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MULTIPLIERS FOR THE L_p-SPACES OF A HYPERGROUP

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ABSTRACT. Let K be a hypergroup with Haar measure. We investigate the properties of the closed convex invariant subsets of $L_p(K)$, $1 \leq p \leq \infty$, and apply the results to the study of the multipliers for $L_p(K)$.

1. Introduction. There are a lot of results in abstract harmonic analysis on locally compact groups regarding multipliers for various spaces of functions. A good deal of attention was paid to the study of multipliers for $L_1(G)$, the classical characterization of Wendel [**20**] describing their structure. The compact multipliers for $L_1(G)$ were first studied by Sakai [**17**] who proved that if G is not compact, then zero is the only weakly compact multiplier of $L_1(G)$. Conversely, Akemann [**1**] showed that if G is compact then every multiplier for $L_1(G)$ is compact. All these results were extended to the hypergroups case by Ghahramani and Medgalchi [**6, 7**]. Multipliers from $L_1(G)$ to $L_p(G)$, $1 \leq p \leq \infty$, were investigated by Brainerd and Edwards [**2**]. In [**13**] Lau studied closed convex sets of $L_p(G)$, $1 \leq p \leq \infty$, applying his approach in order to rediscover the classical above-mentioned results and also to extend them to affine multipliers. Bearing in mind the Lau idea, the purpose of this paper is to obtain some insight into the multipliers problem for the L_p-spaces, $1 \leq p \leq \infty$, of a hypergroup, starting from the study of the invariant subsets of the L_p-spaces of the hypergroup.

Hypergroups generalize locally compact groups. Roughly speaking, they are locally compact spaces, whose regular, complex-valued Borel measures form an algebra, which has properties similar to the convolution algebra $(M(G), *)$ of a locally compact group G. The theory of hypergroups was initiated by Dunkl [**3**], Jewett [**9**] and Spector [**19**]. Throughout our paper, we will consider hypergroups in the sense of

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Jewett [**9**]. In our approach the hypergroup possesses a Haar measure. We notice that it is still unknown if an arbitrary hypergroup admits a Haar measure, but all the known examples, such as commutative hypergroups, compact hypergroups and central hypergroups, do. After the preliminaries, containing notations and some technical lemmas, in the second section we study the convex invariant subsets, in particular the invariant subspaces of the L_p-spaces, $1 \leq p \leq \infty$, of a hypergroup K, the results obtained generalizing the ones from the locally compact group case of Lau [**13**]. We apply these results to extend some classical theorems concerning multipliers for the L_p-spaces, $1 \leq p \leq \infty$, of a locally compact group to multipliers for the L_p-spaces of a hypergroup. We obtain among other things an extension of the multipliers characterization of Brainerd and Edwards for multipliers from $L_1(K)$ to $L_p(K)$.

2. Preliminaries and technical lemmas. For basic definitions and results on hypergroups we shall follow [**9**]. K always stands for a hypergroup with a fixed Haar measure m, symbols like $\int \cdots dx$ will always denote integration with respect to m. The notation generally agrees with [**9**]. However, the following notations are different from [**9**]: $x \mapsto x^\vee$ denotes the involution on K, δ_x the Dirac measure concentrated at x and $\mathcal{C}(K)$ the space of bounded continuous complex-valued functions on K. As usual $\mathcal{C}_o(K)$ is the subspace of $\mathcal{C}(K)$ consisting of all those functions vanishing at infinity. In addition, we use the notation $MP(K)$ for the probability measures on K.

We recall that the cone topology, τ_c, on $M^+(K)$ is the weakest topology such that, for each $f \in \mathcal{C}_c^+(K)$, the mapping $\mu \mapsto \int f\, d\mu$ is continuous and such that the mapping $\mu \mapsto \mu(K)$ is continuous, so it is the trace on $M^+(K)$ of the locally convex topology on $M(K)$ generated by the family of semi-norms $\{p_f | f \in \mathcal{C}(K)\}$. It follows that τ_c is stronger than the ω^*-topology, and that they agree on $MP(K)$. The cone topology and the ω^*-topology are equal if and only if K is compact.

