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REGULAR SETS OF SAMPLING AND INTERPOLATION IN BERGMAN SPACES

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ABSTRACT. Let ρ denote the pseudohyperbolic metric in the unit disk $\mathbf{D}$ in the complex plane. We give examples of analytic functions g satisfying the condition $|g(z)| \simeq \rho(z,\Gamma)(1-|z|)^{-\alpha}$, $z \in \mathbf{D}$, in the case when Γ are A^p zero sets considered by Horowitz and Luecking. This helps to solve directly interpolating and sampling problems for these sequences.

1. Introduction. For $0 < p < \infty$, the Bergman space A^p is the set of functions analytic in the unit disk $\mathbf{D}$ with

$$\|f\|_p = \left(\int_{\mathbf{D}} |f(z)|^p dA(z)\right)^{1/p} < \infty,$$

where dA denotes the normalized Lebesgue area measure on $\mathbf{D}$.

A sequence $\{z_k\}$ of distinct points in $\mathbf{D}$ is an interpolation sequence for A^p, if the interpolation problem

$$f(z_k) = w_k, \quad k = 1, 2, \dots ,$$

has a solution $f \in A^p$ provided

$$\sum_{k=1}^{\infty} (1-|z_k|^2)^2 \, |w_k|^p < \infty.$$

A sequence $\{z_k\}$ of distinct points in $\mathbf{D}$ is a sampling sequence for A^p if there exist positive constants K_1, K_2 such that

$$K_1 \|f\|_p^p \le \sum_{k=1}^{\infty} (1-|z_k|^2)^2 \, |f(z_k)|^p \le K_2 \|f\|_p^p.$$

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Sufficient and necessary conditions for a sequence to be interpolation or sampling for A^p are given in terms of pseudohyperbolic densities. These characterizations are due to Seip for the case $p = 2$. Extensions for general values of p can be found in [**3, 7, 8**] and in the book [**1**]. Let ρ denote the pseudohyperbolic metric in $\mathbf{D}$, that is,

$$\rho(z,\zeta) = \left|\frac{\zeta - z}{1-\bar{\zeta}z}\right|, \quad z,\zeta \in \mathbf{D}.$$

We say that a sequence of points $\Gamma = \{z_n\}$ in $\mathbf{D}$ is uniformly discrete if

$$\delta(\Gamma) = \inf_{j\neq k} \rho(z_j, z_k) > 0.$$

For the uniformly discrete set Γ, the lower uniform density of Γ is

$$D^-(\Gamma) = \liminf_{r\to 1} \frac{\inf_{\zeta\in\mathbf{D}} \int_0^r n(\Gamma,\zeta,s)\,ds}{\log(1/(1-r))}$$

and the upper uniform density of Γ is

$$D^+(\Gamma) = \limsup_{r\to 1} \frac{\sup_{\zeta\in\mathbf{D}} \int_0^r n(\Gamma,\zeta,s)\,ds}{\log(1/(1-r))},$$

where $n(\Gamma,\zeta,s)$ denotes the number of points of Γ that lie in the pseudohyperbolic disk $\{z : \rho(\zeta,z) < s\}$. The following theorems are due to Seip (for the case $p = 2$).

Theorem S1. *For $0 < p < \infty$, a sequence Γ of distinct points in the unit disk is an interpolation sequence for A^p if and only if it is uniformly discrete and $D^+(\Gamma) < 1/p$.*

Theorem S2. *For $0 < p < \infty$, a sequence Γ of distinct points in the unit disk is a sampling sequence for A^p if and only if it is a finite union of uniformly discrete subsequences and it has a uniformly discrete subsequence Γ' for which $D^-(\Gamma') > 1/p$.*

