ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 3, 2007

INTEGRAL CLOSURES, LOCAL COHOMOLOGY AND IDEAL TOPOLOGIES

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ABSTRACT. Let $(R, \mathfrak{m})$ be a formally equidimensional local ring of dimension d. Suppose that Φ is a system of nonzero ideals of R such that, for all minimal prime ideals $\mathfrak{p}$ of R, $\mathfrak{a}+\mathfrak{p}$ is $\mathfrak{m}$-primary for every $\mathfrak{a} \in \Phi$. In this paper, the main result asserts that for any ideal $\mathfrak{b}$ of R, the integral closure $\mathfrak{b}^{*(H_\Phi^d(R))}$ of $\mathfrak{b}$ with respect to the Artinian R-module $H_\Phi^d(R)$ is equal to $\mathfrak{b}_a$, the classical Northcott-Rees integral closure of $\mathfrak{b}$. This generalizes the main result of [**13**] concerning the question raised by D. Rees.

1. Introduction. Let R denote a commutative Noetherian ring (with identity) of dimension d, and let A be an Artinian R-module. We say that the ideal $\mathfrak{a}$ of R is a *reduction* of the ideal $\mathfrak{b}$ of R with respect to A if $\mathfrak{a} \subseteq \mathfrak{b}$ and there exists an integer $s \geq 1$ such that $(0 :_A \mathfrak{a}\mathfrak{b}^s) = (0 :_A \mathfrak{b}^{s+1})$. An element x of R is said to be *integrally dependent on* $\mathfrak{a}$ *with respect to* A if $\mathfrak{a}$ is a reduction of $\mathfrak{a} + Rx$ with respect to A, see [**12**]. Moreover, the set $\mathfrak{a}^{*(A)} := \{x \in R \mid x$ is integrally dependent on $\mathfrak{a}$ with respect to $A\}$ is an ideal of R, *called the integral closure of* $\mathfrak{a}$ with respect to A.

In [**13**] the dual concepts of reduction and integral closure of the ideal $\mathfrak{b}$ with respect to a Noetherian R-module N were introduced; we shall use $\mathfrak{b}_a^{(N)}$ to denote the integral closure of $\mathfrak{b}$ with respect to N. If $N = R$, then $\mathfrak{b}_a^{(N)}$ reduces to that the usual Northcott-Rees integral closure $\mathfrak{b}_a$ of $\mathfrak{b}$.

The purpose of the present paper is to show that, for any system of ideals Φ of a formally equidimensional local ring $(R, \mathfrak{m})$ of dimension d, if $\operatorname{Rad}(\mathfrak{a} + \mathfrak{p}) = \mathfrak{m}$ for all minimal primes $\mathfrak{p}$ of R and for every $\mathfrak{a} \in \Phi$, then $\mathfrak{b}^{*(H_\Phi^d(R))}$, the integral closure of $\mathfrak{b}$ with respect to $H_\Phi^d(R)$, is equal

2000 AMS *Mathematics Subject Classification.* Primary 13D45, 13B20, 13E05.
Key words and phrases. Integral closures, local cohomology, formally equidimensional rings.
This research was been in part supported by a grant from IPM (No. 81130020).
Received by the editors on May 25, 2004, and in revised form on Feb. 5, 2005.

to $\mathfrak{b}_a$, the classical Northcott-Rees integral closure of $\mathfrak{b}$, for each ideal $\mathfrak{b}$ of R.

Throughout this paper, all rings considered will be commutative and Noetherian and will have nonzero identity elements. Such a ring will be denoted by R, and a typical ideal of R will be denoted by $\mathfrak{a}$. Moreover, throughout this paper, let Φ denote a system of nonzero ideals of R, i.e., for all $\mathfrak{a}, \mathfrak{b} \in \Phi$, there exists an $\mathfrak{c} \in \Phi$ such that $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$.