If f is a Borel function on K and $x, y \in K$, the left translate f_x or $L_x(f)$ and the right translate f^y or $R_y(f)$ are defined by

$$L_x(f)(y) = f_x(y) = f^y(x) = R_y(f)(x) = \int f d\delta_x * \delta_y = f(x * y)$$

if the integral exists. The function $f^\vee$ is given by $f^\vee(x) = f(x^\vee)$. Convolution of two functions f and g on K is defined by

$$(f * g)(x) = \int f(x * y) g(y^\vee)\, dy$$

whenever it makes sense. If $\mu \in M(K)$, and f is a Borel function, then the convolutions $\mu * f$ and $f * \mu$ are defined on K by

$$(\mu * f)(x) = \int f(y^\vee * x)\, d\mu(y) \quad \text{and} \quad (f * \mu)(x) = \int f(x * y^\vee)\, d\mu(y).$$

It is immediate that $\delta_{x^\vee} * f = f_x$, for each x in K and f a Borel function.

The spaces $(L_p(K), \|\cdot\|_p)$, $1 \le p \le \infty$, are defined in the usual way with respect to the Haar measure of K, see for example, [**5**, Chapter 6]. If A is a subset of $L_p(K)$, $1 \le p \le \infty$, then $\operatorname{co} A$ will denote the convex hull of A, and $\overline{A}$ will denote the closure of A in the norm topology. Besides the norm topology on $L_p(K)$, $1 \le p < \infty$, we will consider the weak-topology, ω ($\omega = \sigma(L_p(K), L_p^*(K))$, $L_p^*(K) = L_q(K), 1/p + 1/q = 1$). The topology ω on $L_p(K)$, $1 \le p < \infty$, will be considered only occasionally, so unless otherwise specified, we will refer to the topological properties of sets and functions on $L_p(K)$, $1 \le p < \infty$, with respect to the norm topology. Identifying $L_\infty(K)$ to $L_1^*(K)$ (whenever this is possible, for example, requiring m to be σ-finite [**5**, Theorem 6.15]) we will often consider the weak*-topology, ω^* ($\omega^* = \sigma(L_\infty(K), L_1(K))$) on $L_\infty(K) = L_1^*(K)$. If $f \in L_p(K)$, $1 \le p \le \infty$, $x \in K$, then $f_x \in L_p(K)$, $\|f_x\|_p \le \|f\|_p$, and this is in general not an isometry [**9**]. The mapping $x \mapsto f_x$ is continuous from K to $(L_p(K), \|\cdot\|_p)$, $1 \le p < \infty$ [**9**, Lemma 2.2B, Lemma 5.4H]. If $f \in L_1(K)$, $g \in L_p(K)$, then $f * g \in L_p(K)$ and $\|f * g\|_p \le \|f\|_1 \|g\|_p$, [**9**, Theorem 6.2C].

Lemma 1. *Let $\mathcal{V}$ be the family of all neighborhoods of e, regarded as a directed set in the usual way: $U \geqslant V$ if $U \subseteq V$. For each $V \in \mathcal{V}$ choose a function $\varphi_V \in \mathcal{C}_c^+(K)$ such that φ_V vanishes outside of V and $\int \varphi_V(x)\, dx = 1$. Then,*

(i) *The net $(\varphi_V)_{V \in \mathcal{V}}$ is a bounded approximate identity for $L_1(K)$.*

(ii) *For each* $f \in L_p(K)$, $1 \le p < \infty$, *the nets* $(\varphi_V * f)_{V\in\mathcal{V}}$ *and* $(f * \varphi_V^\vee)_{V\in\mathcal{V}}$ *are in* $L_p(K)$ *and they converge (in norm) to* f.

(iii) *For each* $f \in L_\infty(K)$ *the nets* $(\varphi_V * f)_{V\in\mathcal{V}}$ *and* $(f * \varphi_V^\vee)_{V\in\mathcal{V}}$ *are in* $L_\infty(K)$ *and they* ω^**-converge to* f.