Unfortunately, lower and upper uniform densities can be quite difficult to compute. Duren, Schuster and Seip [**2**] calculated directly lower and upper uniform densities of the sequence Γ defined as follows. Let

$$d\mu(z) = \frac{a\,dA(z)}{(1-|z|^2)^2}, \quad a > 0,$$

and divide the unit disk into disjoint annuli

$$R_n = \{z : t_{n-1} \le |z| < t_n\}, \quad n = 1, 2, \dots ,$$

such that $\mu(R_n) = 2^{n-1}$. Next divide each annulus into 2^{n-1} cells Q_{nj} by placing radial segments at angles $j2^{-n+2}\pi$, $j = 1, 2, \dots , 2^{n-1}$, set $\zeta_{nj} = \int_{Q_{nj}} z\, d\mu(z)$ and let Γ be an enumeration of ζ_{nj}. Duren, Schuster and Seip [**2**] proved that

$$D^-(\Gamma) = D^+(\Gamma) = \frac{a}{2}.$$

Next using some additional lemmas they have been able to find the uniform densities of A^p zero sequences considered by Horowitz and Luecking. Horowitz [**4, 5**] considered the sequence consisting of 2^n equally spaced points on the circle $|z| = (1/\mu)^{2^{-n}}$, $\mu > 1$. Luecking [**6**] considered the set consisting of $\lfloor \beta^n \rfloor$ equally spaced points on each circle of radius $r_n = 1 - \gamma\, \beta^{-n}$, $\beta > 1$, $\gamma > 0$.

If $f(z)$ and $g(z)$ are nonnegative functions in $\mathbf{D}$, then we write $f(z) \simeq g(z)$ if there are positive constants C_1 and C_2 such that

$$C_1 f(z) \le g(z) \le C_2 f(z) \quad \text{for all} \quad z \in \mathbf{D}.$$

However, if a uniformly discrete sequence Γ admits an analytic function g with the property

$$|g(z)| \simeq \rho(z, \Gamma)(1 - |z|^2)^{-\alpha}, \quad z \in \mathbf{D} \tag{1}$$

for some $\alpha > 0$, then a sequence Γ is an interpolation sequence for A^p if and only if $\alpha < 1/p$, and Γ is a sampling sequence for A^p if and only if $\alpha > 1/p$. Then also

$$D^+(\Gamma) = D^-(\Gamma) = \alpha.$$

Moreover, in the case when (1) holds with $\alpha < 1/p$, using the function g, one can construct directly the function f solving the interpolation problem for A^p. In the case when (1) holds with $\alpha > 1/p$, any $f \in A^p$ can be represented in terms of g (see, e.g., [**1**] for details).

One example of family of sequences and the corresponding function g satisfying (1) was obtained by Seip in 1993 [**10**]. For $a > 1$ and $b > 0$, Seip considered the set of points in the upper half-plane of the form

$$\Lambda(a,b) = \{a^m(bn+i) : m \in \mathbf{Z},\ n \in \mathbf{Z}\}$$

and

$$\Gamma(a,b) = \psi(\Lambda(a,b)) \subset \mathbf{D},$$

where $\psi(\zeta) = (\zeta - i)/(\zeta + i)$, and constructed a function g such that

$$|g(z)| \simeq (1-|z|^2)^{-\beta}\,\rho(z,\Gamma(a,b)),$$

where $\beta = (2\pi)/(b\log a)$.

Here, we prove that in the case when Γ is the Horowitz sequence, the function g defined by Horowitz in [**5**, p. 330] satisfies (1). We also construct a function that has property (1) for the Luecking sets. Our proofs are independent of results obtained in [**2**].

2. Main results. Let Γ be the Horowitz set of points equally spaced on the circles $|z| = (1/\mu)^{2^{-n}}$, $n = 1, 2, \ldots$, such that $z^{2^n} = 1/\mu$, $\mu > 1$. Set

$$H(z) = \prod_{n=1}^{\infty} \frac{1 - z^{2^n}\mu}{1-(1/\mu)z^{2^n}}, \quad z \in \mathbf{D}. \tag{2}$$

The function H was defined by Horowitz in his paper [**5**]. Horowitz also showed that there is a constant C such that

$$|H(z)| \le \frac{C}{(1-|z|^2)^{\alpha}}, \quad z \in \mathbf{D},$$

where $\alpha = \log\mu/\log 2$.

We will prove the following

Theorem 1. *If H is the function defined by* (2)*, then*

$$|H(z)| \simeq \rho(z,\Gamma)(1-|z|)^{-\alpha}$$

with $\alpha = \log\mu/\log 2$.