Let S be a multiplicatively closed subset of R and $\mathfrak{a}$ an ideal of R. The *nth* (S)*-symbolic power* of $\mathfrak{a}$, denoted by $S(\mathfrak{a}^n)$, is defined to be the ideal $\cup_{s \in S}(\mathfrak{a}^n :_R s)$ of R. In the case $S = R \setminus \cup \{\mathfrak{p} \in m\mathrm{Ass}_R R/\mathfrak{a}\}$, where $m\mathrm{Ass}_R R/\mathfrak{a}$ is the set of the minimal primes of $\mathrm{Ass}_R R/\mathfrak{a}$, the nth (S)-symbolic power of $\mathfrak{a}$ is denoted by $\mathfrak{a}^{(n)}$. Let $\dim R = d$, and let $\mathrm{ht}\, \mathfrak{a} = d - 1$ for all $\mathfrak{a} \in \Phi$. Then the sets Φ and $\Phi^{(1)} := \{\mathfrak{a}^{(1)} : \mathfrak{a} \in \Phi\}$ induce topologies on R, which are called the Φ-*adic* and $\Phi^{(1)}$-*symbolic topology*, respectively. These topologies are said to be *equivalent* if, for every $\mathfrak{a} \in \Phi$ there exists $\mathfrak{b} \in \Phi$ such that $\mathfrak{b}^{(1)} \subseteq \mathfrak{a}$. Equivalence of some topologies have been studied in [**5, 8, 9**] and has led to some interesting results. We shall use $\mathcal{C}(R)$ to denote the category of all R-modules and all R-homomorphisms between them. The system of ideals Φ determines the Φ-torsion functor $\Gamma_\Phi : \mathcal{C}(R) \to \mathcal{C}(R)$. This is the subfunctor of the identity functor on $\mathcal{C}(R)$ for which $\Gamma_\Phi(N) = \{x \in N : \mathfrak{a}x = 0 \text{ for some } \mathfrak{a} \in \Phi\}$ for each R-module N. For each $i \geq 0$, the ith right derived functor of Γ_Φ is denoted by H^i_Φ. Moreover, for any R-module N and for any prime ideal $\mathfrak{p}$ of R, the ideal $\cap_{n\geq 1}\mathfrak{p}^{(n)}$ of R, respectively submodule $\cup_{n \geq 1}(0 :_N \mathfrak{p}^{(n)})$ of N, is denoted by $c(\mathfrak{p})$, respectively $\Gamma_{(\mathfrak{p})}(N)$. We denote $\cup_{\mathfrak{a}\in\Phi} V(\mathfrak{a})$ by $V(\Phi)$, where $V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}\, R : \mathfrak{p} \supseteq \mathfrak{a}\}$. Finally, if $(R, \mathfrak{m})$ is local, then $\hat{R}$, respectively E, denotes the *completion of* R with respect to the $\mathfrak{m}$-*adic topology*, respectively *injective envelope* of the simple R-module $R/\mathfrak{m}$.

In the second section we give a generalization of Lichtenbaum-Hartshorne theorem in the context of general local cohomology modules, whose applications will be used in the proof of the main theorem. More precisely we prove the following:

(1.1) Theorem. *Assume that* $(R, \mathfrak{m})$ *is local of dimension* d. *Then the following statements are equivalent*:

(i) $H^d_\Phi(R) \neq 0$.

(ii) *There exists* $\mathfrak{p} \in \operatorname{Spec} \widehat{R}$ *such that* $\dim \widehat{R}/\mathfrak{p} = d$ *and* $\dim \widehat{R}/(\mathfrak{a}\widehat{R}+\mathfrak{p}) = 0$ *for each* $\mathfrak{a} \in \Phi$.

Let $(R, \mathfrak{m})$ be a local ring. Then R is said to be *equidimensional* if, for any minimal prime ideal $\mathfrak{p}$ of R, $\dim R/\mathfrak{p} = \dim R$. Also, R is called *formally equidimensional* if its completion $\widehat{R}$ is equidimensional, see [**6**, p. 251].

In the third section, for any ideal $\mathfrak{b}$ of R, we examine the equality of the classical Northcott-Rees' integral closure $\mathfrak{b}_a$ and the integral closure $\mathfrak{b}^{*(H^d_\Phi(R))}$ of $\mathfrak{b}$ with respect to the Artinian R-module $H^d_\Phi(R)$, by using the applications of the Lichtenbaum-Hartshorne theorem. More precisely, we shall show that:

(1.2) Theorem. *Let* $(R, \mathfrak{m})$ *be a formally equidimensional local ring of dimension* d *such that for all* $\mathfrak{p} \in m\operatorname{Ass} R$, $\operatorname{Rad}(\mathfrak{a}+\mathfrak{p}) = \mathfrak{m}$ *for each* $\mathfrak{a} \in \Phi$. *Then* $\mathfrak{b}_a = \mathfrak{b}^{*(H^d_\Phi(R))}$ *for every ideal* $\mathfrak{b}$ *of* R.

2. Ideal topologies and the Lichtenbaum-Hartshorne theorem. The main point of this section is to establish a generalization of the Lichtenbaum-Hartshorne theorem in the context of general local cohomology modules. Some applications are given. The following lemma plays a key role in this section.

(2.1) Lemma. *Let* $(R, \mathfrak{m})$ *be a complete Gorenstein local ring of dimension* d, *and let* $V := \{\mathfrak{p} \in V(\Phi) : \operatorname{ht}\mathfrak{p} = d-1\}$. *Then* $H^d_\Phi(R) \cong \operatorname{Hom}_R(\cap_{\mathfrak{p}\in V} c(\mathfrak{p}), E)$.

Proof. Let $\mathcal{E}^{\cdot} : 0 \to E^0 \to E^1 \to \cdots \to E^{d-1} \to E^d \to 0$ be a minimal injective resolution of R. Then, by [**6**, Theorems 18.1 and 18.8], we have $E^i \cong \oplus_{\substack{\mathfrak{p}\in \operatorname{Spec} R\\ \operatorname{ht}\mathfrak{p}=i}} E_R(R/\mathfrak{p})$. Moreover, in view of [**6**, Theorem 18.4],

$$\Gamma_\Phi\Bigg(\bigoplus_{\substack{\mathfrak{p}\in \operatorname{Spec} R\\ \operatorname{ht}\mathfrak{p}=d-1}} E_R(R/\mathfrak{p})\Bigg) = \bigoplus_{\mathfrak{p}\in V} E_R(R/\mathfrak{p}) \quad \text{and} \quad \Gamma_\Phi(E) = E.$$