Proof. (i) and (ii) result immediately using the density of $\mathcal{C}_c(K)$ in $L_p(K)$.

(iii) follows as in the locally compact group case [**22**, Lemma 3.3], taking into account that the ω^*-topology on $L_\infty(K) = L_1^*(K)$ is described in terms of convergence by: $f_\alpha \overset{\omega^*}{\to} f$ if and only if $\int f_\alpha(x)h(x)\,dx \to \int f_\alpha(x)h(x)\,dx$, for all $h \in L_1(K)$, (when $(f_\alpha)_\alpha$, f are in $L_\infty(K)$). $\square$

Lemma 2. *Let* K *be a hypergroup. Then,*

(i) *For each* $f \in L_p(K)$, $1 \le p < \infty$, *the mapping* $\mu \mapsto \mu * f$ *from* $(M^+(K), \tau_c)$ *to* $(L_p(K), \|\cdot\|_p)$ *is continuous.*

(ii) *For each* $f \in L_p(K)$, $1 < p < \infty$, *the mapping* $\mu \mapsto \mu * f$ *from* $(M(K), \omega^*)$ *to* $(L_p(K), \omega)$, *is continuous.*

(iii) *For each* $f \in L_\infty(K)$, *the mapping* $\mu \mapsto \mu * f$ *from* $(M^+(K), \tau_c)$ *to* $(L_\infty(K), \omega^*)$, *is continuous.*

Proof. (i) is proved in [**9**, Lemma 5.4H].

(ii) Let $(\mu_\alpha)_\alpha$ be a net in $M(K)$ ω^*-converging to μ and $f \in L_p(K)$. Consider an arbitrary functional Φ in $L_p^*(K)$, so there exists $h \in L_q(K)$, $1/p + 1/q = 1$, such that $\Phi(f) = \int f(x)h(x)\,dx$. Then,

$$\Phi(\mu_\alpha * f) = \int (\mu_\alpha * f)(x)h(x)\,dx = \mu_\alpha(h * f^\vee).$$

As $h * f^\vee \in \mathcal{C}_o(K)$, [**9**, Theorem 6.2F], it follows that $(\mu_\alpha(h * f^\vee))_\alpha$ converges to $\mu(h * f^\vee) = \Phi(\mu * f)$.

(iii) Let $(\mu_\alpha)_\alpha$ be a net in $M^+(K)$ that converges in the cone topology to μ and $f \in L_\infty(K)$. As for each $h \in L_1(K)$, $h * f^\vee \in \mathcal{C}(K)$, [**9**, Theorem 6.2E], it follows that $(\mu_\alpha(h*f^\vee))_\alpha \to \mu(h*f^\vee)$ or equivalently

$$\int (\mu_\alpha * f)(x)h(x)\,dx \longrightarrow \int (\mu * f)(x)h(x)\,dx.$$

This shows that $(\mu_\alpha * f)_\alpha$ converges in the weak*-topology to $\mu * f$. $\square$

The next lemma follows immediately, with a proof similar to Lemma 2(ii).

Lemma 3. *Let K be a hypergroup. Then,*

i) *For each $f \in L_1(K)$, the mapping $h \mapsto f * h$ from $(L_p(K), \omega)$ into $(L_p(K), \omega)$, $1 < p < \infty$, is continuous.*

(ii) *For each $f \in L_1(K)$, the mapping $h \mapsto f * h$ from $(L_\infty(K), \omega^*)$ into $(L_\infty(K), \omega^*)$ is continuous.*

Notations. Further, we denote by $P(K) = \{\varphi \in L_1(K) \mid \varphi \geqslant 0, \|\varphi\|_1 = 1\}$ and by $E(K) = \{\delta_x \mid x \in K\}$.

