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PERTURBATIONS IN THE SPEISER CLASS

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ABSTRACT. In this paper we study perturbations of maps from a family of expanding entire functions from the Speiser class. Maps in this family, which we denoted by $\mathcal{H}$, have the form $f_a(z) = \sum_{j=0}^{n} a_j e^{(j-k)z}$ where $0 < k < n$ and $a = (a_0, \dots, a_n) \in \mathbf{C}^{n+1}$ is a parameter. Using a known result of Eremenko and Lyubich about structural stability of such maps, perturbation theory (Kato-Rellich theorem) and research of Urbański and Zdunik on perturbations in the exponential family, we shall prove that the Hausdorff dimension of the set of points in the Julia set having nonescaping orbits depends analytically on the parameter $a \in \mathbf{C}^{n+1}$.

1. Introduction. The long-term study of dynamical systems directed many authors work toward the investigation of the dynamics of *families* of mappings. The most popular examples of families of transcendental entire functions of finite singular type include the one-parameter *exponential* family $\{ae^z\}$, the one-parameter *sine family* $\{a \sin z\}$, with $a \in \mathbf{C}$ or the generalized 2-parameter *cosine* family $\{ae^z + be^{-z}\}$ with $(a, b) \in \mathbf{C}^2$.

In this paper we continue our study of the dynamics of maps in the family $\mathcal{H}$ introduced in [**4**] and defined as follows. Let n and k be positive integers, let $a = (a_0, \dots, a_n) \in \mathbf{C}^{n+1}$ be a vector and let P_a, f_a be functions defined by the formulas

$$P_a(z) = a_n z^n + \cdots + a_1 z + a_0 \in \mathbf{C}[z],$$
$$f_a(z) = \frac{P_a(e^z)}{e^{kz}} = \sum_{j=0}^{n} a_j e^{(j-k)z}.$$

Then

$$\mathcal{H} = \Big\{ f_a : 0 < k < \deg P_a \text{ and } \delta_a > 0 \Big\}$$

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where

$$\delta_a = \frac{1}{2} \min\left\{\frac{1}{2}, \operatorname{dist}(J_{f_a}, \mathcal{P}_{f_a})\right\}$$

and J_{f_a} is the Julia set of f_a. Recall that by

$$\mathcal{P}_{f_a} = \overline{\bigcup_{n\geq 0} f_a^n(\operatorname{Crit}(f_a))}$$

we denote the post-critical set of $f_a \in \mathcal{H}$ where $\operatorname{Crit}(f_a) = \{z : f_a'(z) = 0\}$ and

$$\operatorname{dist}(J_{f_a}, \mathcal{P}_{f_a}) = \inf\{|z_1 - z_2| : z_1 \in J_{f_a},\ z_2 \in \mathcal{P}_{f_a}\}.$$

The family $\mathcal{H}$ is a subclass of the class Ent, the class of transcendental entire functions. Moreover, every function f_a from $\mathcal{H}$ has only finitely many asymptotic and critical values which are sometimes called singular values. In other words, for $f \in Ent$, $\omega \in \widehat{\mathbf{C}}$ is a singular value if ω is a singularity of f^{-1}, which means that $f : f^{-1}(V) \to V$ is not a regular covering map for any neighborhood V of ω. Therefore, if ω is a nonsingular value of f, then there exists a neighborhood V of ω where every branch of f^{-1} is well defined and is a conformal map of V. This fact is used extensively throughout the paper. The set of singular values we denote by $\operatorname{Sing}(f^{-1})$.

The set

$$\mathcal{S} = \{f \in Ent : \operatorname{Sing}(f^{-1}) \text{ is a finite set}\}$$

is usually called the class of finite singular type entire functions or, following Eremenko and Lyubich, the Speiser class, and has been studied repeatedly for many years; the reader is referred only to [**10, 12, 14**]. Functions belonging to $\mathcal{S}$ have the property that their Fatou set contains no wandering domain and no Baker domain.

For any $f \in \mathcal{H} \subset S$, denote by B an open disk containing $\operatorname{Sing}(f^{-1})$, and let B^c be the complement of B in $\mathbf{C}$. Then any component T of $f^{-1}(B^c)$ is simply connected and its closure contains infinity. The map $f : T \to B^c$ is a universal covering, and T is called an exponential tract. If f satisfies some conditions, it was shown that the set of points whose orbits remain in T is a Cantor bouquet, see [**10**].

1.1. An additional assumption. In view of the structural stability theorem of Eremenko and Lybich, we additionally assume that, if $f_a \in \mathcal{H}$ and if z is a periodic point of f_a of period m, then $|(f^m)'(z)| \neq 0$. For the sake of generality, we ask also the question about the possibility of proving the main result of this paper without this extra condition. Of course, such maps are not structurally stable, but what we actually need is an existence of quasiconformal conjugacy on some ε-neighborhood of the Julia set of them to all maps with close parameter. Then changing the proofs a bit we could obtain the main result of this paper, that is, the Hausdorff dimension depends analytically on the parameter for any member of "old" $\mathcal{H}$.

Observe that the assumption $0 < k < \deg P_a$ implies that any map $f_a \in \mathcal{H}$ does not have a finite asymptotic value since $P_a(z)/z^k$ converges to infinity when z approaches 0 or ∞. If this condition is not satisfied, then one of the limits is finite and it would be a finite asymptotic value of f_a. Even in this case, the main result of this paper can be established using the proofs from this article with minor changes provided that the maps do not have super-attracting points.