Consequently, the exact sequence $\mathcal{E}^{\cdot}$ induces an exact sequence

$$\bigoplus_{\mathfrak{p}\in V} E_R(R/\mathfrak{p}) \longrightarrow E \longrightarrow H^d_\Phi(R) \longrightarrow 0.$$

Let $\mathfrak{p} \in V$ so that $\operatorname{ht}\mathfrak{p} = d-1$. Since each element of $E_R(R/\mathfrak{p})$ is annihilated by some symbolic power of $\mathfrak{p}$, it follows that $\Gamma_{(\mathfrak{p})}(E^{d-1}) = \Gamma_{(\mathfrak{p})}(E_R(R/\mathfrak{p})) = E_R(R/\mathfrak{p})$. Hence, we obtain the following exact sequence

$$E_R(R/\mathfrak{p}) \longrightarrow \Gamma_{(\mathfrak{p})}(E) \longrightarrow \varinjlim_i \operatorname{Ext}^d_R(R/\mathfrak{p}^{(i)}, R) \longrightarrow 0.$$

On the other hand, since $\mathfrak{p}^{(i)}$ is $\mathfrak{p}$-primary it follows that $\operatorname{Ass}_R R/\mathfrak{p}^{(i)} = \{\mathfrak{p}\}$, and so $\operatorname{depth}_R R/\mathfrak{p}^{(i)} > 0$, note that $\operatorname{ht}\mathfrak{p} = d-1$. Consequently, from [**3**, Lemma 8.1.8] we get $\operatorname{Ext}^d_R(R/\mathfrak{p}^{(i)}, R) = 0$. Therefore, the sequence $E_R(R/\mathfrak{p}) \to \Gamma_{(\mathfrak{p})}(E) \to 0$ is exact, and so $H^d_\Phi(R) \cong E/\sum_{\mathfrak{p}\in V}\Gamma_{(\mathfrak{p})}(E)$. Hence, it is enough to show that

$$E/\Sigma_{\mathfrak{p}\in V}\Gamma_{(\mathfrak{p})}(E) = \operatorname{Hom}_R\bigg(\bigcap_{\mathfrak{p}\in V} c(\mathfrak{p}), E\bigg).$$

To this end, as R is complete, by using the notations of [**14**, Section 5.4] we get

$$\begin{aligned}\Sigma_{\mathfrak{p}\in V}\Gamma_{(\mathfrak{p})}(E) &= \Sigma_{\mathfrak{p}\in V}\Sigma_{n\geq 1}(0:_E \mathfrak{p}^{(n)})\\ &= \bigg(\bigcap_{\mathfrak{p}\in V}\bigcap_{n\geq 1}(0:_E \mathfrak{p}^{(n)})^\lambda\bigg)^\mu\\ &= \bigg(\bigcap_{\mathfrak{p}\in V}\bigcap_{n\geq 1}\mathfrak{p}^{(n)}0^\lambda\bigg)^\mu\\ &= \bigg(0:_E \bigcap_{p\in V} c(\mathfrak{p})\bigg).\end{aligned}$$

The exact sequence $0 \to \cap_{\mathfrak{p}\in V} c(\mathfrak{p}) \to R \to R/\cap_{\mathfrak{p}\in V} c(\mathfrak{p}) \to 0$ induces an exact sequence $0 \to \operatorname{Hom}_R(R/\cap_{\mathfrak{p}\in V} c(\mathfrak{p}), E) \to E \to \operatorname{Hom}_R(\cap_{\mathfrak{p}\in V} c(\mathfrak{p}), E) \to 0$, and the desired result now follows. $\square$

The following corollary will establish the equivalence between the topologies defined by Φ and $\Phi^{(1)}$ in terms of vanishing the local cohomology module $H^d_\Phi(R)$. Recall that for any ideal $\mathfrak{a}$ of R, we use $\mathfrak{a}^{(1)}$ to denote the ideal $\cup_{s\in S}(\mathfrak{a} :_R s)$, where $S = R\backslash \cup\{\mathfrak{p} \in m\mathrm{Ass}_R R/\mathfrak{a}\}$ and $\Phi^{(1)} := \{\mathfrak{b}^{(1)} : \mathfrak{b} \in \Phi\}$.

(2.2) Corollary. *Let* $(R, \mathfrak{m})$ *be a Gorenstein local* (*not necessarily complete*) *ring of dimension* d*, and let* $\mathrm{ht}\,\mathfrak{a} = d - 1$ *for each* $\mathfrak{a} \in \Phi$*. Then the following conditions are equivalent*:

(i) $H^d_\Phi(R) = 0$.