Lemma 4. *Let K be a hypergroup. Then,*

$$MP(K) = \overline{\operatorname{co} E(K)}^{\tau_c} = \overline{P(K)}^{\tau_c}$$

Proof. Clearly, $P(K) \subseteq MP(K)$ and $\operatorname{co} E(K) \subseteq P(K)$. As $MP(K) = \overline{\operatorname{co} E(K)}^{\omega^*}$, for each $\mu \in MP(K)$, there exists a net $(\mu_\alpha)_\alpha$ in $\operatorname{co} E(K)$ such that $(\mu_\alpha)_\alpha$ ω^*-converges to μ. As on $MP(K)$, the weak*-topology and the cone topology agree, it follows that the net $(\mu_\alpha)_\alpha$ τ_c-converges to μ. $\square$

Finally, we mention that the set of almost periodic functions on K will be denoted by $AP(K)$. Following [**11**], we remind that a bounded continuous complex-valued function f on K is said to be almost periodic, if $O(f) = \{f_x \mid x \in K\}$ is relatively compact in $(\mathcal{C}(K), \|\cdot\|_\infty)$.

3. Invariant subsets of $L_p(K)$. In this section we transfer the results of [**13**] concerning the characterization of closed convex invariant subsets of the L_p-spaces of a locally compact group to the hypergroups case. The corresponding proofs of [**13**] apply with the appropriate modifications required by the new context. For the sake of completeness we include them here entirely.

Let K be a hypergroup and $1 \leq p \leq \infty$. We begin with the definition of the left invariant subsets of $L_p(K)$.

Definition 1. A subset C of the space $L_p(K)$ is called left invariant if $f_x \in C$, for each $f \in C$ and $x \in K$.

Theorem 1. *Let K be a hypergroup and C a closed convex subset of $L_p(K)$, $1 \leq p < \infty$. Then C is left invariant if and only if $\varphi * C \subseteq C$ for all $\varphi \in P(K)$.*

Proof. Suppose that C is a closed convex left-invariant subset of $(L_p(K), \|\cdot\|_p)$ and $\varphi \in P(K)$. From Lemma 4 it follows that there exists a net $(\theta_\alpha)_\alpha \subseteq \operatorname{co} E(K)$, $\theta_\alpha = \sum_{k=1}^{n_\alpha} \lambda_k^{(\alpha)} \delta_{x_k^{(\alpha)}}$, such that $(\theta_\alpha)_\alpha$ τ_c-converges to φ. Let f be arbitrary in C. Then, using Lemma 2 (i), it results that $(\theta_\alpha * f)_\alpha$ converges to $\varphi * f$ in $L_p(K)$. As

$$\theta_\alpha * f = \left(\sum_{k=1}^{n_\alpha} \lambda_k^{(\alpha)} \delta_{x_k^{(\alpha)}} \right) * f = \sum_{k=1}^{n_\alpha} \lambda_k^{(\alpha)} f_{x_k^{(\alpha)\vee}}$$

and C is convex and left invariant, the net $(\theta_\alpha * f)_\alpha$ is contained in C. The set C being closed it follows that $\varphi * f \in C$.

Conversely, consider $x \in K$ and $f \in C$. As $\delta_{x^\vee} \in MP(K)$, there exists a net $(\varphi_\alpha)_\alpha \subseteq P(K)$ such that $(\varphi_\alpha)_\alpha$ τ_c-converges to $\delta_{x^\vee}$ (see Lemma 4). Then, as $(\varphi_\alpha * f)_\alpha \subseteq C$ and C is closed, it follows that its limit, $\delta_{x^\vee} * f = f_x$ belongs to C. $\square$

The following consequence is apparent.

Corollary 1. *Let K be a hypergroup and I a closed linear subspace of $L_p(K)$, $1 \leq p < \infty$. Then $L_1(K) * I \subseteq I$ if and only if I is a left invariant subspace of $L_p(K)$.*

Remark. This result generalizes the classical result from the locally compact group case, namely that a closed linear subspace of $L_1(K)$ is a left ideal if and only if it is left invariant.

