)/2$. We also derive formulas for the Euler characteristic for some other special types of finite rings.

2. Preliminaries. Let R be a commutative ring and $\mathcal{P}$ the set of proper ideals of R. For each $n \geq 0$, let $S_n(R)$ be the set of ordered $(n+1)$-tuples $(I_0, \dots, I_n)$, where $I_0, \dots, I_n$ are distinct proper ideals of R and $I_0 I_1 \cdots I_n \neq 0$; let $S_{-1}(R)$ be a set consisting of one element. If there is no danger of ambiguity, we simply write S_n instead of $S_n(R)$. Observe that, for each i, $0 \leq i \leq n$, there is a "face map" $\phi_i^n : S_n \to S_{n-1}$ defined by $\phi_i^n(I_0, \dots, I_n) = (I_0, \dots, \hat{I_i}, \dots, I_n)$. Moreover, $S_0(R) = \varnothing$ if and only if R is a field, so when R is not a field, there is a unique "augmentation" map $\varepsilon : S_0(R) \to S_{-1}(R)$. Now, for each $n \geq -1$, let Z_n be the free abelian group generated by S_n. We denote by $[I_0, \dots, I_n]$ the basis element corresponding to $(I_0, \dots, I_n) \in S_n$. Likewise, the various face maps ϕ_i^n extend $\mathbb{Z}$-linearly to maps $\phi_i^n : Z_n \to Z_{n-1}$; moreover, if $S_0 \neq \varnothing$, there is a unique $\mathbb{Z}$-linear map $\varepsilon : Z_0 \to Z_{-1} = \mathbb{Z}$ defined by $\varepsilon(\sum n_i(I_i)) = \sum n_i$. Thus, there is a semi-simplicial abelian group:

$$\mathbf{Z}_{.}(R): \quad \dots \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} Z_1 \substack{\longrightarrow \\ \longrightarrow} Z_0$$

with augmentation $\varepsilon : Z_0 \to \mathbb{Z}$ if R is not a field.

This in turn gives rise to an (augmented) chain complex in the standard manner by taking an alternating sum of face maps. For each $n \geq 0$, define $\delta_n = \sum_{i=0}^{n}(-1)^i \phi_i^n$; then we have a complex:

$$\mathbf{C}_{.}(R): \quad \dots \xrightarrow{\delta_1} Z_1 \xrightarrow{\delta_0} Z_0$$

of abelian groups.

In practice, the Z_n are too large to be useful invariants; in particular, we chose Z_n to be the free $\mathbb{Z}$-module with basis S_n, which consisted of *ordered* $(n+1)$-tuples of ideals of R having nonzero product. Because multiplication in R is commutative, the order of the ideals in this $(n+1)$-tuple ought not to matter; it might appear more natural to work with *unordered* $(n+1)$-tuples. Unfortunately, the definition of the face maps *does* depend on the ordering within each such tuple, so we resort instead to the following device: for each $n \geq 0$, let R_n denote the subgroup of Z_n generated elements of the form:

$$[I_0, \dots, I_n] - (-1)^{\operatorname{sgn}\sigma}[I_{\sigma(0)}, \dots, I_{\sigma(n)}],$$

where σ in an element of the symmetric group $\mathfrak{S}_{n+1}$ (viewed as permutations of the set $\{0, \dots, n\}$) and $[I_0, \dots, I_n]$ is a basis element of Z_n. Set $T_n = Z_n/R_n$.

We claim that $\delta_n(R_n) \subseteq R_{n-1}$. Thus we must show

$$\delta_n([I_0, \dots, I_n]) \equiv (-1)^{\operatorname{sgn}\sigma}\delta_n([I_{\sigma(0)}, \dots, I_{\sigma(n)}]) \pmod{R_{n-1}}.$$

Since every permutation may be written as a product of transpositions, we may reduce to the case that σ is the transposition which exchanges r and s, where $0 \leq r < s \leq n$. In this case,

$$\begin{aligned}
&(-1)^{\operatorname{sgn}\sigma}\delta_n([I_{\sigma(0)}, \dots, I_{\sigma(n)}]) \\
&= -\sum_{i=0}^{n}(-1)^i[I_{\sigma(0)}, \dots, \hat{I}_{\sigma(i)}, \dots, I_{\sigma(n)}] \\
&= \sum_{i \neq r,s}(-1)^{i+1}[I_0, \dots, I_{r-1}, I_s, I_{r+1}, \dots, \hat{I}_i, \dots, I_{s-1}, I_r, I_{s+1}, \dots, I_n] \\
&\quad + (-1)^{r+1}[I_0, \dots, I_{r-1}, I_{r+1}, \dots, I_{s-1}, I_r, I_{s+1}, \dots, I_n] \\
&\quad + (-1)^{s+1}[I_0, \dots, I_{r-1}, I_s, I_{r+1}, \dots, I_{s-1}, I_{s+1}, \dots, I_n] \\
&\equiv \sum_{i \neq r,s}(-1)^{i}[I_0, \dots, I_{r-1}, I_r, I_{r+1}, \dots, \hat{I}_i, \dots, I_{s-1}, I_s, I_{s+1}, \dots, I_n] \\
&\quad + (-1)^{s}[I_0, \dots, I_{r-1}, I_r, I_{r+1}, \dots, I_{s-1}, I_{s+1}, \dots, I_n] \\
&\quad + (-1)^{2s-r}[I_0, \dots, I_{r-1}, I_{r+1}, \dots, I_{s-1}, I_s, I_{s+1}, \dots, I_n] \pmod{R_{n-1}} \\
&\equiv \sum_{i=0}^{n}(-1)^i[I_0, \dots, \hat{I}_i, \dots, I_n] \pmod{R_{n-1}} \\
&\equiv \delta_n([I_0, \dots, I_n]) \pmod{R_{n-1}}.
\end{aligned}$$

Thus $\delta_n(R_n) \subseteq R_{n-1}$ for all $n \geq 1$, and hence $\mathbf{C}_{\cdot}(R)$ factors through a complex:

$$\overline{\mathbf{C}}_{\cdot}(R): \quad \dots \xrightarrow{\partial_1} T_1 \xrightarrow{\partial_0} T_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

By abuse of notation, we continue to use the symbol $[I_0, \dots, I_n]$ to denote the class of $[I_0, \dots, I_n]$ in T_n; hence the formula for ∂_n (on generators) reads: $\partial_n([I_0, \dots, I_n]) = \sum_{i=0}^{n}(-1)^i[I_0, \dots, \hat{I}_i, \dots, I_n]$.