(ii) *The* $\Phi^{(1)}$*-symbolic topology is equivalent to the* Φ*-adic topology.*

Proof. First we show (i) $\Rightarrow$ (ii). To do this, suppose that $\mathfrak{a} \in \Phi$. Then, we can write $\mathfrak{a} = \mathfrak{a}^{(1)} \cap \mathfrak{q}$ for some $\mathfrak{m}$-primary ideal $\mathfrak{q}$. In view of [**11**, Lemma 1.3], there exists a parameter ideal $\mathfrak{q}_0$ of R such that $\mathfrak{q}_0 \subseteq \mathfrak{q}$. By virtue of [**6**, Theorem 18.1], $\mathfrak{q}_0$ is irreducible; and hence there exists $x \in E$ such that $\mathfrak{q}_0 = \mathrm{Ann}\,(x)$. By assumption and the proof of (2.1), there is an $n \in \mathbf{N}$ such that $x \in \Sigma_{\substack{\mathfrak{p}\in V(\Phi)\\ \mathrm{ht}\,\mathfrak{p}=d-1}} (0 :_E \mathfrak{p}^{(n)})$. Hence, we can write $x = e_1 + \cdots + e_r$, where $e_i \in (0 :_E \mathfrak{p}_i^{(n)})$ and $\mathfrak{p}_i \in V(\Phi)$ with $\mathrm{ht}\,\mathfrak{p}_i = d - 1$ for all $i = 1, \dots, r$. There exists $\mathfrak{b} \in \Phi$ such that $\mathfrak{b} \subseteq \cap_{i=1}^r \mathfrak{p}_i$. Since $\mathrm{ht}\,\mathfrak{b} = d - 1$, it follows that $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are minimal prime ideals of $\mathfrak{b}$. Now, let $b \in \mathfrak{b}^{(n)}$. Then, $sb \in \mathfrak{b}^n$ for some $s \in R\backslash \cup \{\mathfrak{p} \in m\mathrm{Ass}_R R/\mathfrak{b}\}$. Since $\mathfrak{b}^n \subseteq \cap_{i=1}^r \mathfrak{p}_i^{(n)}$, it follows that $sb \in \cap_{i=1}^r \mathfrak{p}_i^{(n)}$, and therefore $b \in \cap_{i=1}^r \mathfrak{p}_i^{(n)}$. Consequently $bx = be_1 + \cdots + be_r = 0$, so that $b \in \mathfrak{q}_0$. Accordingly $\mathfrak{b}^{(n)} \subseteq \mathfrak{q}$. Moreover, it is easily seen that there exists $\mathfrak{c} \in \Phi$ such that $\mathfrak{c}^{(1)} \subseteq \mathfrak{b}^{(n)}$, (note that $\mathrm{ht}\,\mathfrak{a} = d - 1$ for all $\mathfrak{a} \in \Phi$). Therefore we have $\mathfrak{c}^{(1)} \subseteq \mathfrak{q}$. On the other hand, there is an $\mathfrak{d} \in \Phi$ such that $\mathfrak{d} \subseteq \mathfrak{a}\mathfrak{c}$; hence, $\mathfrak{d}^{(1)} \subseteq \mathfrak{a}^{(1)} \cap \mathfrak{c}^{(1)} \subseteq \mathfrak{a}^{(1)} \cap \mathfrak{q}$, as required.

In order to prove that (ii) $\Rightarrow$ (i) suppose that $x \in E$. Then, there is $\mathfrak{a} \in \Phi$ such that $\mathfrak{a}x = 0$. Therefore, by assumption, there exists $\mathfrak{b} \in \Phi$ such that $\mathfrak{b}^{(1)} \subseteq \mathfrak{a}$. Next, let $m\mathrm{Ass}_R R/\mathfrak{b} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$. Then, it is easy to see that $\cap_{i=1}^r \mathfrak{p}_i^{(l)} \subseteq \mathfrak{b}^{(1)}$ for sufficiently large l. Therefore, $(\cap_{i=1}^r \mathfrak{p}_i^{(l)})x = 0$, and hence $x \in \Sigma_{i=1}^r (0 :_E \mathfrak{p}_i^{(l)})$ by [**10**, Lemma 2.2].

Consequently $E = \Sigma_{\substack{\mathfrak{p}\in V(\Phi) \\ \operatorname{ht}\mathfrak{p}=d-1}} \cup_{j\in \mathbf{N}} (0 :_E \mathfrak{p}^{(j)})$; so that, by the proof of (2.1) $H^d_\Phi(R) = 0$, as desired. □

Before obtaining the main result of this section we recall some useful properties of the local cohomology functors $H^i_\Phi(.)$ in the following remark.

(2.3) Remark. Let $\varphi : R \to S$ be a ring homomorphism.

(i) For any system of ideals Φ of R, let $\Phi S := \{\mathfrak{a}S : \mathfrak{a} \in \Phi\}$. Then it is easy to see that ΦS is a system of ideals of S and for any $i \in \mathbf{N}_0$, $H^i_{\Phi S}(N) \cong H^i_\Phi(N)$, for any S-module N, see [**3**, Theorem 4.2.1].

(ii) Suppose that $\varphi : R \to S$ is surjective and Φ is a system of ideals of S. Then one can check easily that $\Phi^\varphi := \{(\varphi^{-1}(\mathfrak{a}))^n : \mathfrak{a} \in \Phi,\ n \in \mathbf{N}\}$ is a system of ideals of R. Consequently, for any S-module N and $i \in \mathbf{N}_0$, there is an isomorphism of R-modules $H^i_{\Phi^\varphi}(N) \cong H^i_\Phi(N)$ as Φ and $\Phi^\varphi S$ introduce the same topology on S.