Basically with the same arguments, using Lemma 2 (iii) and Lemma 4, we can prove a variant of Theorem 1 and of its corollary for $p = \infty$. More precisely, we have:

Theorem 2. *Let K be a hypergroup and C a ω^*-closed convex subset of $L_\infty(K)$. Then C is left invariant if and only if $\varphi * C \subseteq C$ for all $\varphi \in P(K)$.*

Corollary 2. *Let K be a hypergroup and I a ω^*-closed linear subspace of $L_\infty(K)$. Then $L_1(K) * I \subseteq I$ if and only if I is a left-invariant subspace of $L_\infty(K)$.*

Applying the previous theorems to the set $\overline{\mathrm{co}\,\{f_x \mid x \in K\}}$, which is a closed, convex and, with Lemma 4, left-invariant subset of $L_p(K)$ for all $f \in L_p(K)$, we derive the next consequence:

Corollary 3. a) *Let $f \in L_p(K)$, $1 \leq p < \infty$. Then*

$$\overline{\mathrm{co}\,\{f_x \mid x \in K\}} = \overline{\{\varphi * f \mid \varphi \in P(K)\}}.$$

b) *Let $f \in L_\infty(K)$. Then*

$$\overline{\mathrm{co}\,\{f_x \mid x \in K\}}^{\omega^*} = \overline{\{\varphi * f \mid \varphi \in P(K)\}}^{\omega^*}.$$

Theorem 3. *Let K be a hypergroup, $1 < p < \infty$. Then K is noncompact if and only if each closed convex left-invariant nonempty subset of $L_p(K)$ contains the origin.*

Proof. Assume that K is noncompact, and take C a closed convex left-invariant nonempty subset of $L_p(K)$. As K is noncompact for each compact subset L of K, there exists an element $x_L \in K \backslash L$. It follows that the net $(\delta_{x_L})_L$ ω^*-converges to zero in $M(K)$. Using Lemma 2 (ii) we conclude that the net $(\delta_{x_L} * f)_L$ converges to zero in $(L_p(K), \omega)$. The set C is closed and convex, so ω-closed. As $(\delta_{x_L} * f)_L \subseteq C$, it follows that its ω-limit is still in C, that is, C contains the origin. The converse is clear because, if K is compact, the set $C = \{1_K\}$ is a closed, convex left-invariant subset of $L_p(K)$. $\square$

Proposition 1. *Let K be a hypergroup, $1 < p$, $q < \infty$ such that $1/p + 1/q = 1$ and C a compact, convex, left-invariant nonempty subset of $L_p(K)$. Then,*

$$\{f * g^\vee \mid f \in C,\ g \in L_q(K)\} \subseteq AP(K) \cap \mathcal{C}_o(K).$$

Proof. Let f be arbitrary in C and $g \in L_q(K)$. By [**9**, Theorem 6.2F], $f * g^\vee \in \mathcal{C}_o(K)$. In order to justify that $f * g^\vee \in AP(K)$, we notice that, for each $x \in K$, we have:

$$\begin{aligned}(f * g^\vee)_x(y) &= \int f_{x*y}(z) g(z)\, dz = \int f_x(y * z) g(z)\, dz \\ &= (f_x * g^\vee)(y), \quad \forall\, y \in K.\end{aligned}$$

As $\{f_x \mid x \in K\}$ is contained in the compact set C, there is a net $(x_\alpha)_\alpha \subseteq K$ such that $(f_{x_\alpha})_\alpha$ converges in the normed space $L_p(K)$. Since $g \in L_q(K)$, using [**9**, Theorem 6.2E], we have

$$\|f_{x_\alpha} * g^\vee - f_{x_\beta} * g^\vee\|_\infty \le \|f_{x_\alpha} - f_{x_\beta}\|_p \|g\|_q \longrightarrow 0,$$

so $(f_{x_\alpha} * g^\vee)_\alpha$ converges in $(\mathcal{C}(K), \|\cdot\|_\infty)$. $\square$

Remark. As far as we know, almost periodic functions on hypergroups have been investigated by Lasser [**11, 12**] and, in connection with weakly almost periodic functions, by Wolfenstetter [**21**]. It has been already established that, in a general setting, without imposing supplementary conditions on the hypergroup, many of the classical results about almost periodic functions on locally compact groups do not hold in the hypergroups (even abelian) context. For example, different from the locally compact noncompact groups case, for certain hypergroups, the set $AP(K) \cap \mathcal{C}_o(K)$ does not reduce to zero. Almost periodic nonzero functions vanishing at infinity can be found for example on the polynomial hypergroups from Jacobi family. On the other hand there are certain important classes of hypergroups, such as $[FD]_B^- \cap [SIN]_B$

groups and Chebyshev polynomials [**12**], for which any almost periodic function vanishing at infinity is zero. This fact justifies the next theorem that generalizes [**13**, Theorem 4.6].

