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THE GENERALIZED QUASILINEARIZATION FOR INTEGRO-DIFFERENTIAL EQUATIONS OF VOLTERRA TYPE ON TIME SCALES

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ABSTRACT. We apply the method of quasilinearization to integro-differential equations of Volterra type. It is shown that two monotone sequences converge quadratically to a unique solution of our problem.

1. Introduction. Throughout this paper, we denote by $\mathbf{T}$ any time scale (nonempty closed subset of real numbers $\mathbf{R}$). By $J = [0, T]$, we denote a subset of $\mathbf{T}$ such that $[0, T] = \{t \in \mathbf{T} : 0 \le t \le T\}$. By $C(J, \mathbf{R})$, we denote the set of continuous functions $u : J \to \mathbf{R}$.

In this paper, we investigate the following first order integro-differential equations of Volterra type on time scales

$$(1) \qquad \begin{cases} x^{\triangle}(t) = f\left(t, x(t), \displaystyle\int_0^t k(t,s)x(s)\triangle s\right) \equiv (\mathcal{F}x)(t) \quad t \in J, \\ x(0) = x_0 \in \mathbf{R}, \end{cases}$$

where $f \in C(J \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$, $k \in C(J \times J, \mathbf{R})$.

The method of quasilinearization is a well-known technique for obtaining approximate solutions of nonlinear differential equations (for details, see for example [**7**] and references therein). There is a lot of application of this method to ordinary differential equations both with initial and boundary conditions. This technique can also be applied to corresponding problems on time scales (see, for example [**2, 3**]). In this paper, we apply the generalized quasilinearization method for integro-differential problems of Volterra type on time scales. The purpose of this paper is to exploit the recent ideas of this method applied to nonlinear differential equations (see, for example [**7**]). We investigate the

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case when $f+\Phi$ is convex for some convex function Φ. It is shown that two monotone sequences converge quadratically to a unique solution of problem (1). Note that in papers [**2, 3**] for the corresponding function f, the quasilinearization method was applied for $\Phi = 0$. It means that our approach is more general than in [**2,3**]. In the last section, we discuss the application of the generalized quasilinearization method when f in equation (1) is replaced by $f+g$ assuming that $f+\Phi$ is convex for some convex function Φ and $g+\Psi$ is concave for some concave function Ψ.

2. Calculus on time scales. In 1988, Stefan Hilger [**5**] introduced the calculus of measure chains in order to unify continuous and discrete analysis. Major works devoted to the calculus on time scales has been conducted by Agarwal and Bohner [**1**], Bohner and Peterson [**4**], Kaymakçalan et al. [**6**].

We present some definitions and notations which are common in the recent literature. We define the forward jump operator $\sigma : \mathbf{T} \to \mathbf{T}$ by

$$\sigma(t) = \inf\{s \in \mathbf{T} : s > t\},$$

while the backward jump operator $\rho : \mathbf{T} \to \mathbf{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbf{T} : s < t\}.$$

If $\sigma(t) > t$, then we say that t is right-scattered. If $\sigma(t) < t$, then we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $t < \sup \mathbf{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbf{T}$ and $\rho(t) = t$, then t is called left-dense. Finally, the graininess function $\mu : \mathbf{T} \to [0,\infty)$ is defined by $\mu(t) = \sigma(t) - t$. If $\mathbf{T}$ has a left-scattered maximum m, then $\mathbf{T}^k = \mathbf{T} - \{m\}$; otherwise, $\mathbf{T}^k = \mathbf{T}$. Now we consider a function $f : \mathbf{T} \to \mathbf{R}$ and let $t \in \mathbf{T}^k$. Then we define $f^{\triangle}(t)$ to be the number (provided it exists) with the property that, given any $\varepsilon > 0$, there is a neighborhood U of t, i.e., $U = (t-\delta, t+\delta) \cap \mathbf{T}$ for some $\delta > 0$, such that

$$|[f(\sigma(t)) - f(s)] - f^{\triangle}(t)[\sigma(t) - t]| \le \varepsilon|\sigma(t) - s|$$

for all $s \in U$. We call $f^{\triangle}(t)$ the delta (or Hilger) derivative of f at t. If $\mathbf{T} = \mathbf{R}$, then $f^{\triangle} = f'$; if $\mathbf{T} = \mathbf{Z}$ (the integers), then $f^{\triangle}(t) = f(t+1) - f(t)$.