1.2. The quotient space. Let $b = (b_0, \dots, b_n) \in \mathbf{C}^{n+1}$. Since the map $f_b \in \mathcal{H}$ is periodic with period $2\pi i$, we consider its natural action on the quotient space $P = \mathbf{C}/\sim$ (the cylinder) where $z_1 \sim z_2$ if, and only if, $z_1 - z_2 = 2k\pi i$ for some $k \in \mathbf{Z}$. If $\pi : \mathbf{C} \to P$ is the natural projection, then, since the map $\pi \circ f_b : \mathbf{C} \to P$ is constant on equivalence classes of relation $\sim$, it induces a conformal map

$$F_b : P \longrightarrow P.$$

The cylinder P is endowed with Euclidean metric which will be denoted in what follows by the symbol $|w - z|$ for all $z, w \in P$. The Julia set of F_b is defined to be

$$J_{F_b} = \pi(J_{f_b}).$$

Observe that

$$F_b(J_{F_b}) = J_{F_b} = F_b^{-1}(J_{F_b}).$$

1.3. Let $b = (b_0, \dots, b_n) \in \mathbf{C}^{n+1}$. Note that the derivative $f_b^{(s)}(z)$, $s \geq 0$, of a map $f_b \in \mathcal{H}$ has the expression

$$f_b^{(s)}(z) = \sum_{j=0}^{n} b_j (j-k)^s e^{(j-k)z}$$

for $z \in \mathbf{C}$. Also observe that the derivative with respect to the parameter $b \in \mathbf{C}^{n+1}$ of this function has the form

$$\frac{\partial f_b^{(s)}}{\partial b}(z) = \begin{bmatrix} (\partial f_b^{(s)}/\partial b_0)(z) \\ (\partial f_b^{(s)}/\partial b_1)(z) \\ \cdots \\ (\partial f_b^{(s)}/\partial b_j)(z) \\ \cdots \\ (\partial f_b^{(s)}/\partial b_n)(z) \end{bmatrix} = \begin{bmatrix} (-k)^s e^{-kz} \\ (1-k)^s e^{(1-k)z} \\ \cdots \\ (j-k)^s e^{(j-k)z} \\ \cdots \\ (n-k)^s e^{(n-k)z} \end{bmatrix}.$$

Hence,

$$\left\| \frac{\partial f_b}{\partial b}(z) \right\|^2 = \sum_{j=0}^{n} e^{2(j-k)\operatorname{Re} z} \tag{1.1}$$

and

$$\left\| \frac{\partial f_b'}{\partial b}(z) \right\|^2 = \sum_{j=0}^{n} (j-k)^2 e^{2(j-k)\operatorname{Re} z}, \tag{1.2}$$

where $\| \cdot \|$ means the norm on $\mathbf{C}^{n+1}$ defined by the formula

$$||(c_0, \dots, c_n)|| = \sqrt{\sum_{j=0}^{n} |c_j|^2}$$

for $(c_0, \dots, c_n) \in \mathbf{C}^{n+1}$.

1.4. Remark. Let $b \in \mathcal{H}$ and $t > 1$. By f and F we respectively denote f_b and F_b. Then, for every $z \in J_F$, the lower and upper topological pressure is defined by the formulas

$$\underline{P}_z(t) = \liminf_{n\to\infty} \frac{1}{n} \log \sum_{x \in F^{-n}(z)} |(F^n)'(x)|^{-t} = \liminf_{n\to\infty} \frac{1}{n} \log P_z(n,t),$$

$$\overline{P}_z(t) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{x \in F^{-n}(z)} |(F^n)'(x)|^{-t} = \limsup_{n\to\infty} \frac{1}{n} \log P_z(n,t),$$

where $P_z(n,t) = \sum_{x\in F^{-n}(z)} |(F^n)'(x)|^{-t}$. It can be proved that these do not depend on the choice of $z \in J_F$ and that $\underline{P}(t) =$

$\overline{P}(t) = P(t)$. Moreover, the function $P(t)$ for $t > 1$ is finite, convex, continuous, strictly decreasing and $\lim P(t) = -\infty$ as $t \to \infty$. For more explanations we refer the reader to our paper [**4**].

As we already said we study the set $J^r_{f_b}$ consisting of those points of J_{f_b} that do not escape to infinity under positive iterates of f_b. If

$$I_\infty(f_b) = \left\{ z \in \mathbf{C} : \lim_{n\to\infty} f^n(z) = \infty \right\}, \quad \text{then} \quad J^r_{f_b} = J_{f_b} \backslash I_\infty(f_b)$$

and if

$$I_\infty(F_b) = \left\{ z \in P : \lim_{n\to\infty} F^n(z) = \infty \right\}, \quad \text{then} \quad J^r_{F_b} = J_{F_b} \backslash I_\infty(F_b).$$

In [**4**] an important relation was proved between the pressure function $P(t)$ and the set $J^r_{f_b}$ called *Bowen's formula*, i.e., HD $(J^r_{F_b})$ is the unique zero of the function $t \mapsto P(t)$ for $t > 1$.

1.5. Results. Using the methods of Thermodynamic Formalism (for more information about TF and its connection with dynamics, we refer the reader to [**21, 27**]) we are able to prove that the Hausdorff dimension of the subset of the Julia set of such maps, consisting of the points for which the forward orbit does not escape to infinity, i.e., the set

$$J^r_{f_a} = J_{f_a} \backslash I_\infty(f_a),$$

where $I_\infty(f_a) = \{z \in \mathbf{C} : \lim_{n\to\infty} f^n(z) = \infty\}$, depends real-analytically on the parameter $a \in \mathbf{C}^{n+1}$.