Theorem 4. *Let K be a hypergroup such that $AP(K)\cap\mathcal{C}_o(K)=\{0\}$, $1<p<\infty$. Then K is noncompact if and only if each compact, convex, left-invariant nonempty subset of $L_p(K)$ consists only of the origin.*

Proof. Let K be noncompact, C a compact, convex, left-invariant nonempty subset of $L_p(K)$ and f arbitrary in C. Then, taking $(\varphi_V)_V\subseteq L_q(K)$ as in Lemma 1, by Proposition 1, $f*\varphi_V^\vee\in AP(K)\cap\mathcal{C}_o(K)=\{0\}$, so $f*\varphi_V^\vee=0$, for all $V\in\mathcal{V}$. As $\|f*\varphi_V^\vee-f\|_p\to 0$, it follows that $f=0$. Hence the set C consists only of zero.

Conversely, if K is compact, the set $C=\{1_K\}$ is a compact, convex and left-invariant subset of $L_p(K)$. $\square$

Remark. Obviously, the "*if*" part is valid without any restriction on K.

4. Multipliers for the spaces $\mathbf{L}_p(K)$. According to the definition of multipliers for topological linear spaces of functions [**10**, Chapter 3, p. 66], we give the following definition of the left multipliers for $L_p(K)$, $1\le p\le\infty$.

Definition 2. Let $p,q\in[1,\infty]$. A bounded linear operator T from the normed space $L_q(K)$ to the normed space $L_p(K)$ is called a left multiplier for the pair $(L_q(K),L_p(K))$ if $T(f_x)=(Tf)_x$, for all $f\in L_q(K)$, $x\in K$.

Notation. The set of the left multipliers for the pair $(L_q(K),L_p(K))$ will be denoted by $\mathcal{M}(L_q(K),L_p(K))$.

Theorem 5. *Let $q\in[1,\infty)$, $p\in[1,\infty]$, T a bounded linear operator applying $L_q(K)$ into $L_p(K)$. Then T is in $\mathcal{M}(L_q(K),L_p(K))$ if and only if $T(\varphi*f)=\varphi*T(f)$ for all $f\in L_q(K)$, $\varphi\in L_1(K)$.*

Proof. Assume first that T is a left multiplier for the pair $(L_q(K), L_p(K))$. It is enough to prove that $T(\varphi * f) = \varphi * T(f)$ for all $f \in L_q(K)$, for all $\varphi \in P(K)$. Let $\varphi \in P(K)$. By Lemma 4, there exists $(\theta_\alpha)_\alpha \subseteq \operatorname{co} E(K)$, $\theta_\alpha = \sum_{k=1}^{n_\alpha} \lambda_k^{(\alpha)} \delta_{x_k^{(\alpha)}}$, such that $(\theta_\alpha)_\alpha$ τ_c-converges to φ. Using Lemma 2 (i), it results that $(\theta_\alpha * f)_\alpha$ converges to $\varphi * f$ in $L_q(K)$. Then,

$$\begin{aligned} T(\varphi * f) &= \lim_\alpha T(\theta_\alpha * f) = \lim_\alpha T\bigg(\sum_{k=1}^{n_\alpha} \lambda_k^{(\alpha)} \delta_{x_k^{(\alpha)}} * f\bigg) \\ &= \lim_\alpha T\bigg(\sum_{k=1}^{n_\alpha} \lambda_k^{(\alpha)} f_{x_k^{(\alpha)\vee}}\bigg) = \lim_\alpha \sum_{k=1}^{n_\alpha} \lambda_k^{(\alpha)} T(f_{x_k^{(\alpha)\vee}}) \\ &= \lim_\alpha \sum_{k=1}^{n_\alpha} \lambda_k^{(\alpha)} T(f)_{x_k^{(\alpha)\vee}} = \lim_\alpha \sum_{k=1}^{n_\alpha} \lambda_k^{(\alpha)} \delta_{x_k^{(\alpha)}} * T(f) \\ &= \varphi * T(f), \quad \forall\, f \in L_q(K). \end{aligned}$$