Finally we define the *homology groups*:

$$H_n(R) = \begin{cases} \ker(\partial_{n-1})/\mathrm{Im}\,(\partial_n) & \text{if } n > 0 \\ T_0/\mathrm{Im}\,\partial_0 & \text{if } n = 0 \end{cases}.$$

If $\mathrm{rk}\, H_n(R)$ is finite for all n and zero for sufficiently large n, we define the *Euler characteristic* of R:

$$\chi(R) = \sum_{n=0}^{\infty}(-1)^n \mathrm{rk}\, H_n(R).$$

Since a field has no proper ideals, we immediately have:

Proposition 2.1. *Let F be a field. Then $H_n(F) = 0$ for all $n \geq 0$.*

The term "homology" is used somewhat loosely, since neither the complexes $\overline{\mathbf{C}}_{\cdot}(R)$ nor the groups $H_n(R)$ are functorial in R. This is not particularly surprising: given a ring homomorphism $f : R \to S$, if $[I_0, \dots, I_n] \in T_n(R)$, it is possible that $I_0 \cdots I_n = 0$ or one of the $f(I_i)$ may be zero, so it does not necessarily follow that $[f(I_0), \dots, f(I_n))]$ makes sense as an element of $T_n(S)$. Similarly, if $[J_0, \dots, J_n] \in T_n(S)$, it does not follow that $[f^{-1}(J_0), \dots, f^{-1}(J_n)]$ defines an element of $T_n(R)$.

The following well-known device is often useful in computing the Euler characteristic:

Proposition 2.2. *Suppose* $\mathrm{rk}\, T_n$ *is finite for all n and $T_n = 0$ for $n \gg 0$. Then*

$$\chi(R) = \sum_{n=0}^{\infty}(-1)^n \mathrm{rk}\, T_n.$$

Proof. By definition of $H_0(R)$, there is an exact sequence:

$$0 \longrightarrow \operatorname{Im}\partial_0 \longrightarrow T_0 \longrightarrow H_0(R) \longrightarrow 0$$

and, for each $n \geq 1$, there is a short exact sequence:

$$0 \longrightarrow \operatorname{Im}\partial_n \longrightarrow \ker\partial_{n-1} \longrightarrow H_n(R) \longrightarrow 0.$$

Since the rank is additive across exact sequences, we have:

$$\begin{aligned}\chi(R) &= \sum_{n=0}^{\infty}(-1)^n \operatorname{rk} H_n \\ &= \operatorname{rk} T_0 - \operatorname{rk}\operatorname{Im}\partial_0 + \sum_{n=1}^{\infty}(-1)^n(\operatorname{rk}\,\ker\partial_{n-1} - \operatorname{rk}\operatorname{Im}\partial_n).\end{aligned}$$

Furthermore, for any $n \geq 0$, $\operatorname{rk}\operatorname{Im}\partial_n = \operatorname{rk} T_{n+1} - \operatorname{rk}\,\ker\partial_n$, so the above expression for $\chi(R)$ becomes:

$$\begin{aligned}\chi(R) &= \operatorname{rk} T_0 - \operatorname{rk} T_1 + \operatorname{rk}\,\ker(\partial_0) \\ &\quad + \sum_{n=1}^{\infty}(-1)^n(\operatorname{rk}\,\ker\partial_{n-1} - \operatorname{rk} T_{n+1} + \operatorname{rk}\,\ker\partial_n) \\ &= \operatorname{rk} T_0 - \operatorname{rk} T_1 + \sum_{n=1}^{\infty}(-1)^n \operatorname{rk} T_{n+1} = \sum_{n=0}^{\infty}(-1)^n \operatorname{rk} T_n.\end{aligned}$$

3. The group $H_0(R)$.

Let R be a commutative ring with unity. In order to analyze $H_0(R)$, we recall the construction of the so-called *ideal graph* $\mathcal{I}_R$. This is a (simple) graph whose vertices are the proper ideals of R, with $\{I, J\}$ being an edge if and only if $IJ = 0$. We will be more interested in the *complement* graph $\bar{\mathcal{I}}_R$, whose vertices are the same as $\mathcal{I}_R$, but in which $\{I, J\}$ is an edge if and only if $IJ \neq 0$.

If $\sum_{i=1}^{n}[I_i] \in T_0$ is an element whose class in $H_0(R)$ is zero, this means that $\sum_{i=1}^{n}[I_i] = \partial_0(\sum_{j=1}^{m} c_j[A_j, B_j])$ for some integers c_j and proper ideals A_j, B_j. Without loss of generality, we may assume $c_j = \pm 1$. Equality still holds if we replace $[A_j, B_j]$ by $-[B_j, A_j]$, so we may always write $\sum_{i=1}^{n}[I_i] = \partial_0(\sum_{k=1}^{r}[C_k, D_k])$ for some proper ideals C_k, D_k.

Proposition 3.1. *Let I and J be distinct proper ideals of R. Then $[I]$ and $[J]$ have the same class in $H_0(R)$ if and only if I and J lie in the same connected component of the graph $\bar{\mathcal{I}}_R$.*

Proof. If I and J are in the same connected component of $\bar{\mathcal{I}}_R$, then there is some path $I = A_0 - A_1 - \cdots - A_n = J$ connecting I and J, where the A_i are ideals such that for each $i = 0, \ldots, n-1$, $A_iA_{i+1} \neq 0$. This directly implies that $\sum_{i=0}^{n-1}[A_i, A_{i+1}]$ is an element of T_1, and by direct calculation we see that

$$\partial_0\bigg(\sum_{i=0}^{n-1}[A_i, A_{i+1}]\bigg) = [A_0] - [A_n] = [I] - [J].$$

Hence $[I] = [J]$ in $H_0(R)$.

Conversely, suppose $[I]$ and $[J]$ define the same class in $H_0(R)$. Then $[I]-[J] = \partial_0(\sum_{i=0}^{n}[A_i, B_i]) = \sum_{i=0}^{n}[A_i]-[B_i]$ where A_i, B_i are distinct proper ideals of R and $A_iB_i \neq \varnothing$. Let n be the smallest integer for which this is possible. We prove by induction on n that, after suitable reordering of the A_i and B_i, there is a path in $\bar{\mathcal{I}}_R$ from I to J.