In order to do that we first study quasiconformal conjugacies in the family $\mathcal{H}$ and then we introduce well defined Perron-Frobenius operators associated with some special maps. The classical theorem of Hartogs will help us to prove the main tool of this paper (Theorem 3.10) which will allow us to prove Proposition 4.10 which shows that these Perron-Frobenius operators can be embedded into a family of operators which depend holomorphically on the parameter a chosen from a designed open set $G \subset \mathbf{C}^{n+1}$.

Finally, using perturbation theory (Kato-Rellich theorem) and our results from [**4**] where we proved mainly that HD $(J^r_{F_a}) = h$ is the unique zero of the pressure function $t \to P(t)$ for $t > 1$ and $a \in \mathbf{C}^{n+1}$, we get (Theorem 4.17) that the function $a \mapsto HD(J^r_{f_a})$ is real-analytic.

2. Quasiconformal conjugacies in the family $\mathcal{H}$. In this section we present some analytic and geometric properties of the family $\mathcal{H}$. We follow the analysis from [**25**], which in turn follows the more elaborated descriptions from [**12, 18**]. As in [**12**] every $f \in \mathcal{H} \subset S$ is viewed as an element of a finite dimensional complex analytic manifold $M_f = \mathcal{H} \subset S$. In the referred paper [**12**], various analytical and geometrical results are proved on M_f.

Since the domain of all functions from $\mathcal{H}$ is the noncompact complex plane, the most natural topology of $\mathcal{H}$ is the topology of uniform convergence on the compact subset of $\mathbf{C}$. Observe that this topology is equivalent to the Euclidean topology on $\mathbf{C}^{n+1}$ when we identify a parameter a with the function f_a. Therefore, throughout this paper we sometimes write $a \in \mathcal{H}$ with the meaning that $f_a \in \mathcal{H}$. Moreover, whenever we say b is close to a we mean that f_b is close to f_a as well. We also say b is sufficiently close to a whenever we need b to be chosen from a small open neighborhood of $a \in \mathbf{C}^{n+1}$, compare [**12**].

A sense-preserving homeomorphism f of a domain G is called *quasiconformal* if its maximal dilatation $Q(G) \geq 1$ is finite. If $Q(G) \leq Q < \infty$, then f will be called $Q-$*quasiconformal*, see [**17**, p. 16]. Following the terminology used in the conformal case, we also call a quasiconformal homeomorphism a *quasiconformal mapping*.

After this short introduction and the description of the topological structure of $\mathcal{H}$, we can formulate a lemma which follows from [**12**], see also [**18**].

2.1 Lemma. *For $a \in \mathcal{H}$, f_a is* structurally stable*, i.e., if b is sufficiently close to a, then there exists a conjugating quasiconformal homeomorphism $h_b : \mathbf{C} \to \mathbf{C}$ such that*

$$f_b \circ h_b = h_b \circ f_a.$$

Moreover, the map $b \mapsto h_b(z)$ is holomorphic for every $z \in \mathbf{C}$ and the mapping $(b, z) \mapsto h_b(z)$ is continuous. The quasiconformal constant converges to 1 *as b approaches a.*

This is the moment when we need our extra condition since, if f_a has a super-attracting periodic point, then f_a is not structurally stable. This property of stability of the family $\mathcal{H}$ stated in the previous lemma is a crucial fact. But we need to have some kind of control over the changes resulting from the action of the quasiconformal homeomorphism in a neighborhood of a. This is stated in Proposition 2.4. To obtain this result we need to provide some information about quasiconformal maps and give some properties of functions from $\mathcal{H}$.

Let K, $\alpha > 0$. We say that a map $h : \mathbf{C} \to \mathbf{C}$ is (K, α)-Hölder continuous if

$$|h(z_1) - h(z_2)| \le K|z_1 - z_2|^\alpha$$

for all $z_1, z_2 \in \mathbf{C}$ such that $|z_1 - z_2| < 1$.

But what we are really interested in is the distortion of Euclidean distances under normalized K-quasiconformal maps, and we use the classical theorems of Koebe and Mori. For the proof of Koebe's theorems, the reader can see [**11**] and for the proof of Mori's theorem, see, for example, [**17**, p. 66].

Next we formulate two lemmas about functions from the family $\mathcal{H}$, and we generalize a useful result from [**4**]. The first one follows from 1.3 and brings essential information for understanding the dynamics of our maps. We also refer the reader to [**4**].

2.2. Lemma. *For $a \in \mathcal{H}$ there exist positive numbers M_1, M_2, M_3 and r such that for all $b \in B(a, r)$ and for all $z \in \mathbf{C}$ with $|\operatorname{Re} z| > M_3$, the following inequalities hold.*

(i) $M_1 e^{|\operatorname{Re} z| q(z)} \le |f_b'(z)| \le M_2 e^{|\operatorname{Re} z| q(z)}$,

(ii) $M_1 e^{|\operatorname{Re} z| q(z)} \le |f_b''(z)| \le M_2 e^{|\operatorname{Re} z| q(z)}$,

(iii) $M_1 e^{|\operatorname{Re} z| q(z)} \le |\partial f_b' / \partial b(z)| \le M_2 e^{|\operatorname{Re} z| q(z)}$,

where

$$q(z) = \begin{cases} k & \text{if } \operatorname{Re} z < 0 \\ n - k & \text{if } \operatorname{Re} z > 0. \end{cases}$$

Another important observation is that we can maintain the bounds from Lemma 2.2 when we apply the quasiconformal homeomorphism h_b to the points of J_{f_a}. Note the parts (iii) and (iv) follow from the equalities (1.2) and (1.1).