Conversely, let $(\varphi_V)_{V \in \mathcal{V}}$ be a net of functions chosen as in Lemma 1. As $(\varphi_V)_x * f = (\varphi_V * f)_x$ for all $x \in K$, $V \in \mathcal{V}$ and as $T((\varphi_V)_x * f) = T(\varphi_V * f)_x$ for all $f \in L_q(K)$, everything follows from the continuity of T. $\square$

Remarks 1. When $q = p = 1$, Theorem 5 translates to the case of hypergroups the classical result of Wendel [**20**] concerning $(L_1(G), L_1(G))$ multipliers.

2. Extending the concept of *convolutor of the L_p-spaces* from the locally compact group context, see for example, [**16**, Definition 9.1], we may define a convolutor of $L_p(K)$, $1 \le p < \infty$, as a bounded linear operator $T : L_p(K) \to L_p(K)$, enjoying the property that $T(f * g) = f * T(g)$ for all $f, g \in \mathcal{C}_c(K)$. Using Theorem 5, we infer that, just as for the locally compact groups, $\mathcal{M}(L_p(K), L_p(K))$, $1 \le p < \infty$, coincides with the space of convolutors of $(L_p(K))$, $1 \le p < \infty$.

The complete characterization of the elements of $\mathcal{M}(L_1(G), L_p(G))$, $1 \le p \le \infty$, when G is a locally compact group, was obtained by Brainerd and Edwards [**2**, Theorem 2.5]. For hypergroups in our approach (with Haar measure) the structure of elements of $\mathcal{M}(L_1(K), L_1(K))$ can be obtained as a particular case of a result of Ghahramani and Medgalchi concerning multipliers on weighted hypergroup algebras [**6**] and extends the Brainerd and Edwards characterization. The next theorem generalizes it to $\mathcal{M}(L_1(K), L_p(K))$, $1 < p \le \infty$, K a hypergroup with Haar measure.

Theorem 6. *Let $1 < p < \infty$ or $p = \infty$ and $L_\infty(K) = L_1^*(K)$. For T a bounded linear operator from $L_1(K)$ to $L_p(K)$, the following are equivalent*:

(i) $T \in \mathcal{M}(L_1(K), L_p(K))$.

(ii) *There exists $f \in L_p(K)$ such that $T(h) = h * f$, $h \in L_1(K)$.*

Proof. Consider first $T \in \mathcal{M}(L_1(K), L_p(K))$, $1 < p < \infty$. Let $(\varphi_V)_{V \in \mathcal{V}}$ a net of functions chosen as in Lemma 1. As $\|h*\varphi_V - h\|_1 \to 0$ it follows that $(T(h * \varphi_V))_{V \in \mathcal{V}}$ converges (in norm) to $T(h)$ for all $h \in L_1(K)$. On the other hand, the net $(T(\varphi_V))_{V \in \mathcal{V}}$ is bounded in the reflexive space $L_p(K)$, so passing to a subnet, if necessary, we may assume that $(T(\varphi_V))_{V \in \mathcal{V}}$ converges in the weak-topology to some $f \in L_p(K)$. Then, using Theorem 5 and Lemma 3, we have

$$T(h) = \lim_V T(h * \varphi_V) = \lim_V h * T(\varphi_V) = h * f.$$

When $p = \infty$ and $L_\infty(K) = L_1^*(K)$, the arguments are similar since, in $L_\infty(K) = L_1^*(K)$, the bounded net $(T(\varphi_V))_{V \in \mathcal{V}}$ has a weak*-cluster point.