Now we are ready to state and prove the main theorem of this section.

(2.4) Theorem (Lichtenbaum-Hartshorne theorem). *Let* $(R, \mathfrak{m})$ *be local of dimension* d. *Then the following statements are equivalent*:

(i) $H^d_\Phi(R) = 0$.

(ii) *For every* $\mathfrak{p} \in \operatorname{Spec} \widehat{R}$ *with* $\dim \widehat{R}/\mathfrak{p} = d$, *there exists* $\mathfrak{a} \in \Phi$ *such that* $\dim \widehat{R}/\mathfrak{a}\widehat{R} + \mathfrak{p} > 0$.

Proof. First we show (i) $\Rightarrow$ (ii). Suppose the contrary is true. Then there exists $\mathfrak{p} \in \operatorname{Spec} \widehat{R}$ such that $\dim \widehat{R}/\mathfrak{p} = d$ but $\dim \widehat{R}/\mathfrak{a}\widehat{R} + \mathfrak{p} = 0$ for all $\mathfrak{a} \in \Phi$. Suppose that $\Psi = \Phi\widehat{R} := \{\mathfrak{a}\widehat{R} : \mathfrak{a} \in \Phi\}$ and $\Theta = \Psi(\widehat{R/\mathfrak{p}}) := \{\mathfrak{a}\widehat{R/\mathfrak{p}} : \mathfrak{a} \in \Phi\}$ are systems of ideals of $\widehat{R}$ and $\widehat{R/\mathfrak{p}}$, respectively. Then, in view of (2.3), we have $H^d_\Phi(\widehat{R}/\mathfrak{p}) \cong H^d_\Psi(\widehat{R}/\mathfrak{p}) \cong H^d_\Theta(\widehat{R}/\mathfrak{p})$. On the other hand, it is easy to see that two systems of ideals Θ and $\{(\mathfrak{m}\widehat{R}/\mathfrak{p})^n : n \in \mathbf{N}\}$ are 'comparable' in sense [**3**, Proposition 3.1.1]. It follows that $H^d_\Phi(\widehat{R}/\mathfrak{p}) \cong H^d_{\mathfrak{m}\widehat{R}/\mathfrak{p}}(\widehat{R}/\mathfrak{p})$, and so by [**3**, Theorem 6.1.4]), $H^d_\Phi(\widehat{R}/\mathfrak{p}) \neq 0$.

Moreover, in view of [**1**, Proposition 2.1], $H^d_{\Phi}(\widehat{R}/\mathfrak{p}) \cong H^d_{\Psi}(\widehat{R}) \otimes_{\widehat{R}} \widehat{R}/\mathfrak{p}$ and by virtue of (2.3) we have $H^d_{\Psi}(\widehat{R}) \cong H^d_{\Phi}(R) \otimes_R \widehat{R}$. Now, putting this together with the above isomorphism, we obtain a contradiction.

In order to prove (ii) $\Rightarrow$ (i), suppose that $\Phi\widehat{R} := \{\mathfrak{a}\widehat{R} : \mathfrak{a} \in \Phi\}$. Then by virtue of (2.3), $H^d_{\Phi}(R) \otimes_R \widehat{R} \cong H^d_{\Phi\widehat{R}}(\widehat{R})$. Since $\widehat{R}$ is faithfully flat over R, we may assume that R is complete. Now, we use the Cohen structure theorem to see that there is a complete Gorenstein local ring A of dimension d and a surjective ring homomorphism $\psi : A \to R$. Let $\Psi = \Phi^{\psi} := \{(\psi^{-1}(\mathfrak{a}))^n : \mathfrak{a} \in \Phi,\ n \in \mathbf{N}\}$, then by (2.3) $H^i_{\Psi}(R) \cong H^i_{\Phi}(R)$. Therefore, it is enough to show that $H^d_{\Psi}(R) = 0$. To achieve this, suppose $\mathfrak{b} = \operatorname{Ker}\psi$. Then $A/\mathfrak{b}$ and R are isomorphic A-modules, and therefore, it follows from [**1**, Proposition 2.1] that $H^d_{\Psi}(R) \cong H^d_{\Psi}(A) \otimes_A R$. On the other hand, in view of (2.1),

$$H^d_{\Psi}(A) \cong \operatorname{Hom}_A\Bigg(\bigcap_{\substack{\mathfrak{p}\in V(\Psi)\\ \operatorname{ht}\mathfrak{p}=d-1}} c(\mathfrak{p}), E_A(A/\mathfrak{m})\Bigg).$$

Hence, we obtain that

$$H^d_{\Psi}(R) \cong \operatorname{Hom}_A\Bigg(\bigcap_{\substack{\mathfrak{p}\in V(\Psi)\\ \operatorname{ht}\mathfrak{p}=d-1}} c(\mathfrak{p}), E_A(A/\mathfrak{m})\Bigg) \bigotimes_A R.$$