Proof. We first show that there is a positive constant C such that

$$|H(z)| \leq C\rho(z,\Gamma)\,\frac{1}{(1-|z|)^{\alpha}}, \quad z\in \mathbf{D}. \tag{3}$$

To this end, put $\beta = 1/\mu$ and for a positive integer n define

$$H_n(z) = \frac{\beta - z^{2^n}}{1 - z^{2^n}\beta}, \quad z \in \mathbf{D}.$$

Then

$$\frac{H(z)}{H_n(z)} = \mu^n \prod_{k=1}^{n-1} \frac{\beta - z^{2^k}}{1 - z^{2^k}\beta} \prod_{k=n+1}^{\infty} \frac{1-(z^{2^k}/\beta)}{1-z^{2^k}\beta}.$$

Note first that if z is in the annulus $A_n = \{z : \beta^{2^{-n+(1/2)}} \leq |z| \leq \beta^{2^{-n-(1/2)}}\}$, then the modulus of the last product is bounded above by a constant independent of n. Thus

$$\left|\frac{H(z)}{H_n(z)}\right| \leq C\mu^n.$$

Since $z \in A_n$ if and only if

$$\frac{\log \mu}{\sqrt{2}} \cdot 2^{-n} \leq \log \frac{1}{|z|} \leq (\sqrt{2}\log \mu)\cdot 2^{-n},$$

we see that

$$(1-|z|) \leq \log \frac{1}{|z|} \leq (\log \mu)\sqrt{2}\cdot 2^{-n},$$

and consequently, $2^n \leq (\sqrt{2}\log\mu)/(1-|z|)$. This implies that if $z\in A_n$, then

$$\left|\frac{H(z)}{H_n(z)}\right| \leq C\mu^n = C2^{n(\log\mu/\log 2)} \leq C\left(\frac{\log \mu}{1-|z|}\right)^{\log\mu/\log 2}$$

with a constant C independent of n.

Let $z\in\mathbf{D}$ be arbitrarily chosen. Then there is an n such that $z\in A_n$ and there is a $z_k \in \Gamma$ such that $\rho(z,\Gamma) = |(z-z_k)/(1-\bar{z}_k z)| = \rho(z,z_k)$.

If z_k is in A_n, then z_k is one of the roots of the equation $z^{2^n} = \beta$. Let $\beta_1, \beta_2, \dots, \beta_{2^n}$ denote the distinct roots of this equation. Then

$$|H_n(z)| = \left| \frac{(z-\beta_1)\cdots(z-\beta_{2^n})}{(1-\bar{\beta}_1 z)\cdots(1-\bar{\beta}_{2^n} z)} \right| \le \left| \frac{z-\beta_i}{1-\bar{\beta}_i z} \right|, \quad i = 1, 2, \dots, 2^n,$$

and (3) follows from the last two inequalities. Now note that each annulus A_n contains the pseudohyperbolic disks with centers at β_i and a positive radius δ. (One can show that $\delta > \beta/7$). So, if $z \in A_n$ and z_k is not equal to any β_i, then $\rho(z, \Gamma) = \rho(z, z_k) > \delta$ and consequently

$$|H(z)| \le C \frac{1}{(1-|z|)^\alpha} \le \frac{C}{\delta}\, \delta \frac{1}{(1-|z|)^\alpha} \le \frac{C}{\delta}\, \rho(z, \Gamma) \frac{1}{(1-|z|)^\alpha}.$$

Our aim is now to prove the other inequality

$$|H(z)| \ge C\rho(\Gamma, z) \frac{1}{(1-|z|)^\alpha}, \quad z \in \mathbf{D}. \tag{4}$$

We first show that for $z \in A_n$,

$$\left| \frac{H(z)}{H_n(z)} \right| \ge \frac{C}{(1-|z|)^\alpha} \tag{5}$$

with a constant C independent of n. As above we write

$$\frac{H(z)}{H_n(z)} = \mu^n \prod_{k=1}^{n-1} \frac{\beta - z^{2^k}}{1 - z^{2^k}\beta} \prod_{k=n+1}^{\infty} \frac{1-(z^{2^k}/\beta)}{1-z^{2^k}\beta}$$

and claim that for $z \in A_n$ the modulus of each of the last two products is bounded below. Indeed, for $|z| \le \beta^{2^{-n-(1/2)}}$,