Conversely, the next equalities

$$(T(h))_x = \delta_{x^\vee} * (h * f) = (\delta_{x^\vee} * h) * f = h_x * f = T(h_x),$$
$$\forall\, x \in K, \quad \forall\, h \in L_1(K),$$

show that $T \in \mathcal{M}(L_1(K), L_p(K))$. □

Next we give generalizations of some of the classical results on compact multipliers for group*al* algebras of locally compact groups. More specifically, Akemann [**1**] proved that, if G is a compact group, then $\mathcal{M}(L_1(G), L_1(G))$ consists only in compact operators, a result that was extended to the (weighted) compact hypergroups case in [**6**]. The next theorem generalizes this statement to $\mathcal{M}(L_1(K), L_p(K))$, $1 < p < \infty$, K a compact hypergroup.

Theorem 7. *Let K be a compact hypergroup. Then, each $T \in \mathcal{M}(L_1(K), L_p(K))$, $1 < p < \infty$ is a compact operator.*

Proof. We consider $1 < p < \infty$. Let $T \in \mathcal{M}(L_1(K), L_p(K))$, so, using Theorem 6, there exists $f \in L_p(K)$ such that $T(h) = h * f$, for all $h \in L_1(K)$. We will prove that T takes the unit ball of $L_1(K)$ into a relative compact set of $L_p(K)$. Since K is compact and the mapping $x \mapsto f_x$ is continuous, the set $\overline{\mathrm{co}\{f_x \mid x \in K\}} \subseteq L_p(K)$ is compact, so using Corollary 2 it follows that $\{\varphi * f \mid \varphi \in P(K)\}$ is relatively compact in $L_p(K)$. Let h be in the unit ball of $L_1(K)$. Then $h = (h_1 - h_2) + i(h_3 - h_4)$, where $h_j > 0$ and $\|h_j\|_1 < 1$, $j = 1, 2, 3, 4$. It follows that $T(h_j)$ is in $C = \{\lambda\varphi * f \mid \lambda \in [0, 1]\}$. As C is relatively compact, it results that $T(h)$ lies into the relative compact set $(C - C) + i(C - C)$. $\square$

In [**17**], Sakai proved that the only compact multiplier for the pair $(L_1(G), L_1(G))$ when G is a locally compact noncompact group is zero. The extension of this result to compact multipliers when K is a noncompact (weighted) hypergroup is discussed in [**6**]. Following our approach we show that if $1 \le q \le \infty$, $1 < p < \infty$, and if the hypergroup K has the property that $AP(K) \cap \mathcal{C}_o(K) = \{0\}$, then $\mathcal{M}(L_q(K), L_p(K)) = \{0\}$.

Theorem 8. *Let K be a noncompact hypergroup such that $AP(K) \cap \mathcal{C}_o(K) = \{0\}$. Then the only compact element T in $\mathcal{M}(L_q(K), L_p(K))$, $1 \le q \le \infty$, $1 < p < \infty$, is $T = 0$.*

Proof. Let T be a compact left multiplier for the pair $(L_q(K), L_p(K))$. Then, the closure of the set $\{T(f) \mid \|f\|_q \le 1\}$ is a compact convex left-invariant nonempty set of $L_p(K)$, so by Theorem 4, it reduces to zero. Hence $T = 0$. $\square$

Remark. Right multipliers for the L_p-spaces of K and *right* invariant subsets can be defined in a similar way. More precisely, for $p, q \in [1, \infty]$, a bounded linear operator T from the normed space $L_q(K)$ to the normed space $L_p(K)$ is called a *right* multiplier for the pair $(L_q(K), L_p(K))$ if $T(f^x) = (Tf)^x$ for all $f \in L_q(K)$, $x \in K$, and, respectively, a set C of the space $L_p(K)$ is called *right* invariant if $f^x \in C$, for each $f \in C$ and $x \in K$. Following the locally compact group approach of [**13**], we notice that the above results regarding left-invariant subsets and left multipliers can be transferred to the ones that are right invariant, with the natural modifications of the technical details involving the modular function of K.

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