We may assume without loss of generality that $A_0 = I$ and $B_n = J$. If $B_0 = J$, then $IJ \neq 0$ and we are done. Otherwise, assume $B_0 \neq J$; that is, $n > 0$. Since

$$[I] - [J] = [I] - [B_0] + [A_1] - [B_1] + \cdots + [A_n] - [B_n]$$

is a relation in a free abelian group, we may assume without loss of generality that $A_1 = B_0$. Then, adding $[B_0] - [I]$ to both sides of this equation, we get

$$[B_0] - [J] = [A_1] - [B_1] + \cdots + [A_n] - [B_n],$$

so by induction there is a path in $\bar{\mathcal{I}}_R$ from B_0 to J. Since $A_0B_0 \neq 0$, this means that $\{A_0, B_0\}$ is an edge in $\bar{\mathcal{I}}_R$, and hence that there is a path from $A_0 = I$ to J. ☐

Proposition 3.2. *Let $I_1, \ldots, I_n$ be distinct proper ideals of R lying in mutually distinct connected components of $\bar{\mathcal{I}}_R$. Then the classes of $[I_1], \ldots, [I_n]$ are linearly independent in $H_0(R)$.*

Proof. If R is a field, the assertion is trivial. Otherwise, let $C_1, \dots, C_r$ be the components of $\bar{\mathcal{I}}_R$. Suppose the class of $\sum_{i=1}^n c_i[I_i]$ in $H_0(R)$ is 0. We may assume that each I_i lies in component C_i of $\bar{\mathcal{I}}_R$. Now

$$\sum_{i=1}^{n} c_i[I_i] = \partial_0\bigg(\sum_{j=1}^{m}[A_j, B_j]\bigg)$$

for some distinct proper ideals A_j, B_j such that $A_jB_j \neq 0$. Since $[A_j, B_j] \in T_1$, A_j and B_j must lie in the same component of $\bar{\mathcal{I}}_R$. For each k, $1 \leq k \leq r$, let $\mathcal{J}_k = \{j : 1 \leq j \leq m : A_j \in C_k\}$. Then it follows from the above equation that

$$c_k[I_k] = \partial_0\bigg(\sum_{j \in \mathcal{J}_k}[A_j] - [B_j]\bigg).$$

Applying ε to both sides of this equation, we have $c_k = 0$ for all k. □

Combining the previous two propositions, we have:

Corollary 3.3. *Let R be a ring, and r the number of connected components of $\bar{\mathcal{I}}_R$. Then*

$$H_0(R) \cong \mathbb{Z}^r.$$

Corollary 3.3 is a useful tool for calculating $H_0(R)$ in particular cases; nevertheless, using only elementary facts about ideals, one can prove even more. We begin with an elementary lemma:

Lemma 3.4. *Let R be a ring and $\mathfrak{m}_1$, $\mathfrak{m}_2$ distinct maximal ideals of R. If $\mathfrak{m}_1\mathfrak{m}_2 = 0$, then R is isomorphic to a product of two fields.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R. Then $\mathfrak{p} \supseteq \mathfrak{m}_1\mathfrak{m}_2 = 0$, so $\mathfrak{p} \supseteq \mathfrak{m}_1$ or $\mathfrak{p} \supseteq \mathfrak{m}_2$, i.e., $\mathfrak{p} = \mathfrak{m}_1$ or $\mathfrak{p} = \mathfrak{m}_2$. Hence $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the only prime ideals of R and so R is an Artin ring with two maximal ideals. By the structure theorem for Artin rings, $R \cong R_1 \times R_2$, where R_1, R_2 are Artin local rings with respective maximal ideals $\mathfrak{n}_1$, $\mathfrak{n}_2$. Then without loss of generality, $\mathfrak{m}_1 = \mathfrak{n}_1 \times R_2$ and $\mathfrak{m}_2 = R_1 \times \mathfrak{n}_2$. Thus, $0 = \mathfrak{m}_1\mathfrak{m}_2 = \mathfrak{n}_1 \times \mathfrak{n}_2$ so $\mathfrak{n}_1 = 0$, $\mathfrak{n}_2 = 0$ and so R_1, R_2 are fields. $\square$

Proposition 3.5. *Let R be a nonlocal ring which is not isomorphic to the product of two fields. Then $H_0(R) \cong \mathbb{Z}$.*

Proof. By Corollary 3.3 it suffices to prove that $\bar{\mathcal{I}}_R$ is connected. Indeed, let $\mathfrak{m}_1$, $\mathfrak{m}_2$ be distinct maximal ideals of R. If I is any other proper ideal of R, then ann (I) is a proper ideal of R, so ann (I) does not contain both $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Hence for each such I, at least one of $\{I, \mathfrak{m}_1\}$, $\{I, \mathfrak{m}_2\}$ is an edge in $\bar{\mathcal{I}}_R$. If $\mathfrak{m}_1\mathfrak{m}_2 = 0$, then it follows from Lemma 3.4 that R is isomorphic to a product of two fields. Thus $\mathfrak{m}_1\mathfrak{m}_2 \neq 0$, $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ is an edge of $\bar{\mathcal{I}}_R$, and it follows that $\bar{\mathcal{I}}_R$ is connected. $\square$

We have seen that $H_0(F) = 0$ when F is a field and $H_0(R) \cong \mathbb{Z}$ for a large class of rings. Direct computation shows that if F_1 and F_2 are fields, then $H_0(F_1 \times F_2) \cong \mathbb{Z}^2$ and $H_n(F_1 \times F_2) = 0$ for all $n > 0$. A natural question that arises is: given any integer $s \geq 0$, is there a ring R such that $H_0(R) \cong \mathbb{Z}^s$? The discussion above shows that when $s \geq 3$, any such R must necessarily be local. Following an idea supplied to us by Dennis Keeler, we show below that the rank of $H_0(R)$ may be arbitrarily large.

Let k be a field and $x_1, \dots, x_s$ independent indeterminates. Let S be the localization of $k[x_1, \dots, x_s]$ with respect to the maximal ideal $(x_1, \dots, x_s)$. Now let I be the ideal of $k[x_1, \dots, x_s]$ generated by all products $x_i x_j$, where $i \leq j$. Since $I \subseteq (x_1, \dots, x_s)$, I corresponds, in the usual manner, to an ideal $\tilde{I} \subseteq S$. Now let $R = S/\tilde{I}$. Observe now that the proper ideals of R correspond bijectively to ideals $(x_{i_1}, \dots, x_{i_\nu}) \subseteq k[x_1, \dots, x_s]$, where $1 \leq \nu \leq s$ and $1 \leq i_1 < \dots < i_\nu \leq s$. Furthermore, each such ideal (of R), when multiplied by any other, yields 0. Thus $\bar{\mathcal{I}}_R$ is a completely disconnected graph on $2^s - 2$ vertices, and so $H_0(R) \cong \mathbb{Z}^{2^s-2}$.