Theorem 1 [**4**]. *Assume f, $g : \mathbf{T} \to \mathbf{R}$, and let $t \in \mathbf{T}^k$. Then we have the following*

(1) *If f is differentiable at t, then f is continuous at t.*

(2) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^{\triangle}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(3) *If f is differentiable at t and t is right-dense, then*

$$f^{\triangle}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(4) *If f is differentiable at t, then*

$$f^{\sigma}(t) = f(t) + \mu(t) f^{\triangle}(t), \quad \text{where} \quad f^{\sigma} = f \circ \sigma.$$

(5) *If f and g are differentiable at t, then so is fg with*

$$(fg)^{\triangle}(t) = f^{\triangle}(t) g(t) + f^{\sigma}(t) g^{\triangle}(t).$$

A function $F : \mathbf{T} \to \mathbf{R}$ is called an antiderivative of $f : \mathbf{T} \to \mathbf{R}$ provided $f^{\triangle}(t) = f(t)$ for all $t \in \mathbf{T}^k$. In this case, we define the integral of f by

$$\int_s^t f(r) \Delta r = F(t) - F(s) \quad \text{for} \quad s, t \in \mathbf{T}.$$

3. Some lemmas. Below we cite two lemmas from [**8**].

Lemma 1 [8]. *Suppose that*

(H_1) *there is a continuous function $k : J \times J \to \mathbf{R}_+$ and $K_0 = \max\{k(t,s) :\ t, s \in J\} > 0$,*

(H_2) *there exist two positive functions m, n continuous on J such that $\alpha = \sup_{t \in J}[\mu(t) m(t)] < 1$ and*

$$\frac{\alpha N_0 K_0 P}{m_0} \leq 1 - \alpha, \tag{2}$$

where

$$N_0 = \max_{t\in J} n(t), \quad m_0 = \min_{t\in J} m(t), \quad P = e_{\ominus\{-m\}}(a,0) - 1.$$

Let

$$(3) \qquad \begin{cases} x^{\triangle}(t) \geq -m(t)x(t) - n(t)\displaystyle\int_0^t k(t,s)x(s)\Delta s, \\ x(0) \geq 0. \end{cases}$$

Then $x(t) \geq 0$, $t \in J$.

Lemma 2 [**8**]. *Assume that assumptions* $(H_1), (H_2)$ *are satisfied. Then, for any* $h \in C(J, \mathbf{R})$, *the initial problem*

$$(4) \qquad \begin{cases} x^{\triangle}(t) = -m(t)x(t) - n(t)\displaystyle\int_0^t k(t,s)x(s)\Delta s + h(t), \\ x(0) = x_0 \end{cases}$$

has a unique solution x_h.

Put

$$\Omega = \Big\{(t,u,v) : t \in J,\ y_0(t) \leq u \leq z_0(t), \\ \int_0^t k(t,s)y_0(s)\Delta s \leq v \leq \int_0^t k(t,s)z_0(s)\Delta s\Big\}.$$

Using a mean value theorem, we have

Lemma 3. *Let* $u \geq \bar{u}$, $v \geq \bar{v}$. *Assume that* $F, \Phi \in C(\Omega, \mathcal{R})$. *Assume that* F_x, F_y, Φ_x, Φ_y *exist and* F_x, Φ_x, Φ_y *are nondecreasing in the second variable and* F_x, F_y, Φ_y *are nondecreasing in the third variable. Then, for* $F = f + \Phi$, *we have*

$$f(t,u,v) - f(t,\bar{u},\bar{v}) \geq [F_x(t,\bar{u},\bar{v}) - \Phi_x(t,u,v)][u - \bar{u}] \\ + [F_y(t,\bar{u},\bar{v}) - \Phi_y(t,u,v)][v - \bar{v}].$$

Note that F_x denotes the derivative of F with respect to second variable and F_y denotes the derivative of F with respect to the last variable.

Put

$$(F_x u)(t) = F_x\left(t, u(t), \int_0^t k(t,s)u(s)\Delta s\right),$$
$$(\Phi_x u)(t) = \Phi_x\left(t, u(t), \int_0^t k(t,s)u(s)\Delta s\right).$$

In a similar way, we define $(F_y u)(t)$, $(\Phi_y u)(t)$.