2.3. Lemma. *For $a \in \mathcal{H}$ there exists M_1, M_2, M_3 and r such that for all $b \in B(a,r)$ and for all $z \in \mathbf{C}$ with $|\mathrm{Re}\, z| > M_3$, the following inequalities hold.*

(i) $M_1 e^{|\mathrm{Re}\, z| q(z)} \le |f_b'(h_b(z))| \le M_2 e^{|\mathrm{Re}\, z| q(z)}$,

(ii) $M_1 e^{|\mathrm{Re}\, z| q(z)} \le |f_b''(h_b(z))| \le M_2 e^{|\mathrm{Re}\, z| q(z)}$,

(iii) $M_1 e^{|\mathrm{Re}\, z| q(z)} \le |\partial f_b'/\partial b(h_b(z))| \le M_2 e^{|\mathrm{Re}\, z| q(z)}$,

(iv) $M_1 e^{|\mathrm{Re}\, z| q(z)} \le |\partial f_b/\partial b(h_b(z))| \le M_2 e^{|\mathrm{Re}\, z| q(z)}$,

where

$$q(z) = \begin{cases} k & \text{if } \mathrm{Re}\, z < 0 \\ n-k & \text{if } \mathrm{Re}\, z > 0. \end{cases}$$

Consequently, we can also generalize Proposition 2.2 from [**4**], and we obtain that for a fixed parameter $a \in \mathcal{H}$, a map $f \in \mathcal{H}$ is expanding on its Julia set uniformly over a small neighborhood $B(a,r) \subset \mathcal{H}$; more precisely, for every $a \in \mathcal{H}$ there exist $c > 0$, $\gamma > 1$, $r > 0$ such that, for all $b \in B(a,r)$,

$$|(f_b^n)'(z)| > c\gamma^n \tag{2.1}$$

for every $z \in J_{f_b}$.

We state now the principal result of this section.

2.4. Proposition. *Fix $a \in \mathcal{H}$. For b sufficiently close to a, we can choose $h_b : \mathbf{C} \to \mathbf{C}$, the quasiconformal conjugacy homeomorphism, such that the following three properties hold.*

(i) $\sup_{z \in J_{f_a}} \{|(dh_b/db)(z)|\}$ *is bounded.*

(ii) $h_b : \mathbf{C} \to \mathbf{C}$ *is$(K(Q), 1/Q)$-Hölder continuous, where Q is quasiconformal constant of h_b, and $K : [1,\infty) \to (0,\infty)$ is increasing.*

(iii) *For every $z \in \mathbf{C}$ we have $h_b(z + 2\pi i) = h_b(z) + 2\pi i$. This shows that h_b is well defined on the cylinder P.*

Proof. Part (i). Let f_a, f_b be as above. Also consider J_{f_b}, J_{f_a} and $h_b : \mathbf{C} \to \mathbf{C}$ with $|a - b| < \varepsilon$ for a small $\varepsilon > 0$. We need to show that

$$\sup_{z \in J_{f_a}, b \in B(a,\varepsilon)} \left| \frac{dh_b(z)}{db} \right| < \infty$$

By the conjugacy relation we have that $h_b \circ f_a(z) = f_b \circ h_b(z)$ for every $z \in \mathbf{C}$. Therefore, for every $n \geq 0$, we have that

$$h_b(f_a^n(z)) = f_b^n(h_b(z)). \tag{2.2}$$

Let the function $f^j : \mathbf{C}^n \times \mathbf{C} \to \mathbf{C}$ be defined by the formula

$$f^j(b, z) = f_b^j(z)$$

for $j \geq 0$, and let $z \in J_{f_a}$ be a periodic point with period $n \geq 1$, i.e, $f_a^n(z) = z$. Then by (2.2) we obtain that

$$f^n(b, h_b(z)) = h_b(z)$$

for every $b \in B(a, \varepsilon)$. Differentiating the above relation with respect to the variable b, we get

$$D_1 f^n(b, h_b(z)) + D_2 f^n(b, h_b(z)) \cdot \frac{dh_b}{db}(z) = \frac{dh_b}{db}(z).$$

Since periodic points from the Julia set are not parabolic, this implies that

$$\frac{dh_b}{db}(z) = \frac{D_1 f^n(b, h_b(z))}{1 - D_2 f^n(b, h_b(z))}.$$

From this and (2.1) it follows that, if the period n of z is large enough, then

$$\left|\frac{dh_b}{db}(z)\right| \leq \frac{|D_1 f^n(b, h_b(z))|}{|D_2 f^n(b, h_b(z))| - 1} \leq 2\,\frac{|D_1 f^n(b, h_b(z))|}{|D_2 f^n(b, h_b(z))|}. \tag{2.3}$$

Let w denote $h_b(z)$. Then, using the equality $f_b^n(w) = f_b(f_b^{n-1}(w))$ (which is equivalent to $f^n(b, w) = f(b, f_b^{n-1}(w))$) we can estimate D_1 in terms of D_2 as follows. First write

$$\begin{aligned} D_1 f^n(b, w) &= D_1\Big(f(b, f^{n-1}(b, w))\Big) \\ &= D_1 f(b, f^{n-1}(b, w)) + D_2 f(b, f^{n-1}(b, w)) \cdot D_1 f^{n-1}(b, w). \end{aligned}$$