4. Calculation of $H_1(\mathbb{Z}/p^r\mathbb{Z})$.

In this section, we compute the group $H_1(\mathbb{Z}/p^r\mathbb{Z})$ where p is a prime number and $r \geq 1$ an integer. It is easy to see by direct calculation that if $r \leq 3$, then $H_1(\mathbb{Z}/p^r\mathbb{Z}) = 0$.

We assume henceforth that $r \geq 4$.

Recall first that

$$H_1(R) = \frac{\ker(\partial_0 : T_1 \longrightarrow T_0)}{\mathrm{Im}\,(\partial_1 : T_2 \longrightarrow T_1)}$$

where

$$\partial_0\biggl(\sum_j [A_j, B_j]\biggr) = \sum_j [A_j] - [B_j]$$

and

$$\partial_1\biggl(\sum_j [A_j, B_j, C_j]\biggr) = \sum_j [B_j, C_j] - \sum_j [A_j, C_j] + \sum_j [A_j, B_j].$$

Definition 4.1. Let $n \geq 0$ be an integer. An element $\alpha \in T_1$ is called an n-circuit (or simply a *circuit*) if there exist proper ideals $I_1, \dots, I_n$ of R such that

$$\alpha = [I_1, I_2] + \cdots + [I_{n-1}, I_n] + [I_n, I_1].$$

A 3-circuit is called a *triangle*.

Clearly the definition has been chosen to reflect the fact that, in the above context, $I_1 - I_2 - \cdots I_n - I_1$ is a circuit in the graph $\bar{I}_{\mathbb{Z}/p^r\mathbb{Z}}$. The analysis of $\ker \partial_0$ proceeds by a sequence of lemmas.

Lemma 4.2. *Every element $\beta \in \ker \partial_0$ may be written*

$$\beta = \sum_{k=1}^{m} \alpha_k$$

where each α_k is a circuit.

Proof. The proof is by induction on the number of symbols in β. If $\beta = 0$, the claim is clear. Otherwise, let $\beta = \sum_{j=1}^{r} [A_j, B_j]$ with r

chosen to be as small as possible. We may assume that there is no pair of integers (j_1, j_2), $1 \leq j_1 < j_2 \leq r$ such that $A_{j_1} = B_{j_2}$ and $A_{j_2} = B_{j_1}$, for then we may use the relation $[I, J] = -[J, I]$ in T_1 to simplify the expression for β and obtain a relation with smaller r.

Since $\beta \in \ker \partial_0$, we have:

$$0 = \partial_0(\beta) = \partial_0\bigg(\sum_{j=1}^{r}[A_j, B_j]\bigg) = \sum_{j=1}^{r}[A_j] - [B_j].$$

Since this is a relation in the (free abelian) group T_0, it follows that there is some j such that $B_1 = A_j$. Without loss of generality, we may assume that $j = 2$. By the previous discussion, it follows that $A_1 \neq B_2$. Now it must be the case that there is some j such that $B_2 = A_j$; without loss of generality, we assume that $j = 3$. Continue this procedure until one reaches $s \leq r$ such that $B_s = A_1$. Then

$$\beta_1 = [A_1, B_1] + [B_1, B_2] + \cdots + [B_{s-2}, B_{s-1}] + [B_{s-1}, A_1]$$

is a circuit in T_1. By induction, $\beta - \beta_1$ is a sum of circuits in T_1; hence, β itself is a sum of circuits. $\square$

Lemma 4.3. *Every nonzero circuit in $T_1 = T_1(\mathbb{Z}/p^r\mathbb{Z})$ may be written as a sum of triangles.*

Proof. Let $\alpha = \sum_{j=1}^{r-1}[A_j, A_{j+1}] + [A_r, A_1]$ be a circuit in T_1. If α is a 3-circuit, there is nothing to prove. By induction, it suffices to prove that α has a chord, i.e., there exist distinct integers i, j, $1 \leq i < j \leq r$ such that $[A_i, A_j] \in T_1$ and $j - i > 1$. Suppose α is an n-circuit, with $n > 3$. For each k, $1 \leq k \leq r-1$, let I_k denote the ideal of $\mathbb{Z}/p^r\mathbb{Z}$ generated by (the class of) p^k. Let $\mathcal{S} = \{I_k : 1 \leq k < r/2\}$. Observe that if $C, D \in \mathcal{S}$, then $[C, D] \in T_1$. Furthermore, if $[C, D] \in T_1$ and $C \notin \mathcal{S}$, then D must be in $\mathcal{S}$.

If all the A_i appearing in the cycle α are members of $\mathcal{S}$, then by the above observation $[A_1, A_2] + [A_2, A_3] + [A_3, A_1]$ is a triangle. If not, then we may assume without loss of generality that $A_2 \notin \mathcal{S}$. Since $[A_1, A_2] \in T_1$ and $[A_2, A_3] \in T_1$, we must have $A_1 \in \mathcal{S}$, $A_3 \in \mathcal{S}$. This forces $[A_1, A_3] \in T_1$, which completes the proof. $\square$

Lemma 4.4. *Every triangle in* $T_1(\mathbb{Z}/p^r\mathbb{Z})$ *may be written as a sum of triangles of the form* $\tau_{ij} = [I_1, I_i] + [I_i, I_j] + [I_j, I_1]$, *where* $1 < i$, $j < r$.

Proof. This follows immediately from the formal identity:

$$\begin{aligned}[I_h, I_i] + [I_i, I_j] + [I_j, I_h] &= ([I_1, I_h] + [I_h, I_i] + [I_i, I_1]) \\ &\quad + ([I_1, I_i] + [I_i, I_j] + [I_j, I_1]) \\ &\quad + ([I_1, I_j] + [I_j, I_h] + [I_h, I_1]) \\ &= \tau_{hi} + \tau_{ij} + \tau_{jh}. \qquad \square\end{aligned}$$

Lemma 4.5. *The set of triangles* $\mathcal{T} = \{\tau_{ij} : 1 < i < j < r\}$ *is* $(\mathbb{Z})$*-linearly independent in* T_1.