For $n = 0, 1, \ldots$, let us define two sequences $\{y_n, z_n\}$, by relations

(5)
$$\begin{cases} y_{n+1}^{\triangle}(t) = (\mathcal{F}y_n)(t) + [(F_x y_n)(t) - (\Phi_x z_n)(t)][y_{n+1}(t) - y_n(t)] \\ \qquad\qquad + [(F_y y_n)(t) - (\Phi_y z_n)(t)]\displaystyle\int_0^t k(t,s)[y_{n+1}(s) - y_n(s)]\Delta s, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in J, \\ y_{n+1}(0) = x_0, \end{cases}$$

(6)
$$\begin{cases} z_{n+1}^{\triangle}(t) = (\mathcal{F}z_n)(t) + [(F_x y_n)(t) - (\Phi_x z_n)(t)][z_{n+1}(t) - z_n(t)] \\ \qquad\qquad + [(F_y y_n)(t) - (\Phi_y z_n)(t)]\displaystyle\int_0^t k(t,s)[z_{n+1}(s) - z_n(s)]\Delta s, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in J, \\ z_{n+1}(0) = x_0. \end{cases}$$

A function y_0 is said to be a lower solution of problem (1) if

$$\begin{cases} y_0^{\triangle}(t) \le f\left(t, y_0(t), \displaystyle\int_0^t k(t,s)y_0(s)\Delta s\right) \quad t \in J, \\ y_0(0) \le x_0. \end{cases}$$

Similarly, a function z_0 is an upper solution of (1) if the above inequalities are reversed.

Lemma 4. *Suppose $f \in C(J \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$. Assume that y_0, z_0 are lower and upper solutions of problem* (1)*, respectively, and $y_0(t) \le$*

$z_0(t)$, $t \in J$. Let u, v be lower and upper solutions of (1), respectively, and moreover $y_0(t) \leq u(t) \leq v(t) \leq z_0(t)$, $t \in J$. In addition, we assume that F_x, F_y, Φ_x, Φ_y exist and F_x, F_y, Φ_x, Φ_y are nondecreasing in the second variable and F_x, F_y, Φ_x, Φ_y are nondecreasing in the third variable (here $F = f + \Phi$). Let assumptions $(H_1), (H_2)$ hold with

$$(7)\quad m(t) = -(F_x y_0)(t) + (\Phi_x z_0)(t), \quad n(t) = -(F_y y_0)(t) + (\Phi_y z_0)(t).$$

Put

$$h(t,w) = (\mathcal{F}w)(t) + M(t)w(t) + N(t)\int_0^t k(t,s)w(s)\Delta s$$

with

$$M(t) = -[(F_x u)(t) - (\Phi_x v)(t)], \quad N(t) = -[(F_y u)(t) - (\Phi_y v)(t)].$$

Then

(i) *the initial problems*

$$\begin{cases} y^{\triangle}(t) = -M(t)y(t) - N(t)\displaystyle\int_0^t k(t,s)y(s)\Delta s + h(t,u) & t \in J, \\ y(0) = x_0, \end{cases}$$

$$\begin{cases} z^{\triangle}(t) = -M(t)z(t) - N(t)\displaystyle\int_0^t k(t,s)z(s)\Delta s + h(t,v) & t \in J, \\ x(0) = x_0 \end{cases}$$

have their unique solutions y, z, respectively,

(ii) $u(t) \leq y(t) \leq z(t) \leq v(t)$, $t \in J$,

(iii) *y, z are lower and upper solutions of problem (1), respectively.*

Proof. In view of the monotonicity of F_x, F_y, Φ_x, Φ_y, we have $M(t) \leq m(t)$, $N(t) \leq n(t)$, $t \in J$. This and Lemma 2 show that part (i) holds. To show part (ii) we put $p = y - u$. Then $p(0) \geq 0$, and

$$\begin{aligned} p^{\triangle}(t) &\geq (\mathcal{F}u)(t) - M(t)[y(t) - u(t)] \\ &\quad - N(t)\int_0^t k(t,s)[y(s) - u(s)]\Delta s - (\mathcal{F}u)(t) \\ &= -M(t)p(t) - N(t)\int_0^t k(t,s)p(s)\Delta s. \end{aligned}$$