From this and [**3**, Lemma 10.2.16], it is sufficient for us to show that

$$\operatorname{Ass}_A \operatorname{Hom}_A\Bigg(R, \bigcap_{\substack{\mathfrak{p}\in V(\Psi)\\ \operatorname{ht}\mathfrak{p}=d-1}} c(\mathfrak{p})\Bigg) = \varnothing.$$

To this end, by [**2**, Section 2.1, Proposition 10], an associated prime ideal of the A-module $\operatorname{Hom}_A(R, \cap_{\substack{\mathfrak{p}\in V(\Psi)\\ \operatorname{ht}\mathfrak{p}=d-1}} c(\mathfrak{p}))$ must contain $\mathfrak{b}$, and belong to $\operatorname{Ass} A$ (and so have dimension d), and cannot be contained in any prime ideal $\mathfrak{p}$ of $V(\Psi)$ with $\operatorname{ht}\mathfrak{p} = d-1$. Therefore, the hypotheses show that there is no such associated prime ideal, and so the proof is complete. $\square$

(2.5) Corollary. *Suppose that* $(R,\mathfrak{m})$ *is local and has dimension* d *and that for every prime ideal* $\mathfrak{p}$ *of* $\widehat{R}$ *with* $\dim \widehat{R}/\mathfrak{p} = d$ *there exists* $\mathfrak{a} \in \Phi$ *such that* $\dim \widehat{R}/(\mathfrak{a}\widehat{R}+\mathfrak{p}) > 0$*. Then* $H^i_\Phi(N) = 0$ *for all* $i \geq d$ *and for every* R*-module* N*.*

Proof. As $H^i_\Phi(N) = \varinjlim_{\mathfrak{a}\in\Phi} H^i_{\mathfrak{a}}(N)$, the result follows from Grothendieck's vanishing theorem, [**3**, Exercise 6.1.9], and (2.4). □

(2.6) Corollary. *Let* $(R,\mathfrak{m})$ *be local, and let* N *be a nonzero finitely generated* R*-module of dimension* d*. Assume that for every prime ideal* $\mathfrak{p} \in \operatorname{Supp} \widehat{N}$ *with* $\dim \widehat{R}/\mathfrak{p} = d$ *there exists* $\mathfrak{a} \in \Phi$ *such that* $\dim \widehat{R}/(\mathfrak{a}\widehat{R}+\mathfrak{p}) > 0$*. Then* $H^i_\Phi(N) = 0$ *for all* $i \geq d$*.*

Proof. In view of [**1**, Lemma 2.3], we may assume that R is complete. Let $\Psi = \Phi(R/\mathrm{Ann}_R N) := \{\mathfrak{a}(R/\mathrm{Ann}_R N) : \mathfrak{a} \in \Phi\}$, then, by (2.3) and [**1**, Proposition 2.1], we have

$$H^d_\Phi(N) \cong H^d_\Psi(N) \cong H^d_\Psi(R/\mathrm{Ann}_R N) \bigotimes_{R/\mathrm{Ann}_R N} N.$$

The result now follows by using (2.4) and [**3**, Theorem 6.1.2]). □

The following proposition will be one of our main tools in Section 3. Before we state it, let us recall that, if R is local with dimension d, then the general local cohomology module $H^d_\Phi(R)$ is an Artinian R-module, see [**1**, Theorem 3.1], and so has a natural structure as an $\widehat{R}$-module.

(2.7) Proposition. *Suppose that* $(R,\mathfrak{m})$ *is local of dimension* d*. Then*

$$\begin{aligned}\mathrm{Att}_{\widehat{R}}(H^d_\Phi(R)) = \{\mathfrak{p} \in \operatorname{Spec} \widehat{R} : \dim \widehat{R}/\mathfrak{p} = d,\\ \textit{and } \operatorname{Rad}(\mathfrak{a}\widehat{R}+\mathfrak{p}) = \mathfrak{m}\widehat{R} \textit{ for all } \mathfrak{a} \in \Phi\}.\end{aligned}$$

Proof. In view of (2.3) without loss of generality, we may assume that R is complete. First, suppose that $\mathfrak{p} \in \mathrm{Att}_R(H^d_\Phi(R))$. Then, by virtue of [**3**, Proposition 7.2.11], we have $H^d_\Phi(R) \neq \mathfrak{p}H^d_\Phi(R)$, whence $H^d_\Phi(R) \otimes_R R/\mathfrak{p} \neq 0$. Therefore $H^d_\Phi(R/\mathfrak{p}) \neq 0$ by [**1**, Proposition 2.1]. Let $\Psi = \Phi(R/\mathfrak{p}) := \{\mathfrak{a}(R/\mathfrak{p}) : \mathfrak{a} \in \Phi\}$, then by (2.3), $H^d_\Psi(R/\mathfrak{p}) \neq 0$. It follows that $\dim R/\mathfrak{p} = d$. On the other hand, in view of (2.4), we have $\dim R/\mathfrak{a} + \mathfrak{p} = 0$ for all $\mathfrak{a} \in \Phi$. That is $\operatorname{Rad}(\mathfrak{a} + \mathfrak{p}) = \mathfrak{m}$, as desired.