$$\left| \prod_{k=n+1}^{\infty} \frac{1-(z^{2^k}/\beta)}{1-z^{2^k}\beta} \right| \ge \prod_{k=1}^{\infty} \frac{1-\beta^{2^{k-1/2}-1}}{1-\beta^{2^{k-1/2}+1}},$$

and the last product converges. On the other hand, if $|z| \ge \beta^{2^{-n+(1/2)}}$, then

$$\begin{aligned} \left| \prod_{k=1}^{n-1} \frac{z^{2^k} - \beta}{1 - z^{2^k}\beta} \right| &\ge \prod_{k=1}^{n-1} \frac{\beta^{2^{k-n+1/2}} - \beta}{1 - \beta\beta^{2^{k-n+1/2}}} = \prod_{k=1}^{n-1} \frac{\beta^{2^{-k+1/2}} - \beta}{1 - \beta\beta^{2^{-k+1/2}}} \\ &\ge \frac{\beta^{1/\sqrt{2}} - \beta}{1 - \beta\beta^{1/\sqrt{2}}} \prod_{k=1}^{n-2} \frac{\beta^{2^{-k}} - \beta}{1 - \beta\beta^{2^{-k}}}. \end{aligned}$$

Put $n_0 = \lfloor \log \mu / \log 2 \rfloor$, and write

$$\prod_{k=1}^{n-2} \frac{\beta^{2^{-k}} - \beta}{1 - \beta\beta^{2^{-k}}} = \prod_{k=1}^{n_0} \frac{\beta^{2^{-k}} - \beta}{1 - \beta\beta^{2^{-k}}} \prod_{k=n_0+1}^{n-2} \frac{\beta^{2^{-k}} - \beta}{1 - \beta\beta^{2^{-k}}}.$$

Since

$$\frac{\beta^{2^{-k}} - \beta}{1 - \beta\beta^{2^{-k}}} = 1 - \frac{(1+\beta)(1-\beta^{2^{-k}})}{1 - (\beta^{2^{-k}})^{(2^k+1)}} > 1 - \frac{1+\beta}{(1+2^k)\beta},$$

we get

$$\prod_{k=n_0+1}^{n-2} \frac{\beta^{2^{-k}} - \beta}{1 - \beta\beta^{2^{-k}}} = e^{\sum_{k=n_0+1}^{n-2} \log\left(1-(1+\beta)/((1+2^k)\beta)\right)}$$
$$> e^{-C\sum_{k=n_0+1}^{n-2} (1+\beta)/((1+2^k)\beta)},$$

and our claim is the consequence of the convergence of the series $\sum_k (1+\beta)/((1+2^k)\beta)$. Now, to obtain (5) similar reasoning to that in the proof of the first inequality can be applied.

Let $z \in \mathbf{D}$ be arbitrarily chosen. Then there is an n such that $\beta^{2^{-n+(1/2)}} \le |z| \le \beta^{2^{-n-(1/2)}}$ and β_i, where β_i is a root of $z^{2^n} = \beta$, such that $|\arg z - \arg \beta_i| \le 2\pi/2^{n+1}$. Let $z_k \in \Gamma$ be such that $\rho(z,\Gamma) = |(z-z_k)/(1-\bar{z}_k z)| = \rho(z, z_k)$. If $z_k = \beta_i$, then note that

$$\lim_{z\to\beta_i} \frac{|H_n(z)|}{\left|(z-\beta_i)/(1-\bar{\beta}_i z)\right|} = \frac{2^n \beta(1-\beta^{2^{-n+1}})}{\beta^{2^{-n}}(1-\beta^2)}$$

and

$$\frac{2^n \beta(1-\beta^{2^{-n+1}})}{\beta^{2^{-n}}(1-\beta^2)} > \beta^{-2^{-n}+1} > \beta.$$

It is also clear that the function $H_n(z)/[(z-\beta_i)/(1-\bar{\beta}_i z)]$ is analytic and nonvanishing in the cell

$$\left\{ z : \beta^{2^{-n+(1/2)}} \le |z| \le \beta^{2^{-n-(1/2)}},\ |\arg z - \arg \beta_i| \le \frac{\pi}{2^n} \right\}.$$