Proof. This follows readily from the fact that τ_{ij} is the only member of $\mathcal{T}$ involving the symbol $[I_i, I_j]$. $\square$

It follows from the sequence of lemmas above that:

Corollary 4.6. *The group* $\ker \partial_0$ *is a free abelian group with basis* $\mathcal{T}$.

In fact, $\tau_{ij} \in \mathcal{T}$ if and only if $i + j < r$, so an elementary counting argument gives:

Corollary 4.7. *The rank of* $\ker \partial_0$ *is* $(r-4)^2/4$ *if* r *is even or* $((r-4)^2 - 1)/4$ *if* r *is odd.*

We now examine the group $\operatorname{Im} \partial_1$. Observe that:

$$\gamma = \partial_1([I_i, I_j, I_k]) = [I_i, I_j] - [I_i, I_k] + [I_j, I_k] = [I_i, I_j] + [I_j, I_k] + [I_k, I_i]$$

is a triangle of T_1.

Since $I_i I_j I_k \neq 0$ and I_1 contains I_i, I_j and I_k, it follows readily that each of the symbols $[I_1, I_i, I_j]$, $[I_1, I_i, I_k]$ and $[I_1, I_j, I_k]$ are in T_2;

furthermore,

$$\begin{aligned}\gamma = \partial_1([I_i, I_j, I_k]) &= \partial_1([I_1, I_i, I_j]) + \partial_1([I_1, I_j, I_k]) + \partial_1([I_1, I_k, I_i]) \\ &= \tau_{ij} + \tau_{jk} + \tau_{ki},\end{aligned}$$

so in fact $\operatorname{Im} \partial_1$ is generated by those elements $\tau_{ij} \in \mathcal{T}$ such that $1 + i + j < r$, i.e., $i + j < r - 1$.

By the same computation as used to derive Corollary 4.7, we obtain:

Corollary 4.8. *The group* $\operatorname{Im} \partial_1$ *is a free abelian group of rank* $((r-5)^2 - 1)/4$ *if* r *is even or* $((r-5)^2)/4$ *if* r *is odd.*

In particular, we observe that the basis elements τ_{ij} for $\operatorname{Im}(\partial_1)$ identified in the previous discussion are a subset of those identified as a basis for $\ker(\partial_0)$. Thus, we have:

Corollary 4.9. *Suppose* $r \geq 4$. *Then* $H_1(\mathbb{Z}/p^r\mathbb{Z})$ *is a free abelian group of rank* $(r-4)/2$ *if* r *is even or* $(r-5)/2$ *if* r *is odd.*

5. Acyclicity. In this section, we make a general study of the higher homology groups $H_n(R)$, $n > 0$; in particular, we give various conditions sufficient for these groups to be zero.

Towards this end, it is convenient to introduce some notation: if $I_{j_0}, \ldots, I_{j_m}$ $(j = 1 \ldots, r)$ and $J_0, \ldots, J_n$ are mutually distinct ideals of a ring R such that $[I_{j_0}, \ldots, I_{j_m}] \in T_m(R)$ for each j and $[J_0, \ldots, J_n] \in T_n(R)$, and also $I_{j_0} \cdots I_{j_m} J_0 \cdots J_n \neq 0$, for each j, we write:

$$\sum_{j=1}^{r} [I_{j_0}, \ldots, I_{j_m}] \times [J_0, \ldots, J_n] = \sum_{j=1}^{r} [I_{j_0}, \ldots, I_{j_m}, J_0, \ldots, J_n].$$

Lemma 5.1 (Acyclicity lemma). *Suppose* $n > 0$ *and* $\alpha = \sum_{j=1}^{r} [I_{j_0}, \ldots, I_{j_n}] \in \ker(\partial_{n-1})$. *If there exists an ideal* $J \notin \{I_{j_k} : 1 \leq j \leq r,\ 0 \leq k \leq n\}$ *such that* $J I_{j_0} \cdots I_{j_n} \neq 0$ *for all* j, $1 \leq j \leq r$, *then* $\alpha \in \operatorname{Im}(\partial_n)$. *Thus the class of* α *in* $H_n(R)$ *is zero.*

Proof. If such J exists, then

$$\begin{aligned}\partial_n((-1)^{n+1}\sum_{j=1}^{r}[I_{j_0},\dots,I_{j_n}]\times[J])&\\ &= (-1)^{n+1}\sum_{i=0}^{n}\sum_{j=1}^{r}(-1)^n[I_{j_0},\dots,\hat{I}_{j_i},\dots,I_{j_n},J]+\alpha\\ &= -\partial_{n-1}(\alpha)\times[J]+\alpha=\alpha.\end{aligned}$$

So indeed $\alpha\in\operatorname{Im}(\partial_n)$, as desired. □

Theorem 5.2. *Let R be a ring satisfying at least one of the following conditions*:

- *There exists a nonzero element $x\in R$ which is neither a unit nor a zero-divisor.*
- *R has infinitely many maximal ideals.*
- *R is reduced, Noetherian, and of positive* (*Krull*) *dimension.*

Then $H_n(R)=0$ for all $n>0$.

Proof. First, suppose $x\in R$ is a nonzero element which is neither a unit nor a zero-divisor. Then it is easy to see that x^i and x^j are associate if and only if $i=j$. Thus,

$$(x)\supset(x^2)\supset(x^3)\supset\cdots$$

is a descending chain of distinct ideals. Furthermore, if I is a nonzero ideal, then $(x^i)I\neq 0$, for any $i\geq 1$ because x (and hence x^i) is not a zero-divisor. Given any $n>0$ and $\alpha=\sum_{j=1}^{r}[I_{j_0},\dots,I_{j_n}]\in\ker(\partial_{n-1})$ as in Lemma 5.1, choose m such that $(x^m)\neq I_{j_k}$ for all j, k. Then $J=(x^m)$ satisfies the hypotheses of the lemma and the assertion follows.