Therefore, repeating these computations for $n, n-1, \ldots, 1, 0$ and using the chain rule, we obtain

$$D_1 f^n(b, w) = \sum_{k=0}^{n-1} D_2 f^k(b, f^{n-k}(b, w)) \cdot D_1 f(b, f^{n-k-1}(w)).$$

With ∂-notation, for $w = h_b(z)$, it looks like this:

$$\begin{aligned} D_1 f^n(b, h_b(z)) &= \frac{\partial f^n}{\partial b}(b, w) \\ &= \sum_{k=0}^{n-1} (f_b^k)'(f_b^{n-k}(w)) \frac{\partial f_b}{\partial b}\, (f_b^{n-k-1}(w)). \end{aligned}$$

Then

$$\begin{aligned} \frac{D_1 f^n(b, h_b(z))}{D_2 f^n(b, h_b(z))} &= \frac{(\partial f^n/\partial b)(b, w)}{(f_b^n)'(w)} \\ &= \sum_{k=0}^{n-1} \frac{(\partial f_b/\partial b)(f_b^{n-k-1}(w)) \cdot (f_b^k)'(f_b^{n-k}(w))}{(f_b^n)'(w)} \\ &= \sum_{k=0}^{n-1} \frac{(\partial f_b/\partial b)(f_b^{n-k-1}(w))}{(f_b^{n-k})'(w)} \\ &= \sum_{k=0}^{n-1} \frac{(\partial f_b/\partial b)(f_b^{n-k-1}(w))}{(f_b')(f_b^{n-k-1}(w))} \cdot \frac{1}{(f_b^{n-k-1})'(w)}. \end{aligned} \tag{2.4}$$

Next we would like to show that

$$\left| \frac{(\partial f_b/\partial b)(f_b^{n-k-1}(w))}{(f_b')(f_b^{n-k-1}(w))} \right|$$

is uniformly (with respect to b) bounded from above. It is worth reminding that $f_b^{n-k-1}(w) \in J_{f_b}$ and both functions $(\partial f_b/\partial b)(\cdot)$ and $f_b'(\cdot)$ are periodic with period $2\pi i$. Therefore, it is enough to prove that there exists a constant C such that

$$\frac{|(\partial f_b/\partial b)(z)|}{|(f_b')(z)|} \le C \tag{2.5}$$

for b sufficiently close to a and for $z \in J_{f_b} \cap \{z \in \mathbf{C} : \operatorname{Im} z \in [0, 2\pi]\}$. To do this we split the set $J_{f_b} \cap \{z \in \mathbf{C} : \operatorname{Im} z \in [0, 2\pi]\}$ into two sets, a compact one $\{z \in J_{f_b} : x \in [-M_3, M_3] \times [0, 2\pi]\}$ and its complement. By (2.1)

$$C' = \sup \left\{ \frac{|(\partial f_b/\partial b)(x)|}{|(f_b')(x)|} : b \in B(a, \varepsilon),\ x \in J_{f_b}, \right. \\ \left. x \in [-M_3, M_3] \times [0, 2\pi] \right\} < \infty$$

for ε small enough. Moreover, by Lemma 2.2 (i) and (iii),

$$\frac{|(\partial f_b/\partial b)(x)|}{|(f_b')(z)|} \leq \frac{M_2}{M_1}$$

if $|\mathrm{Re}\, x| \geq M_3$. Therefore (2.5) is proved with $C = \max\{C', M_2/M_1\}$.

To finish the proof of part (i) note that, by (2.1), we can assume that $\varepsilon > 0$ satisfies the condition

$$\sum_{j=0}^{l} \frac{1}{|(f_b^j)'(w)|} \leq \frac{1}{c(1-(1/\gamma))}$$

for all n and $b \in B(a, \varepsilon)$. Then, putting (2.3), (2.4) and (2.5) together, we get

$$\sup_{\substack{z \in \mathrm{Per} \\ b \in B(a,\varepsilon)}} \left| \frac{dh_b}{db}(z) \right| < \infty.$$

Hence, since $\overline{\mathrm{Per}} = J_{f_a}$ and since $b \mapsto h_b(z)$ is analytic, part (i) follows.

Part (ii). Obviously we want to use Mori's theorem and the result obtained before. Point (i) shows, in particular, that for small $\varepsilon > 0$,

$$\sup_{\substack{z \in J_{f_a} \\ b \in B(a,\varepsilon)}} |z - h_b(z)| < 1. \tag{2.6}$$

Then let $\varepsilon > 0$ be small. Fix $x \in J_{f_a}$ and consider the open disk $D(x, 1)$ of radius 1 with center at x. Then $G_b = h_b(B(x, 1))$ is an open simply connected set for every $b \in B(a, \varepsilon)$. Let $R_b : D(0, 1) \to G_b$ be the conformal representation (Riemann map) of G_b such that $R(0) = h_b(x)$. Then the map

$$g_b = R_b^{-1} \circ h_b : D(x, 1) \longrightarrow D(0, 1)$$

is a Q-quasiconformal homeomorphism between two disks of radius 1.

Now let χ_x be a path in J_{F_a} which joins x and infinity. The existence of such a path is a consequence of the fact that all Fatou components are simply connected, see e.g., [**18**, p. 90]. Let $\chi_x^\omega \subset \chi_x \cap \overline{D(x, 1)}$ be an arc inside $B(x,1)$ joining x with a point on the boundary $\partial B(x,1)$, call it ω. Then $h_b(\chi_x^\omega)$ is an arc joining $h_b(x)$ and $h_b(\omega) \in \partial G_b$.