Conversely, let $\mathfrak{p} \in \operatorname{Spec} R$ be such that $\dim R/\mathfrak{p} = d$ and $\operatorname{Rad}(\mathfrak{a} + \mathfrak{p}) = \mathfrak{m}$ for all $\mathfrak{a} \in \Phi$. Let $\Psi = \Phi(R/\mathfrak{p}) := \{\mathfrak{a}(R/\mathfrak{p}) : \mathfrak{a} \in \Phi\}$, then in view of (2.3), $H^d_\Psi(R/\mathfrak{p}) \cong H^d_\Phi(R/\mathfrak{p})$, and it is straightforward to check that two systems of ideals Ψ and $\{(\mathfrak{m}/\mathfrak{p})^n : n \in \mathbf{N}\}$ introduce the same topology on $R/\mathfrak{p}$. Consequently, we have $H^d_\Psi(R/\mathfrak{p}) \cong H^d_{\mathfrak{m}/\mathfrak{p}}(R/\mathfrak{p})$. Thus, by [**3**, Theorems 6.1.4 and 7.1.3], $H^d_\Psi(R/\mathfrak{p})$ is a nonzero Artinian $R/\mathfrak{p}$-module which is 0-secondary. Therefore, $H^d_\Phi(R/\mathfrak{p})$ is a nonzero Artinian R-module with $\operatorname{Rad}(0 :_R H^d_\Phi(R/\mathfrak{p})) = \mathfrak{p}$. But, by virtue of [**1**, Proposition 2.1], $H^d_\Phi(R/\mathfrak{p})$ is a homomorphic image of $H^d_\Phi(R)$, and so it follows that $\mathfrak{p} \in \operatorname{Att}_R(H^d_\Phi(R))$. This completes the proof. □

(2.8) Remark. Let the situation be as in (2.7). Then

$$\operatorname{Att}_{\widehat{R}}(H^d_\Phi(R)) = \operatorname{Ass}_{\widehat{R}} D((H^d_\Phi(R))) = \operatorname{Ass}_{\widehat{R}}\Bigg(\operatorname{Hom}_A\Big(R, \bigcap_{\substack{\mathfrak{p}\in V(\Psi)\\ \operatorname{ht}\mathfrak{p}=d-1}} c(\mathfrak{p})\Big)\Bigg),$$

in which the second equation follows from the proof of Theorem 2.4 and $D := \operatorname{Hom}_{\widehat{R}}(-, E(\widehat{R}/\mathfrak{m}\widehat{R}))$ represents Matlis duality.

(2.9) Corollary. *Let the situation be as in* (2.7). *Then*

$$\operatorname{Att}_R(H^d_\Phi(R)) = \{\mathfrak{p} \cap R : \mathfrak{p} \in \operatorname{Spec} \widehat{R}, \dim \widehat{R}/\mathfrak{p} = d, \\ \textit{and } \operatorname{Rad}(\mathfrak{a}\widehat{R} + \mathfrak{p}) = \mathfrak{m}\widehat{R} \textit{ for all } \mathfrak{a} \in \Phi\}.$$

3. Local cohomology and integral closures. The purpose of this section is to prove that, for any system of ideals Φ of a formally equidimensional local ring $(R, \mathfrak{m})$ of dimension d, the integral closure $\mathfrak{b}^{*(H^d_\Phi(R))}$ of $\mathfrak{b}$ with respect to the Artinian R-module $H^d_\Phi(R)$ is equal to the classical Northcott-Rees integral closure $\mathfrak{b}_a$ for every ideal $\mathfrak{b}$ of R provided $\operatorname{Rad}(\mathfrak{a}+\mathfrak{p}) = \mathfrak{m}$ for all minimal primes $\mathfrak{p}$ of R and for every

$\mathfrak{a} \in \Phi$. The main goal of this section is Theorem 3.3. Before we state Theorem 3.3, we give a couple of lemmas that we will use in the proof of this theorem.

(3.1) Lemma. *Let* $(R, \mathfrak{m})$ *be a local ring of dimension* d. *Then the following statements are equivalent*:

(i) $\operatorname{Ann}_R H_\Phi^d(R)$ *is a nilpotent ideal.*

(ii) R *is equidimensional and* $\operatorname{Rad}(\mathfrak{a}\widehat{R}+\mathfrak{q}) = \mathfrak{m}\widehat{R}$ *for all* $\mathfrak{q} \in m\operatorname{Ass}\widehat{R}$ *and for every* $\mathfrak{a} \in \Phi$.

Proof. First we show (i) $\Rightarrow$ (ii). To do this, suppose $\mathfrak{p}$ is an arbitrary minimal prime of R. In view of [**3**, Proposition 7.2.11] and (2.9), it follows that

$$\begin{aligned}\operatorname{Rad}(\operatorname{Ann}_R(H_\Phi^d(R))) = \cap\big\{\mathfrak{q} \cap R : \mathfrak{q} \in \operatorname{Spec}\widehat{R},\ \dim \widehat{R}/\mathfrak{q} = d, \\ \text{and } \operatorname{Rad}(\mathfrak{a}\widehat{R} + \mathfrak{q}) = \mathfrak{m}\widehat{R} \text{ for all } \mathfrak{a} \in \Phi\big\}.\end{aligned}$$

Hence, there exists $\mathfrak{q} \in \operatorname{Spec}\widehat{R}$ such that $\dim \widehat{R}/\mathfrak{q} = d$, $\mathfrak{q} \cap R = \mathfrak{p}$; and, for every $\mathfrak{a} \in \Phi$, $\operatorname{Rad}(\mathfrak{a}\widehat{R} + \mathfrak{q}) = \mathfrak{m}\widehat{R}$. Now, it is easy to see that $\dim R/\mathfrak{p} = d$, and so (ii) holds.