Thus its modulus attains minimum on the boundary. Moreover,

$$\frac{|H_n(z)|}{\big|(z-\beta_i)/(1-\bar{\beta}_i z)\big|} \geq |H_n(z)|,$$

and one can easily show that on the boundary of the cell $|H_n(z)| > \beta/7$. So, in the case when $\rho(z,\Gamma) = \rho(z,\beta_i)$, inequality (4) holds. If $z_k \neq \beta_i$, then $\rho(z,z_k) < \rho(z,\beta_i)$, so (4) also holds. This ends the proof of Theorem 1. □

For $\beta > 1$ and $\gamma \in (0,1)$, set

$$r_k = 1 - \gamma\beta^{-k}, \quad N_k = \lfloor \beta^k \rfloor,$$

and let Λ consist of N_k equally spaced points on each circle $|z| = r_k$, $k = 1, 2, \dots$. Then for each k there is θ_k such that points in Λ lying on the circle $|z| = r_k$ are of the form $z_{kj} = r_k e^{i\theta_k}\zeta_j$, $j = 1, \dots, N_k$, where ζ_j are the distinct N_kth roots of unity. Analysis similar to that in the proof of Theorem 1 can be applied to obtain the following

Theorem 2. *If Λ is as above and*

$$G(z) = \prod_{k=1}^{\infty} \frac{r_k^{N_k} - z^{N_k} e^{-iN_k\theta_k}}{r_k^{N_k}\left(1 - r_k^{N_k} z^{N_k} e^{-iN_k\theta_k}\right)}, \quad z \in \mathbf{D}, \tag{6}$$

then

$$|G(z)| \simeq \rho(z,\Lambda)(1-|z|)^{-\alpha}$$

with $\alpha = \gamma/\log\beta$.

We start with showing the following

Lemma 1. *If the function G is defined by* (6)*, then there is a positive constant C such that*

$$|G(z)| \leq \frac{C}{(1-|z|)^{\alpha}}, \quad z \in \mathbf{D}, \tag{7}$$

with $\alpha = \gamma/\log\beta$.

Proof. Assume that $\theta_k = 0$, $k = 1, 2, \ldots$. We first show that (7) holds for $|z| = r_n = 1 - \gamma\beta^{-n}$. We have

$$|G(z)| = \prod_{k=1}^{n} \frac{1}{r_k^{N_k}} \left| \frac{r_k^{N_k} - z^{N_k}}{1 - r_k^{N_k} z^{N_k}} \right| \cdot \prod_{k=n+1}^{\infty} \frac{1}{r_k^{N_k}} \left| \frac{r_k^{N_k} - z^{N_k}}{1 - r_k^{N_k} z^{N_k}} \right|$$
$$\leq \prod_{k=1}^{n} \frac{1}{r_k^{N_k}} \cdot \prod_{k=n+1}^{\infty} \frac{1}{r_k^{N_k}} \left| \frac{r_k^{N_k} - z^{N_k}}{1 - r_k^{N_k} z^{N_k}} \right|.$$

Now note that

$$\log \prod_{k=1}^{n} \frac{1}{r_k^{N_k}} = -\sum_{k=1}^{n} N_k \log\left(1 - \gamma\beta^{-k}\right) \leq \sum_{k=1}^{n} N_k \frac{\gamma\beta^{-k}}{1 - \gamma\beta^{-k}}$$
$$\leq \sum_{k=1}^{n} \frac{\gamma}{1 - \gamma\beta^{-k}} = n\gamma + \gamma^2 \sum_{k=1}^{n} \frac{\beta^{-k}}{1 - \gamma\beta^{-k}}$$
$$\leq n\gamma + \frac{\gamma^2}{(1-\gamma)(\beta - 1)}.$$

Thus there is a constant $C > 0$ such that

$$\prod_{k=1}^{n} \frac{1}{r_k^{N_k}} \leq Ce^{n\gamma}.$$

On the other hand, a calculation shows that

$$\frac{1}{(1 - r_n)^{\alpha}} = \gamma^{-(\gamma/\log\beta)} \cdot e^{n\gamma}.$$