Note that there exists $z \in D(0,1)$ with $|z| = 1/2$ and $y \in D(x,1) \cap J_{F_a}$ such that $R_b(z) = h_b(y) \in J_{F_b}$ (or equivalently $g_b(y) = z$). From (2.6), for $|a-b| < \varepsilon$, it follows that

$$\begin{aligned} |R_b(z) - R_b(0)| &= |h_b(y) - h_b(x)| \\ &\le |h_b(y) - y| + |y - x| + |x - h_b(x)| \\ &= |h_b(y) - h_a(y)| + |y - x| + |x - h_b(x)| \\ &\le 2\varepsilon \sup \left\{ \left| \frac{\partial h_b}{\partial b} \right| : z \in J_{f_a}, b \in B(a, \varepsilon) \right\} + 1 \\ &\le 2\varepsilon + 1. \end{aligned}$$

It follows that $R_b(D(0,1/2))$ does not contain the ball $D(h_b(x), 2\varepsilon+1)$ since $R_b(D(0,1/2))$ does not contain any ball centered at $h_b(x)$ with radius greater than $|R_b(z) - h_b(x)|$. Then, using Koebe's distortion theorem, we get

$$|R_b'(0)| \le 4(2\varepsilon + 1). \tag{2.7}$$

Next, applying Mori's Theorem to the quasiconformal mapping g_b and to points $z_1, z_2 \in D(x,1)$, we get

$$|g_b(z_1) - g_b(z_2)| < 16|z_1 - z_2|^{1/Q}.$$

If additionally $z_1, z_2 \in B(x, 1/(32)^Q)$, then, using Koebe's theorem with $K = K(1/2)$ for the function R_b together with (2.7), we get

$$\begin{aligned} |h_b(z_1) - h_b(z_2)| &= |R_b(g_b(z_1) - R_b(g_b(z_2))| \\ &\le K|R'(0)||g_b(w) - g_b(z)| \le 4K(2\varepsilon + 1)|w - z|^{1/Q}. \end{aligned}$$

From the above computations it follows that h_b is $(4K(2\varepsilon+1), 1/Q)$-Hölder continuous on $1/(32)^Q$-neighborhood of J_{f_a}. But note that there exists r, such that the $r/(32)^Q$-neighborhood of J_{f_a} contains the whole plane $\mathbf{C}$. Therefore, considering the map $g_b^r(z) = g_b(rz)$ instead of g_b (we have to increase the domain of g_b to $D(x,r)$), we can repeat the computations to prove that h_b is $(K(Q), 1/Q)$-Hölder continuous on $\mathbf{C}$ for some $K(Q)$.

Part (iii). Consider the map $b \mapsto k_z(b) = h_b(z+2\pi i) - h_b(z) \in \mathbf{C}$. Since $b \mapsto h_b(z)$ is continuous, the map k_z is continuous as well. If $b \in B(a, \varepsilon)$ for some small ε, then as before we get from the conjugacy relation that

$$f_b(h_b(z+2\pi i)) = h_b(f_b(z+2\pi i)) = h_b(f_b(z)) = f_b(h_b(z)). \tag{2.8}$$

Then, for every $b \in B(a, \varepsilon)$, the set of possible values of k_z is a discrete subset of $\mathbf{C}$ (in particular has a finite intersection with the stripe $\{z \in \mathbf{C} : \operatorname{Im} z \in [0, 2\pi)\}$). If $h_b(z+2\pi i)$ and $h_b(z)$ are regular points of f_b, then, for c sufficiently close to b, $h_c(z+2\pi i)$ and $h_c(z)$ are regular points of f_c and, if $k_z(b) \neq 2\pi i$, then $k_z(c) \neq 2\pi i$, and, if $k_z(b) = 2\pi i$, then $k_z(c) = 2\pi i$. Since $k_z(a) = 2\pi i$ for $z \in \mathbf{C}$ and since the set of critical points of f_a is discrete, k_z is the constant function $2\pi i$. This finishes the proof. □

3. The main tool. In this section we fix $a \in \mathcal{H}$ and we denote f_a, F_a and δ_a respectively by f, F and δ. Let $\mathrm{CB}\,(J_F, \mathbf{C})$ be the Banach space of all bounded continuous functions $g : J_F \to \mathbf{C}$ with the norm $||\cdot||_\infty$. For $\alpha \in (0,1]$ and for $g \in \mathrm{CB}\,(J_F, \mathbf{C})$ we denote by $v_\alpha(g)$ the *α-variation* of the function g which is

$$\inf\{L \geq 0 : \forall_{x,y \in J_F}\ |x-y| \leq \delta \implies |g(x) - g(y)| \leq L|x-y|^\alpha\}.$$

Let

$$||g||_\alpha = v_\alpha(g) + ||g||_\infty,$$

and define

$$H_\alpha = H_\alpha(J_F) = \{g \in \mathrm{CB}\,(J_F, \mathbf{C}) : ||g||_\alpha < \infty\}.$$

Then the set H_α with the norm $||\cdot||_\alpha$ is a Banach space and H_α is a dense subset of $\mathrm{CB}\,(J_F, \mathbf{C})$.