Now suppose R has infinitely many maximal ideals, and suppose α is as above. For each j, let $A_j=\operatorname{ann}(I_{j_0}\cdots I_{j_n})$; A_j is a proper ideal of R, so choose some maximal ideal $\mathfrak{m}_j$ such that $A_j\subseteq m_j$. For each j, $1\leq j\leq r$ and k, $1\leq k\leq n$, choose a maximal ideal $\mathfrak{m}_{jk}$ such that $I_{j_k}\subseteq\mathfrak{m}_{jk}$. Now let

$$D=\bigcup_{j=1}^{r}\mathfrak{m}_j\cup\bigcup_{j=1}^{r}\bigcup_{k=1}^{n}\mathfrak{m}_{jk}.$$

Let $\mathfrak{m}$ be some other maximal ideal of R not equal to any $\mathfrak{m}_j$ or $\mathfrak{m}_{jk}$. By [**4**, Proposition 1.11], $\mathfrak{m} \not\subseteq D$. Choose $x \in \mathfrak{m} - D$. Evidently, (x) is a proper ideal of R. Furthermore, since $x \notin \mathfrak{m}_{jk}$, $(x) \neq I_{j_k}$ for any j, k. Finally, $x \notin \mathfrak{m}_j \supseteq A_j$ implies that $(x)I_{j_0} \cdots I_{j_n} \neq 0$ for all j. Thus, $J = (x)$ satisfies the hypotheses of Lemma 5.1, and the assertion is proved.

Last, suppose R is reduced, Noetherian, and $\dim R > 0$. Let $\mathfrak{p}_0$ be a minimal prime ideal of R which is not also maximal. Then $\dim(R/\mathfrak{p}_0) > 0$, so in particular $R/\mathfrak{p}_0$ is not Artinian. Thus, there is a strictly descending sequence of ideals of R:

$$R \supsetneq J_1 \supsetneq J_2 \supsetneq \cdots$$

each of which strictly contains $\mathfrak{p}_0$.

Let $\mathfrak{p}_0, \dots, \mathfrak{p}_n$ be the minimal prime ideals of R; there are only finitely many of them because R is Noetherian ([**4**, Chapter 6, Exercise 9]). It is well-known (cf. [**4**, Proposition 1.8]) that the nilradical of R is the intersection of the prime ideals of R, hence also of the minimal prime ideals of R. Thus in our case, $\cap_{i=0}^n \mathfrak{p}_i = 0$.

We claim that $IJ_m \neq 0$ for any nonzero ideal I and any $m \geq 1$. Suppose to the contrary that $IJ_m = 0$. Since $\cap_{i=0}^n \mathfrak{p}_i = 0$, this means $\mathfrak{p}_i \supseteq IJ_m$ for each i. Since $\mathfrak{p}_i$ is prime, $\mathfrak{p}_i \supseteq I$ or $\mathfrak{p}_i \supseteq J_m$. In the latter case, $\mathfrak{p}_i \supseteq J_m \supseteq \mathfrak{p}_0$, so by minimality of $\mathfrak{p}_i$, we must have $\mathfrak{p}_i = J_m = \mathfrak{p}_0$. However, J_m strictly contains $\mathfrak{p}_0$, so this is impossible. Thus, we must have $\mathfrak{p}_i \supseteq I$ for each i; hence, $0 = \cap_{i=0}^n \mathfrak{p}_i \supseteq I$ and so $I = 0$.

Continuing with the proof of Theorem 5.2, suppose $n > 0$ and $\alpha = \sum_{j=1}^r [I_{j_0}, \dots, I_{j_n}] \in \ker(\partial_{n-1})$ as in Lemma 5.1. Choose $m \geq 1$ such that $J_m \notin \{I_{j_k} : 1 \leq j \leq r,\ 0 \leq k \leq n\}$. Then the previous paragraph shows that for any j, $1 \leq j \leq r$, $JI_{j_0} \cdots I_{j_n} \neq 0$; thus we may take $J = J_m$ and apply Lemma 5.1 to conclude.

6. χ for finite rings.

Theorem 5.2 establishes that the higher homology groups are uninteresting for a large class of rings. Finite rings, on the other hand, satisfy none of the conditions of the theorem; in this section we examine these rings more closely. While the prospect of computing the actual homology groups seems daunting, the Euler characteristic turns out to be a much more tractable object. In particular, if R is a finite ring, hence having only finitely many ideals, it is clear from the definition that each $T_n(R)$ has finite rank and that $T_n(R) = 0$ for sufficiently large n. Hence the hypotheses of Proposition 2.2 are satisfied and we may use it to compute the Euler characteristic. In particular, let $U_n = U_n(R)$ denote the number of *unordered* $(n+1)$-tuples $\{I_0, \dots, I_n\}$ of distinct ideals whose product is nonzero. Then we have the convenient formula

$$\chi(R) = \sum_{n=0}^{\infty} (-1)^n |U_n|.$$

Throughout this section, if a set is denoted by an uppercase letter, we will use the corresponding lower case letter for the number of elements in that set. For example, we will write u_n for $|U_n|$ as defined above.

We begin by examining the same rings encountered in Section 4, namely those of the form $R = \mathbb{Z}/p^r\mathbb{Z}$ where p is a prime and $r \geq 1$ is some integer. Recall that for each i, $1 \leq i \leq r-1$, there is an ideal I_i of R generated by (the class of) (p^i) and that these are all the proper ideals of R. In the following, we implicitly identify the ideal I_i with the integer i. Since U_n is the set of unordered $(n+1)$-tuples $\{I_0, \dots, I_n\}$ of distinct proper ideals of R, we have

$$u_n = \sum_{k=1}^{r-1} P(k, n+1)$$

where $P(k, n+1)$ represents the number of partitions of k into $(n+1)$ distinct positive integer parts. Hence

$$\begin{aligned}\chi(R) &= \sum_{n=0}^{\infty} (-1)^n s_n = \sum_{n=0}^{\infty} (-1)^n \sum_{k=1}^{r-1} P(k, n+1) \\ &= \sum_{k=1}^{r-1} \sum_{n=1}^{\infty} (-1)^{n+1} P(k, n).\end{aligned}$$

We may interpret the inner sum

$$\sum_{n=1}^{\infty} (-1)^{n+1} P(k, n) = -\sum_{n=1}^{\infty} (-1)^n P(k, n)$$

as the coefficient of x^k in the power series:

$$-(1-x)(1-x^2)(1-x^3)\cdots.$$

By Euler's pentagonal theorem, we have:

$$\begin{aligned}&-(1-x)(1-x^2)(1-x^3)\cdots\\&\qquad=-1+x+x^2-x^5-x^7+x^{12}+x^{15}-x^{22}-x^{26}+\cdots,\end{aligned}$$

where the pattern of signs on the right (from the second term forth) is $++--$ and the exponents alternate between the "pentagonal" numbers of the form

$$P_m=\frac{m(3m-1)}{2}$$

and the related numbers

$$Q_m=\frac{m(3m+1)}{2},$$

where $m=1,2,3,\ldots$.