In order to prove that (ii) $\Rightarrow$ (i), suppose that R is equidimensional and $\mathfrak{p}$ is an arbitrary minimal prime of R. Then, there exists $\mathfrak{q} \in \operatorname{Ass}_{\widehat{R}}\widehat{R}/\mathfrak{p}\widehat{R}$ such that $\dim \widehat{R}/\mathfrak{q} = \dim \widehat{R}/\mathfrak{p}\widehat{R} = \dim R/\mathfrak{p} = d$. Therefore, by assumption $\operatorname{Rad}(\mathfrak{a}\widehat{R}+\mathfrak{q}) = \mathfrak{m}\widehat{R}$ for every $\mathfrak{a} \in \Phi$. Moreover, by using the Going Down theorem it follows that $\mathfrak{p} \cap R = \mathfrak{q}$. Consequently, in view of (2.9) and [**3**, Proposition 7.2.11], we deduce that $\operatorname{Ann}_R H_\Phi^d(R)$ is nilpotent, as desired. $\square$

(3.2) Lemma. *Suppose that* $(R, \mathfrak{m})$ *is a local ring and that* A *is an Artinian* R*-module. Then, for any ideal* $\mathfrak{b}$ *of* R, $(\mathfrak{b}\widehat{R})^{*(A)} \cap R = \mathfrak{b}^{*(A)}$.

Proof. Follows from the definition and [**6**, Theorem 7.11]. $\square$

We are now ready to state and prove the main theorem of this paper.

(3.3) Theorem. *Let* $(R, \mathfrak{m})$ *be a formally equidimensional local ring of dimension d such that, for all minimal primes* $\mathfrak{p}$ *of* R, $\operatorname{Rad}(\mathfrak{a}+\mathfrak{p}) = \mathfrak{m}$ *for every* $\mathfrak{a} \in \Phi$. *Then, for every ideal* $\mathfrak{b}$ *of* R, $\mathfrak{b}^{*(H^d_\Phi(R))} = \mathfrak{b}_a$.

Proof. First of all, let us assume $\mathfrak{q}$ is an arbitrary minimal prime ideal of $\widehat{R}$. Then it is readily checked that $\mathfrak{p} := \mathfrak{q} \cap R$ is a minimal prime ideal of R, so that we have $\operatorname{Rad}(\mathfrak{a}\widehat{R} + \mathfrak{q}) = \mathfrak{m}\widehat{R}$, for every $\mathfrak{a} \in \Phi$. Therefore, in view of [**7**, Lemma 3.15] and (3.2), without loss of generality we may assume that $(R, \mathfrak{m})$ is complete. Let $D := \operatorname{Hom}_R(-, E)$, then according to the Matlis' duality theorem [**3**, Theorem 10.2.12], $D(H^d_\Phi(R))$ is a Noetherian R-module and $H^d_\Phi(R) \cong DD(H^d_\Phi(R))$. Now, let $\mathfrak{b}$ be an arbitrary ideal of R, then by [**13**, Theorem 2.1] we have $(\mathfrak{b})_a^{D(H^d_\Phi(R))} = \mathfrak{b}^{*(H^d_\Phi(R))}$. Furthermore, because $\operatorname{Ann}_R D(H^d_\Phi(R)) = \operatorname{Ann}_R H^d_\Phi(R)$ from (3.1), it follows that $\operatorname{Ann}_R D(H^d_\Phi(R))$ is a nilpotent ideal. Thus using [**13**, Remark 1.6], it is straightforward to check that $\mathfrak{b}_a = (\mathfrak{b})_a^{D(H^d_\Phi(R))}$. Therefore now the assertion follows. $\square$

As an application of Theorem 3.3, we provide the following result which generalizes the main theorem of [**13**] concerning the question raised by D. Rees.

(3.4) Corollary (see [**13**, Corollary 3.5]). *Let* $(R, \mathfrak{m})$ *be a formally equidimensional local ring of dimension d and* $\mathfrak{a}$ *an ideal of* R *such that, for every minimal prime ideal* $\mathfrak{p}$ *of* R, $\mathfrak{a} + \mathfrak{p}$ *is* $\mathfrak{m}$*-primary. Then, for each ideal* $\mathfrak{b}$ *of* R, $\mathfrak{b}^{*(H^d_\mathfrak{a}(R))} = \mathfrak{b}_a$.

Acknowledgments. The author is deeply grateful to the referee for his or her valuable suggestions on the paper and for drawing the author's attention to Remarks 2.3 and 2.8. Also, I would like to thank Prof. P. Schenzel for his useful suggestions and the many helpful discussions during the preparation of this paper.

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