3.1. Observe that it follows immediately from [**4**, Proposition 2.2] $L > 0$ and $0 < \beta < 1$ exist such that, for every $n \geq 0$, every $v \in J_F$ and every $z \in B(F^n(v), \delta)$

$$|(F_v^{-n})'(z)| \leq L\beta^n, \tag{3.1}$$

where F_v^{-n} is a branch of F^{-n} such that $F_v^{-n}(F^n(v)) = v$.

3.2. We say a continuous function $\phi : J_F \to \mathbf{C}$ is *dynamically Hölder with an exponent* $\alpha > 0$ if there exists $c_\phi > 0$ such that

$$|\phi_n(F_v^{-n}(y)) - \phi_n(F_v^{-n}(x))| \le c_\phi |\phi_n(F_v^{-n}(x))||y-x|^\alpha$$

for all $n \ge 1$, all $x, y \in J_F$ with $|x-y| \le \delta$ and all $v \in F^{-n}(x)$, where

$$\phi_n(z) = \phi(z)\phi(F(z)) \cdots \phi(F^{n-1}(z)). \tag{3.2}$$

We say that a continuous function $\phi : J_F \to \mathbf{C}$ is *summable* if

$$\sup_{z \in J_F} \left\{ \sum_{v \in F^{-1}(z)} ||\phi \circ F_v^{-1}||_\infty \right\} < \infty.$$

Next define

$$H_\alpha^s = \{\phi : J_F \longrightarrow \mathbf{C} : \phi \text{ is a Hölder continuous summable function}\}.$$

If the function $\phi \in H_\alpha^s$, then the formula

$$\mathcal{L}_\phi g(z) = \sum_{x \in F^{-1}(z)} \phi(x) g(x)$$

defines a bounded operator $\mathcal{L}_\phi : \mathrm{CB}\,(J_F, \mathbf{C}) \to \mathrm{CB}\,(J_F, \mathbf{C})$ called *the Perron-Frobenius operator associated with the function* ϕ.

Observe now that (see [**25**, Lemma 4.1]) if we let $\phi : J_F \to \mathbf{C}$ be a summable dynamically Hölder function with an exponent $\alpha > 0$, then

$$Ł_\phi(H_\alpha) \subset H_\alpha$$

and then, if $\phi(J_F) \subset [0, \infty)$ and $\sup_{n \ge 1}\{||\mathcal{L}_\phi^n(\mathbf{1})||_\infty\} < \infty$, there exists a constant $c_1 > 0$ such that

$$||\mathcal{L}_\phi^n g||_\alpha \le \frac{1}{2}||g||_\alpha + c_1||g||_\infty \tag{3.3}$$

for all n large enough and for every $g \in H_\alpha$.

3.3. We say (see also [**25**]) that a summable dynamically Hölder potential $\phi : J_F \to (0, \infty)$ *satisfies the* Q*-condition* if

$$Q_\phi = \sup_{n \ge 1}\{||\mathcal{L}_\phi^n(\mathbf{1})||_\infty\} < \infty,$$

and we say that ϕ is *rapidly decreasing* if

$$\lim_{|\mathrm{Re}\, z|\to\infty} \mathcal{L}(\mathbf{1})(z) = 0.$$

As in [**25**, Lemma 4.2] we also observe that if we let $\phi : J_F \to (0,\infty)$ be a rapidly decreasing summable dynamically Hölder potential satisfying the $Q-$condition and, if B is a bounded subset of $(H_\alpha, ||\cdot||_\alpha)$, then $\mathcal{L}_\phi(B)$ is a pre-compact subset of the space $(\mathrm{CB}\,(J_F, \mathbf{C}), ||\cdot||_\infty)$.

Combining all the above results we observe that the assumptions of Ionescu-Tulcea and Marinescu's ergodic theorem [**15**, Theorem 1.5] are satisfied for Banach spaces H_α and for the bounded operator $\mathcal{L}_\phi : \mathrm{CB} \to \mathrm{CB}$. All this shows the following theorem, see [**25**, Theorem 4.3].

3.4. Theorem. *If $\phi : J_F \to (0,\infty)$ is a rapidly decreasing summable dynamically Hölder potential satisfying the Q-condition, then there exist $p \in \mathbf{N}$ numbers $\gamma_1, \ldots, \gamma_p \in S_1 = \{z \in \mathbf{C} : |z| = 1\}$, finitely dimensional bounded operators $Q_1, \ldots, Q_p : H_\alpha \to H_\alpha$ and an operator $S : H_\alpha \to H_\alpha$ such that, for all $n \geq 1$,*

$$\mathcal{L}_\phi^n = \sum_{i=1}^{p} \gamma_i^n Q_i + S^n,$$

$$Q_i^2 = Q_i, \quad Q_i \circ Q_j = 0, \quad (i = j), \quad Q_i \circ S = S \circ Q_i = 0$$

and

$$||S^n||_\alpha \leq C\xi^n$$

for some constant $C > 0$ and some constant $\xi \in (0,1)$. In particular all numbers $\gamma_1, \ldots, \gamma_p$ are isolated eigenvalues of the operator $\mathcal{L}_\phi : H_\alpha \to H_\alpha$, and this operator is quasicompact.