Hence

$$\chi(R)=-\sum_{k=1}^{r-1}\sum_{n=1}^{\infty}(-1)^nP(k,n)$$

is the sum of the coefficients of the terms $x,x^2,\ldots,x^{r-1}$ appearing in the above series. It is clear from the sign pattern that this sum is either 0, 1, or 2, depending on the value of r in relation to the numbers P_m and Q_m.

We summarize our findings in the following:

Theorem 6.1. *Let p be a prime and $r\geq 1$ an integer. Then $\chi(\mathbb{Z}/p^r\mathbb{Z})$ is equal to 0, 1, or 2, depending on the value of r in relation to the various pentagonal numbers $m(3m-1)/2$ and the associated numbers $m(3m+1)/2$.*

By being careful with counting methods, we can prove the following theorem, whose proof is facilitated by the paucity of ideals in a field.

Theorem 6.2. *Let R be a finite ring and F a field. Then*

$$\chi(R \times F) = 2 - \chi(R).$$

Proof. Let π_1, π_2 denote the projection maps onto the respective factors of $R \times F$. Recall that for any $n \geq 0$, the typical element $U_n(R \times F)$ is an unordered $(n+1)$-tuple $\{I_0, \dots , I_n\}$ where $I_0 \cdots I_n \neq 0$. Moreover, each $I_i = A_i \times B_i$, with $A_i = \pi_1(I_i)$ being an ideal of R and $B_i = \pi_2(I_i)$ an ideal of F, i.e. $B_i = 0$ or $B_i = fF$. In order to have $I_0 \cdots I_n \neq 0$, at least one of $\prod_{i=0}^n A_i \neq 0$ or $\prod_{i=0}^n B_i \neq 0$. Define:

$$\begin{aligned}
U_n^1(R \times F) &= \left\{\{I_0, \dots , I_n\} \in U_n(R \times F) : \prod_{i=0}^n A_i \neq 0\right\} \\
U_n^2(R \times F) &= \left\{\{I_0, \dots , I_n\} \in U_n(R \times F) : \prod_{i=0}^n B_i \neq 0\right\} \\
&= \big\{\{I_0, \dots , I_n\} \in U_n : B_i = F \text{ for each } i\big\} \\
U_n^3(R \times F) &= U_n^1(R \times F) \cap U_n^2(R \times F) \\
&= \big\{\{I_0, \dots , I_n\} \in U_n(R \times F) : B_i = F \\
&\qquad \text{for each } i \text{ and } (A_0, \dots , A_n) \in U_n(R)\big\}.
\end{aligned}$$

Thus we have $u_n = u_n^1 + u_n^2 - u_n^3$.

It is clear from the above description that $u_n^3(R \times F) = u_n(R)$ and furthermore that if $\{I_0, \dots , I_n\} \in U_n^2(R \times F)$, then $A_0, \dots , A_n$ are allowed to be any (mutually distinct) proper ideals of R; hence, $u_n^2(R \times F) = \binom{\rho}{n+1}$, where ρ is the number of proper ideals in R.

The set U_n^1 is slightly more difficult to analyze: define

$$\begin{aligned}
U_n^{1,0}(R \times F) &= \big\{\{I_0, \dots , I_n\} \in U_n^1(R \times F) : I_i \neq R \times 0 \\
&\qquad\qquad \text{for all } \ i, \ 0 \leq i \leq n\big\} \\
U_n^{1,1}(R \times F) &= U_n^1(R \times F) - U_n^{1,0}(R \times F).
\end{aligned}$$

Clearly $u_n^{1,0}(R \times F) + u_n^{1,1}(R \times F) = u_n^1(R \times F)$. Somewhat more subtly, there is a natural bijective map $U_n^{1,0}(R \times F) \to U_{n+1}^{1,1}(R \times F)$

sending $\{I_0, \dots, I_n\} \mapsto \{I_0, \dots, I_n, R \times 0\}$, so it is also true that $u_n^{1,0}(R \times F) = u_{n+1}^{1,1}(R \times F)$.

Combining all these relations, we have:

$$\begin{aligned}
&\chi(R \times F)\\
&= \sum_{n=0}^{\infty} (-1)^n u_n(R \times F)\\
&= \sum_{n=0}^{\infty} (-1)^n (u_n^1(R \times F) + u_n^2(R \times F) - u_n^3(R \times F))\\
&= \sum_{n=0}^{\infty} (-1)^n \left(u_n^{1,0}(R \times F) + u_n^{1,1}(R \times F) + \binom{\rho}{n+1} - u_n(R)\right)\\
&= \sum_{n=0}^{\infty} (-1)^n u_n^{1,0}(R \times F) + \sum_{n=0}^{\infty} (-1)^n u_n^{1,1}(R \times F)\\
&\quad + \sum_{n=0}^{\infty} (-1)^n \binom{\rho}{n+1} - \sum_{n=0}^{\infty} (-1)^n u_n(R))\\
&= \sum_{n=0}^{\infty} (-1)^n u_{n+1}^{1,1}(R \times F) + \sum_{n=0}^{\infty} (-1)^n u_n^{1,1}(R \times F) + 1 - \chi(R)\\
&= u_0^{1,1}(R \times F) + 1 - \chi(R)\\
&= 2 - \chi(R) \qquad \square
\end{aligned}$$

Corollary 6.3. *Let $F_1, \dots, F_n$ be fields. Then*

$$\chi(F_1 \times \cdots \times F_n) = 1 + (-1)^n.$$

We have not yet found a general method for computing $\chi(\mathbb{Z}/n\mathbb{Z})$, where $n > 0$ is an arbitrary integer. However, it is possible to analyze some specific examples using idiosyncratic counting methods:

Theorem 6.4. *Let p, q be primes and $r \geq 2$ an integer. Then*

$$\chi(\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/q^2\mathbb{Z}) = 2 - \chi(\mathbb{Z}/p^r\mathbb{Z}) + \sum_{k=1}^{r-1} \chi(\mathbb{Z}/p^k\mathbb{Z}).$$

Proof. For convenience, set $R = \mathbb{Z}/p^r\mathbb{Z}$ and $S = \mathbb{Z}/q^2\mathbb{Z}$; to ease notation, we denote the unique proper ideal of S by (q). As in Theorem 6.2, let π_1, π_2 be the projection maps onto the respective factors of $R\times S$. As before, for any $n \geq 0$, the typical element $U_n(R\times S)$ is an unordered $(n+1)$-tuple $\{I_0, \dots, I_n\}$ where $I_0 \cdots I_n \neq 0$ and $I_i = A_i \times B_i$, where $A_i = \pi_1(I_i)$ an ideal of R and $B_i = \pi_2(I_i)$ an ideal of S. In this situation, B_i may either be 0, (q) or S. As before, $\prod_{i=0}^n A_i \neq 0$ or $\prod_{i=0}^n B_i \neq 0$.