Hence $y(t) \geq u(t)$, $t \in J$, in view of Lemma 1. In a similar way, we have $v(t) \geq z(t)$, $t \in J$. Now, we put $p = z - y$, so $p(0) = 0$. In view of Lemma 3, we have

$$\begin{aligned} p^{\triangle}(t) &= (\mathcal{F}v)(t) - (\mathcal{F}u)(t) - M(t)[z(t) - v(t) - y(t) + u(t)] \\ &\quad - N(t) \int_0^t k(t,s)[z(s) - v(s) - y(s) + u(s)]\Delta s \\ &\geq - M(t)[v(t) - u(t)] - N(t) \int_0^t k(t,s)[v(s) - u(s)]\Delta s \\ &\quad - M(t)[z(t) - v(t) - y(t) + u(t)] \\ &\quad - N(t) \int_0^t k(t,s)[z(s) - v(s) - y(s) + u(s)]\Delta s \\ &= - M(t)p(t) - N(t) \int_0^t k(t,s)p(s)\Delta s. \end{aligned}$$

By Lemma 1, $z(t) \geq y(t)$, $t \in J$. It proves that (ii) holds.

In the next step, we show that z is an upper solution of problem (1). Note that

$$\begin{aligned} z^{\triangle}(t) &= (\mathcal{F}v)(t) - (\mathcal{F}z)(t) + (\mathcal{F}z)(t) - M(t)[z(t) - v(t)] \\ &\quad - N(t) \int_0^t k(t,s)[z(s) - v(s)]\Delta s \\ &\geq [(F_x z)(t) - (\Phi_x v)(t)][v(t) - z(t)] \\ &\quad + [(F_y z)(t) - (\Phi_y v)(t)] \int_0^t k(t,s)[v(s) - z(s)]\Delta s \\ &\quad - M(t)[z(t) - v(t)] - N(t) \int_0^t k(t,s)[z(s) - v(s)]\Delta s \\ &\geq (\mathcal{F}z)(t)) \end{aligned}$$

in view of Lemma 3 and the monotonicity of F_x, F_y, Φ_x, Φ_y. In the same way, we can show that y is a lower solution of problem (1). This ends the proof. $\square$

4. Main results.

Theorem 2. *Suppose that* $f \in C(J \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$. *Assume that* y_0, z_0 *are lower and upper solutions of problem* (1), *respectively, and*

$y_0(t) \le z_0(t)$, $t \in J$. In addition, assume that F_{xx}, F_{xy}, F_{yy}, Φ_{xx}, Φ_{xy}, Φ_{yy} exist for $F = f + \Phi$, are continuous and

$$F_{xx}(t,u,v) \ge 0, \quad F_{xy}(t,u,v) \ge 0, \quad F_{yy}(t,u,v) \ge 0,$$
$$\Phi_{xx}(t,u,v) \ge 0, \quad \Phi_{xy}(t,u,v) \ge 0, \quad \Phi_{yy}(t,u,v) \ge 0$$

for $(t,u,v) \in \Omega$. Let assumptions $(H_1), (H_2)$ hold with functions m and n defined by (7). Then problem (1) has a unique solution being the limit of sequences $\{y_n, z_n\}$ defined by (5)–(6) and this convergence is quadratic.

Proof. In view of Lemma 4, y_1, z_1 are well defined and $y_0(t) \le y_1(t) \le z_1(t) \le z_0(t)$, $t \in J$. Moreover, y_1, z_1 are lower and upper solutions of problem (1), respectively. By induction, we have the following relation

$$y_0(t) \le \cdots \le y_n(t) \le y_{n+1}(t) \le z_{n+1}(t) \le z_n(t) \le \cdots \le z_0(t), \quad t \in J$$

for $n = 0, 1, \dots$. Since the interval J is compact and the convergence is monotone and bounded, sequences $\{y_n,\, z_n\}$ converge uniformly to some limit functions y and z, respectively. Indeed, functions y and z satisfy the equations

$$y(t) = x_0 + \int_0^t (\mathcal{F}y)(s)\Delta s, \quad z(t) = x_0 + \int_0^t (\mathcal{F}z)(s)\Delta s$$

and $y_0(t) \le y(t) \le z(t) \le z_0(t)$, $t \in J$. Put

$$\begin{aligned}\overline{M}(t) = -\bigg[F_x\bigg(t, \xi_1(t), \int_0^t k(t,s)z(s)\Delta s\bigg)& \\ - \Phi_x\bigg(t, \xi_1(t), \int_0^t k(t,s)z(s)\Delta s\bigg)\bigg]&, \\ \overline{N}(t) = -\left[F_y(t, y(t), \xi_2(t)) - \Phi_y(t, y(t), \xi_2(t))\right]&,\end{aligned}$$