Moreover, if $|z| = r_n$, then

$$\left| \prod_{k \geq n+1} \frac{r_k^{N_k} - z^{N_k}}{r_k^{N_k}\left(1 - r_k^{N_k} z^{N_k}\right)} \right|$$
$$\leq \prod_{k \geq n+1} \frac{r_k^{N_k} + r_n^{N_k}}{r_k^{N_k}\left(1 + r_k^{N_k} r_n^{N_k}\right)} \leq \prod_{k \geq n+1} \left(1 + \left(\frac{r_n}{r_k}\right)^{N_k}\right)$$
$$= e^{\sum_{k \geq n+1} \log\left(1 + (r_n/r_k)^{N_k}\right)} \leq e^{\sum_{k \geq n+1} (r_n/r_k)^{N_k}}$$
$$\leq e^{C \sum_{k \geq n+1} r_n^{N_k}} \leq e^{(C/(1-\gamma)) \sum_{k=1}^{\infty} e^{-\gamma\beta^k}},$$

where the one before the last inequality follows from the fact that $\{r_n^{N_n}\}$ converges asymptotically to $e^{-\gamma}$. In the case when $r_n \le |z| \le r_{n+1}$, we have

$$\begin{aligned}|G(z)| &\le \sup_{|z|=r_{n+1}} |G(z)| \le \frac{C}{(1-r_{n+1})^\alpha} = C\gamma^{-(\gamma/\log\beta)} \cdot e^{(n+1)\gamma} \\ &= \frac{Ce^\gamma}{(1-r_n)^\alpha} \le \frac{Ce^\gamma}{(1-|z|)^\alpha}.\end{aligned}$$

It is also clear that the same proof can be applied for a general case when not all θ_k are zeros. □

Proof of Theorem 2. Without loss of generality, we can assume that all θ_k are zeros. For a positive integer n, put

$$G_n(z) = \frac{r_n^{N_n} - z^{N_n}}{1 - r_n^{N_n} z^{N_n}}$$

and $r_{n-1/2} = 1 - \gamma\beta^{-n+1/2}$. We will show that if $z \in L_n = \{z : r_{n-1/2} \le |z| \le r_{n+1/2}\}$, then there is a positive constant C independent of n such that

$$\left|\frac{G(z)}{G_n(z)}\right| \le \frac{C}{(1-|z|)^\alpha}$$

with $\alpha = \gamma/\log\beta$. Since there are positive constants C_1 and C_2 independent of n such that for $z \in L_n$,

$$\frac{C_1}{(1-|z|)^\alpha} \le e^{\gamma n} \le \frac{C_2}{(1-|z|)^\alpha},$$

to prove this claim the reasoning similar to that used in the proof of Lemma 1 can be used. Now our aim is to prove that

$$\left|\frac{G(z)}{G_n(z)}\right| \ge \frac{C}{(1-|z|)^\alpha} \quad \text{for} \quad z \in L_n.$$

To this end we write

$$\left|\frac{G(z)}{G_n(z)}\right| = \prod_{k=1}^{n} \frac{1}{r_k^{N_k}} \cdot \prod_{k=1}^{n-1} \left|\frac{r_k^{N_k} - z^{N_k}}{1 - r_k^{N_k} z^{N_k}}\right| \cdot \prod_{k=n+1}^{\infty} \frac{1}{r_k^{N_k}} \left|\frac{r_k^{N_k} - z^{N_k}}{1 - r_k^{N_k} z^{N_k}}\right|. \tag{8}$$

We first note that

$$\log \prod_{k=1}^{n} \frac{1}{r_k^{N_k}} \geq n\gamma - \frac{\gamma}{\beta - 1},$$

which means that

$$\prod_{k=1}^{n} \frac{1}{r_k^{N_k}} \geq \frac{C}{(1-|z|)^{\alpha}},$$

provided that $z \in L_n$. Now we observe that for $z \in L_n$ each factor in the second product in (8) is bounded below by a constant dependent only on β and γ. Indeed, for $k = 1, 2, \ldots, n-1$,

$$\begin{aligned} \left| \frac{r_k^{N_k} - z^{N_k}}{1 - r_k^{N_k} z^{N_k}} \right| &\geq \frac{r_{n-1/2}^{N_k} - r_{n-1}^{N_k}}{1 - r_{n-1/2}^{N_k} r_{n-1}^{N_k}} \geq (1-\gamma)\, \frac{r_{n-1/2} - r_{n-1}}{1 - r_{n-1/2} r_{n-1}} \\ &\geq (1-\gamma)\, \frac{\sqrt{\beta} - 1}{\sqrt{\beta} + 1}. \end{aligned}$$