$$\begin{aligned}
U_n^1(R \times S) &= \Big\{\{I_0, \dots, I_n\} \in U_n(R \times S) : \prod_{i=0}^n A_i \neq 0\Big\}\\
U_n^2(R \times S) &= \Big\{\{I_0, \dots, I_n\} \in U_n(R \times S) : \prod_{i=0}^n B_i \neq 0\Big\}\\
&= \big\{\{I_0, \dots, I_n\} \in U_n : \text{ there exists some } i_0 \text{ such that}\\
&\qquad B_{i_0} = S \text{ or } B_{i_0} = (q) \text{ and } B_i = S \text{ for all } i \neq i_0\big\}\\
U_n^3(R \times S) &= U_n^1(R \times S) \cap U_n^2(R \times S).
\end{aligned}$$

Now define

$$\begin{aligned}
U_n^{1,0}(R \times S) &= \big\{\{I_0, \dots, I_n\} \in U_n^1(R \times S) : I_i \neq R \times 0 \text{ for all } i,\\
&\qquad 0 \leq i \leq n\big\}\\
U_n^{1,1}(R \times S) &= U_n^1(R \times S) - U_n^{1,0}(R \times S)\\
U_n^{3,q}(R \times S) &= \big\{\{I_0, \dots, I_n\} \in U_n^3(R \times S) : \text{ there exists } i_0 \text{ such that}\\
&\qquad B_{i_0} = (q) \text{ and } B_i = S \text{ for all } i \neq i_0\big\}\\
U_n^{3,S}(R \times S) &= U_n^3(R \times S) - U_n^{3,q}(R \times S)\\
&= \big\{\{I_0, \dots, I_n\} \in U_n^3(R\times S) : B_i = S \text{ for all } i,\ 0 \leq i \leq n\big\}.
\end{aligned}$$

It follows immediately from the above definitions that $u_n(R \times S) = u_n^1(R \times S) + u_n^2(R \times S) - u_n^3(R \times S)$.

The map $U_n^{1,0}(R \times S) \to U_{n+1}^{1,1}(R \times S)$ sending $\{I_0, \dots, I_n\} \mapsto \{I_0, \dots, I_n, R\times 0\}$ establishes a bijection, so $u_n^{1,0}(R\times S) = u_{n+1}^{1,1}(R\times S)$.

Now let ρ denote the number of proper ideals in R. Evidently, by the description given above,

$$u_n^2(R \times S) = \rho\binom{\rho}{n} + \binom{\rho}{n+1}.$$

Finally, it is clear that $u_n^{3,S}(R\times S)=u_n(R)$. Observe that, given a typical element $\{I_0,\dots,I_n\}$ of $U_n^{3,q}(R\times S)$, we may assume without loss of generality that $B_j=S$ for all $j>0$ and that $B_0=(p^k)\times(q)$ for some k, $1\le k\le r-1$. (This is the only place in the proof where we use the fact that R has the form $\mathbb{Z}/p^r\mathbb{Z}$.) Thus, in order to have $\prod_{i=0}^n A_i\ne 0$, we must have $\{A_1,\dots,A_n\}\in U_{n-1}(\mathbb{Z}/p^{r-k}\mathbb{Z})$. Hence, $u_n^{3,q}(R\times S)=\sum_{k=1}^{r-1}u_{n-1}(\mathbb{Z}/p^k\mathbb{Z})$.

Collecting this information together, we have:

$$\begin{aligned}
&\chi(R\times S)\\
&\quad=\sum_{n=0}^{\infty}(-1)^n u_n(R\times S)\\
&\quad=\sum_{n=0}^{\infty}(-1)^n(u_n^1(R\times S)+u_n^2(R\times S)-u_n^3(R\times S))\\
&\quad=\sum_{n=0}^{\infty}(-1)^n\Big(u_n^{1,0}(R\times S)+u_n^{1,1}(R\times S)\\
&\qquad\qquad\qquad+\rho\binom{\rho}{n}+\binom{\rho}{n+1}-u_n(R)-\sum_{k=1}^{r-1}u_{n-1}(\mathbb{Z}/p^k\mathbb{Z})\Big)\\
&\quad=\sum_{n=0}^{\infty}(-1)^n(u_n^{1,0}(R\times S)+u_{n+1}^{1,1}(R\times S))\\
&\qquad+\sum_{n=0}^{\infty}(-1)^n\Big(\rho\binom{\rho}{n}+\binom{\rho}{n+1}\Big)\\
&\qquad-\sum_{n=0}^{\infty}(-1)^n u_n(R)-\sum_{n=1}^{\infty}(-1)^n\sum_{k=1}^{r-1}u_{n-1}(\mathbb{Z}/p^k\mathbb{Z}))\\
&\quad=u_0^{1,1}(R\times S)+1-\chi(R)+\sum_{k=1}^{r-1}\sum_{n=1}^{\infty}(-1)^{n-1}u_{n-1}(\mathbb{Z}/p^k\mathbb{Z})\\
&\quad=2-\chi(R)+\sum_{k=1}^{r-1}\chi(\mathbb{Z}/p^k\mathbb{Z}).
\end{aligned}$$

Thus,

$$\chi(\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/q^2\mathbb{Z}) = 2 - \chi(\mathbb{Z}/p^r\mathbb{Z}) + \sum_{k=1}^{r-1} \chi(\mathbb{Z}/p^k\mathbb{Z}). \qquad \square$$

From Theorem 6.4 and Theorem 6.1, we see that the value of $\chi(\mathbb{Z}/p^r\mathbb{Z})$ may be made arbitrary large by choosing r large enough. By Theorem 6.2, we see that by taking the product with a field, we can obtain a ring whose Euler characteristic is arbitrary large and negative. Summarizing, we have:

Corollary 6.5. *The value of* $\chi(R)$ *is unbounded in both the positive and negative directions as* R *ranges over the set of finite rings.*

It is not difficult to develop *ad hoc* counting methods along similar lines to compute $\chi(\mathbb{Z}/p^r\mathbb{Z}\times\mathbb{Z}/q^3\mathbb{Z})$, but it is not clear how to generalize this method to compute $\chi(\mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/q^s\mathbb{Z})$ for arbitrary $s \geq 1$.

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