where $y(t) \le \xi_1(t) \le z(t)$, $\int_0^t k(t,s)y(s)\Delta s < \xi_2(t) < \int_0^t k(t,s)z(s)\Delta s$ and ξ_1, ξ_2 are continuous functions. Let $p(t) = z(t) - y(t)$, $t \in J$. Then, using the mean value theorem, we have

$$p^{\triangle}(t) = -\overline{M}(t)p(t) - \overline{N}(t)\int_0^t k(t,s)p(s)\Delta s, \quad p(0) = 0. \tag{8}$$

Note that $\overline{M}(t) \le m(t)$, $\overline{N}(t) \le n(t)$, $t \in J$. This and Lemma 2 show that $p(t) = 0$, $t \in J$, is a unique solution of problem (8). Hence, $y(t) = z(t)$ on J is a unique solution of problem (1).

Now we need to show that the convergence of y_n and z_n to y is quadratic. Put

$$p_{n+1}(t) = y(t) - y_{n+1}(t) \ge 0, \quad q_{n+1}(t) = z_{n+1}(t) - y(t) \ge 0, \quad t \in J.$$

We see that

$$\begin{aligned}
&p_{n+1}(t)\\
&= \int_0^t [(\mathcal{F}y)(s) - (\mathcal{F}y_n)(s)]\Delta s\\
&\quad - \int_0^t [(F_x y_n)(s) - (\Phi_x z_n)(s)][y_{n+1}(s) - y_n(s)]\Delta s\\
&\quad - \int_0^t \Big\{[(F_y y_n)(s) - (\Phi_y z_n)(s)] \int_0^s k(s,\tau)[y_{n+1}(\tau) - y_n(\tau)]\Delta\tau\Big\}\Delta s\\
&\le \int_0^t [(F_x y)(s) - (\Phi_x y_n)(s)]\, p_n(s)\Delta s\\
&\quad + \int_0^t \Big\{[(F_y y)(s) - (\Phi_y y_n)(s)] \int_0^s k(s,\tau)\, p_n(\tau)\Delta\tau\Big\}\Delta s\\
&\quad - \int_0^t [(F_x y_n)(s) - (\Phi_x z_n)(s)][p_n(s) - p_{n+1}(s)]\Delta s\\
&\quad - \int_0^t \Big\{[(F_y y_n)(s) - (\Phi_y z_n)(s)] \int_0^s k(s,\tau)[p_n(\tau) - p_{n+1}(\tau)]\Delta\tau\Big\}\Delta s\\
&= \int_0^t [(F_x y)(s) - (F_x y_n)(s) + (\Phi_x z_n)(s) - (\Phi_x y_n)(s)]\, p_n(s)\Delta s\\
&\quad + \int_0^t \Big\{[(F_y y)(s) - (F_y y_n)(s) + (\Phi_y z_n)(s) - (\Phi_y y_n)(s)]\\
&\qquad\qquad \times \int_0^s k(s,\tau)\, p_n(\tau)\Delta\tau\Big\}\Delta s\\
&\quad + \int_0^t [(F_x y_n)(s) - (\Phi_x z_n)(s)]\, p_{n+1}(s)\Delta s\\
&\quad + \int_0^t \Big\{[(F_y y_n)(s) - (\Phi_y z_n)(s)] \int_0^s k(s,\tau)\, p_{n+1}(\tau)\Delta\tau\Big\}\Delta s.
\end{aligned}$$

Let

$$|F_{xx}(t,x,y)| \le A_1, \quad |F_{xy}(t,x,y)| \le A_2, \quad |F_{yy}(t,x,y)| \le A_3,$$
$$|\Phi_{xx}(t,x,y)| \le B_1, \quad |\Phi_{xy}(t,x,y)| \le B_2, \quad |\Phi_{yy}(t,x,y)| \le B_3$$