Consequently, there is a constant $C > 0$ such that

$$\begin{aligned} \log \prod_{k=1}^{n-1} \frac{1}{|G_k(z)|} &\leq C \sum_{k=1}^{n-1} (1 - |G_k(z)|) \leq C \sum_{k=1}^{n-1} \frac{(1 + r_k^{N_k})(1 - r_{n-1/2}^{N_k})}{1 - r_{n-1/2}^{N_k} r_k^{N_k}} \\ &\leq C \sum_{k=1}^{n-1} (1 - (1 - \gamma\beta^{-n+1/2})^{N_k}) \leq C \sum_{k=1}^{n-1} N_k \beta^{-n+1/2} \\ &\leq \frac{C\gamma\sqrt{\beta}}{\beta - 1}, \end{aligned}$$

where we have used the fact that $r_k^{N_k}$ is bounded away from 1. This proves our claim. Finally, to see that the third product in (8) is bounded below in the annulus L_n, note first that each factor in this product is bounded below by a positive constant independent of n for all $z \in L_n$. Indeed, if $z \in L_n$, then

$$\frac{1}{r_k^{N_k}} \left| \frac{r_k^{N_k} - z^{N_k}}{1 - r_k^{N_k} z^{N_k}} \right| \geq \frac{1 - \left(r_{n+1/2}/r_k\right)^{N_k}}{1 - r_k^{N_k} r_{n+1/2}^{N_k}} \geq 1 - \left(\frac{r_{n+1/2}}{r_{n+1}}\right)^{N_{n+1}},$$

and since $\lim_{n\to\infty} r_{n+1/2}^{N_{n+1}} = e^{-\gamma\sqrt{\beta}}$ and $\lim_{n\to\infty} r_{n+1}^{N_{n+1}} = e^{-\gamma}$, our claim follows. Consequently,

$$\prod_{k=n+1}^{\infty} \frac{1}{r_k^{N_k}} \left| \frac{r_k^{N_k} - z^{N_k}}{1 - r_k^{N_k} z^{N_k}} \right| \\ \geq e^{-C\sum_{k=n+1}^{\infty} \left(1-(1/r_k^{N_k})[(r_k^{N_k}-|z|^{N_k})/(1-r_k^{N_k}|z|^{N_k})]\right)}.$$

Moreover,

$$\sum_{k=n+1}^{\infty} \left(1 - \frac{r_k^{N_k} - |z|^{N_k}}{r_k^{N_k}(1 - r_k^{N_k}|z|^{N_k})}\right) = \sum_{k=n+1}^{\infty} \frac{|z|^{N_k}(1 - r_k^{N_k})}{r_k^{N_k}(1 - r_k^{N_k}|z|^{N_k})} \\ \leq \sum_{k=n+1}^{\infty} \frac{|z|^{N_k}}{r_k^{N_k}} \leq C \sum_{k=n+1}^{\infty} r_{n+1/2}^{N_k} = C \sum_{k=1}^{\infty} (1 - \gamma\beta^{-n-1/2})^{\lfloor \beta^{k+n} \rfloor} < \infty.$$

Now, since an annulus L_n contains pseudohyperbolic disks with centers $r_n\zeta_j$, where ζ_j are N_nth roost of unity, and radius $(\sqrt{\beta} - 1)/(\sqrt{\beta}+1)$, the inequality

$$|G(z)| \leq \frac{C}{(1-|z|)^{\alpha}}\, \rho(z,\Lambda)$$

can be derived from the proved inequality in much the same way as it is in the proof of Theorem 1. To see that the inequality

$$|G(z)| \geq \frac{C}{(1-|z|)^{\alpha}}\, \rho(z,\Lambda)$$

also holds, notice that

$$\lim_{z\to r_n\zeta_j} \frac{|G_n(z)|}{\left|(z - r_n\zeta_j)/(1 - z r_n \bar{\zeta}_j)\right|} \geq 1 - \gamma,$$

and that $|G_n(z)|$ is bounded below by a constant independent of n and $j = 1, \ldots, N_n$ on the boundary of a cell

$$\{z : r_{n-1/2} \leq |z| \leq r_{n+1/2},\ |\arg z - \arg \zeta_j| \leq \pi/N_n\}. \qquad \square$$

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