for $(t,x,y) \in \Omega$. Then

$$\begin{aligned}
(F_x y)(s) &- (F_x y_n)(s) + (\Phi_x z_n)(s) - (\Phi_x y_n)(s) \\
&= F_{xx}\Big(s, \xi_3(s), \int_0^s k(s,\tau) y(\tau)\Delta\tau\Big) p_n(s) \\
&\quad + F_{xy}(s, y_n(s), \xi_4(s)) \int_0^s k(s,\tau)\, p_n(\tau)\Delta\tau \\
&\quad + \Phi_{xx}\Big(s, \xi_5(s), \int_0^s k(s,\tau) z_n(\tau)\Delta\tau\Big)[z_n(s) - y_n(s)] \\
&\quad + \Phi_{xy}(s, y_n(s), \xi_6(s)) \int_0^s k(s,\tau)[z_n(\tau) - y_n(\tau)]\Delta\tau \\
&\le A_1 p_n(s) + A_2 \int_0^s k(s,\tau)\, p_n(\tau)\Delta\tau + B_1[q_n(s) + p_n(s)] \\
&\quad + B_2 \int_0^s k(s,\tau)[q_n(\tau) + p_n(\tau)]\Delta\tau,
\end{aligned}$$

where

$$\begin{gathered}
y_n(s) < \xi_3(s) < y(s), \\
\int_0^s k(s,\tau) y_n(\tau)\Delta\tau < \xi_4(s) < \int_0^s k(s,\tau) y(\tau)\Delta\tau, \\
y_n(s) < \xi_5(s) < z_n(s), \\
\int_0^s k(s,\tau) y_n(\tau)\Delta\tau < \xi_6(s) < \int_0^s k(s,\tau) z_n(\tau)\Delta\tau.
\end{gathered}$$

In a similar way we can show that

$$\begin{aligned}
(F_y y)(s) &- (F_y y_n)(s) + (\Phi_y z_n)(s) - (\Phi_y y_n)(s) \\
&\le A_2 p_n(s) + A_3 \int_0^s k(s,\tau)\, p_n(\tau)\Delta\tau \\
&\quad + B_2[q_n(s) + p_n(s)] + B_3 \int_0^s k(s,\tau)[q_n(\tau) + p_n(\tau)]\Delta\tau.
\end{aligned}$$

Using the above inequalities, we have

$$p_{n+1}(t) \le D + D_1 \int_0^t p_{n+1}(s)\Delta s + D_2 K_0 \int_0^t \int_0^s p_{n+1}(\tau)\Delta\tau\Delta s, \tag{9}$$

where

$$\begin{gathered}
D = T[A\|p_n\|^2 + B\|q_n\|^2], \\
\|p_n\| = \max_{t\in J} |p_n(t)|, \quad \|q_n\| = \max_{t\in J} |q_n(t)|, \\
A = A_1 + TK_0(2A_2 + A_3K_0T) + \frac{3}{2}\,[B_1 + TK_0(2B_2 + B_3K_0T)], \\
B = \frac{1}{2}\,[B_1 + TK_0(2B_2 + B_3TK_0)], \\
D_1 = D_{11} + D_{12}, \quad D_2 = D_{21} + D_{22}, \\
|F_x(t,x,y)| \le D_{11}, \quad |\Phi_x(t,x,y)| \le D_{12}, \\
|F_y(t,x,y)| \le D_{21}, \quad |\Phi_y(t,x,y)| \le D_{22}.
\end{gathered}$$

To obtain the formula for D we applied the property that $2ab \le a^2 + b^2$ for nonnegative a, b. By w we denote the right-hand side of equation (9). Then

$$w^{\triangle}(t) = D_1 p_{n+1}(t) + D_2 K_0 \int_0^t p_{n+1}(\tau)\Delta\tau.$$

Note that $w^{\triangle}(t) \ge 0$ on J, so w is nondecreasing. This yields

$$\begin{cases} w^{\triangle}(t) \le \alpha w(t) & t \in J, \\ w(0) = D \end{cases}$$

with $\alpha = D_1 + D_2K_0T$. The constant α is positive, so α is positive regressive, i.e., $1 + \mu(t)\alpha > 0$. This and Theorem 6.1 of [**4**] yield

$$w(t) \le De_\alpha(t,0), \quad t \in J.$$

Hence,

$$p_{n+1}(t) \le w(t) \le De_\alpha(t,0), \tag{10}$$

so

$$\|p_{n+1}\| \le \alpha_1[A\|p_n\|^2 + B\|q_n\|^2] \tag{11}$$

with $\alpha_1 = T \max_{t\in J} e_\alpha(t,0)$.

In a similar way, we can show that

$$\|q_{n+1}\| \le \alpha_2\|p_n\|^2 + \alpha_3\|q_n\|^2$$

for some positive α_2, α_3. This and (11) prove the assertion of Theorem 1. It ends the proof. $\square$

Remark 1. If $\mathbf{T} = \mathbf{R}$, then $e_\alpha(t,0) = \exp(\alpha t)$, but if $\mathbf{T} = \mathbf{Z}$, then $e_\alpha(t,0) = (1+\alpha)^t$.

We can also discuss the case when f in equation (1) is replaced by $f+g$. Then problem (1) takes the form

$$\begin{cases} x^{\triangle}(t) = (\mathcal{F}x)(t) + (\mathcal{G}x)(t) & t \in J, \\ x(0) = x_0 \in \mathbf{R} \end{cases} \tag{12}$$

with

$$(\mathcal{F}x)(t) \equiv f\left(t, x(t), \int_0^t k(t,s)x(s)\triangle s\right),$$
$$(\mathcal{G}x)(t) \equiv g\left(t, x(t), \int_0^t k(t,s)x(s)\triangle s\right).$$

For $n = 0, 1, \dots$, let us define two sequences $\{y_n, z_n\}$, by relations

(13)
$$\begin{cases} y_{n+1}^{\triangle}(t) = (\mathcal{F}y_n + \mathcal{G}y_n)(t) + V(t,y_n,z_n)[y_{n+1}(t) - y_n(t)] \\ \qquad\qquad + W(t,y_n,z_n)\displaystyle\int_0^t k(t,s)[y_{n+1}(s) - y_n(s)]\Delta s \qquad t \in J, \\ y_{n+1}(0) = x_0, \end{cases}$$

(14)
$$\begin{cases} z_{n+1}^{\triangle}(t) = (\mathcal{F}z_n + \mathcal{G}z_n)(t) + V(t,y_n,z_n)[z_{n+1}(t) - z_n(t)] \\ \qquad\qquad + W(t,y_n,z_n)\displaystyle\int_0^t k(t,s)[z_{n+1}(s) - z_n(s)]\Delta s \qquad t \in J, \\ z_{n+1}(0) = x_0, \end{cases}$$

where $F = f + \Phi$, $G = g + \Psi$ and

$$\begin{aligned}V(t, y_n, z_n) &= (F_x y_n)(t) - (\Phi_x z_n)(t) + (G_x z_n)(t) - (\Psi_x y_n)(t)\\ W(t, y_n, z_n) &= (F_y y_n)(t) - (\Phi_y z_n)(t)] + (G_y z_n)(t) - (\Psi_y y_n)(t).\end{aligned}$$

Theorem 3. *Suppose that* $f, g \in C(J \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$. *Assume that* y_0, z_0 *are lower and upper solutions of problem* (12), *respectively and* $y_0(t) \leq z_0(t)$, $t \in J$. *In addition assume that* F_{xx}, F_{xy}, F_{yy}, Φ_{xx}, Φ_{xy}, Φ_{yy}, G_{xx}, G_{xy}, G_{yy}, Ψ_{xx}, Ψ_{xy}, Ψ_{yy}, *exist for* $F = f + \Phi$, $G = g + \Psi$, *are continuous and*

$$\begin{array}{lll} F_{xx}(t,u,v) \geq 0, & F_{xy}(t,u,v) \geq 0, & F_{yy}(t,u,v) \geq 0,\\ \Phi_{xx}(t,u,v) \geq 0, & \Phi_{xy}(t,u,v) \geq 0, & \Phi_{yy}(t,u,v) \geq 0,\\ G_{xx}(t,u,v) \leq 0, & G_{xy}(t,u,v) \leq 0, & G_{yy}(t,u,v) \leq 0,\\ \Psi_{xx}(t,u,v) \leq 0, & \Psi_{xy}(t,u,v) \leq 0, & \Psi_{yy}(t,u,v) \leq 0 \end{array}$$

for $(t, u, v) \in \Omega$. *Let Assumptions* $(H_1), (H_2)$ *hold with functions*

$$m(t) = -V(t, y_0, z_0), \quad n(t) = -W(t, y_0, z_0).$$

Then problem (12) *has a unique solution being the limit of sequences* $\{y_n,\, z_n\}$ *defined by* (13)–(14), *and this convergence is quadratic.*

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