3.5. Lemma. (i) *The function $\phi_1(z) = -t \log |F_a'(z)|$ is 1-Hölder (Lipshitz).*

(ii) *The function $\phi_2(z) = |F_a'(z)|^{-t}$ is 1-Hölder.*

Proof. Observe that $\phi_2(z) = e^{\phi_1(z)}$. To prove (i) we use Koebe's distortion theorem. Let $|z - w| < \delta$ and $|z - w| = \eta 2\delta$. Then

$$\frac{(1-\eta)^3}{(1+\eta)^3} \leq \frac{|F_a'(z)|}{|F_a'(w)|} \leq \frac{(1+\eta)^3}{(1-\eta)^3}.$$

Therefore

$$\begin{aligned}\Big| - t\log|F_a'(z)| + t\log|F_a'(w)|\Big| &= |t|\cdot \Big|\log \frac{|F_a'(z)|}{|F_a'(w)|}\Big| \\ &\leq |t|(3\log(1+\eta) - 3\log(1-\eta)) \\ &\leq |t|\,\frac{9}{2\delta}\,|z-w|\end{aligned}$$

since, for $\eta \in (0,1/2)$, $\log(1+\eta) \leq \eta$ and $\log(1-\eta) < 2\eta$. This finishes the proof of part (i).

To get (ii), first observe that if $\operatorname{Re} t \geq 0$

$$\Big||F_a'(z)|^{-t}\Big| \leq \frac{1}{\Delta_a^{\operatorname{Re} t}},$$

where

$$\Delta_a = \min(1/2, \inf\{|f_a'(z)| : \operatorname{dist}(z, \operatorname{Crit}(f_a)) > \delta\}). \tag{3.4}$$

Second, note that there exists $M_t > 0$ such that, if $|x| \leq 2|t|\log(1/\Delta_a)$, then

$$|e^x - 1| \leq M_t|x|.$$

Therefore

$$\begin{aligned}\Big||F_a'(z)|^{-t} - |F_a'(w)|^{-t}\Big| &= |e^{-t\log|F_a'(z)|} - e^{-t\log|F_a'(w)|}| \\ &= \Big|e^{-t\log|F_a'(w)|}\Big|\cdot\Big|e^{-t\log|F_a'(z)|+t\log|F_a'(w)|} - 1\Big| \\ &\leq \frac{1}{\Delta_a^{\operatorname{Re} t}}\, M_t\, \frac{9}{2\delta}\,|z-w|. \qquad \square\end{aligned}$$

3.6. Let $\phi_t := \phi_2$ be the function from the previous lemma. Since

$$\mathcal{L}_{\phi_t} g(z) = \sum_{x\in F^{-1}(z)} |F'(z)|^{-t} g(z),$$

the operator $\widehat{\mathcal{L}}_t$ from [**4**] can be defined by the formula

$$\widehat{\mathcal{L}}_t = \alpha_t^{-1}\mathcal{L}_{\phi_t},$$

where $\alpha_t = e^{P(t)}$ and $P(t)$ is the topological pressure from Remark 1.4. Moreover, in [**4**] (see also [**25**]) it was proved that, for $t > 1$, there exists a unique (α, t)-conformal measure and $\alpha = \alpha_t$. Moreover, the measure is a fixed point of the operator $\widehat{\mathcal{L}}_t^*$ dual to $\widehat{\mathcal{L}}_t$. Next we shall prove more properties of the operator $\widehat{\mathcal{L}}_t = \alpha_t^{-1}\mathcal{L}_{\phi_t}$.

3.7. Lemma. *If* $\operatorname{Re} t > 1$*, then* $\phi_3(z) := e^{-P(t)}\phi_t(z) = e^{-P(t)}|(F_a)'(z)|^{-t}$ *is a rapidly decreasing summable dynamically Hölder function satisfying the* Q*-condition. Therefore the operator* $\mathcal{L}_{\phi_3} = \widehat{\mathcal{L}}_t$ *satisfies Theorem* 3.4.

Proof. Since $|(F_a^n)'(z)| = |F_a'(z)| \cdot |F_a^\omega(F_a(z))| \cdots |F_a'(F_a^{n-1}(z))|$, then $\phi_n(z)$ for the potential $|(F_a)'(z)|^{-t}$ is equal to $|(F_a^n)'(z)|^{-t}$. Therefore,

$$\phi_n(F_v^{-n}(y)) = |(F_a^{-n})'(F_v^{-n}(y))|^{-t} = |(F_v^{-n})'(y)|^t.$$

Using the same argument as in the proof of Lemma 3.5, we have that

$$|-t\log|(F_v^{-n})'(x)| + t\log|(F_v^{-n})'(y)|| \le |t|\,\frac{9}{2\delta}\,|y-x|. \tag{3.5}$$

Then

$$\begin{aligned} &||(F_v^{-n})'(x)|^t - |(F_v^{-n})'(y)|^t| \\ &\qquad = |e^{t\log|(F_v^{-n})'(y)|}| \cdot |e^{t\log|(F_v^{-n})'(x)| - t\log|(F_v^{-n})'(y)|} - 1| \\ &\qquad \le |(F_v^{-n})'(y)|^{\operatorname{Re} t} M_t' |x-y| \end{aligned}$$

for some constant M_t'. Observe that, from the estimation above and Proposition 3.1 from [**4**], it follows that $\phi_3(z)$ is a rapidly decreasing summable dynamically Hölder function satisfying the Q-condition. □

3.8. Corollary. *If* $t > 1$*, then* 1 *is an isolated simple eigenvalue of* $\widehat{\mathcal{L}}_t : H_\alpha \to H_\alpha$ *and the eigenspace of the eigenvalue* 1 *is generated by the nowhere vanishing function* $\psi_t \in H_\alpha$ *such that*

$$\int \psi_t \, dm_t = 1$$

and

$$\lim_{|\mathrm{Re}\, z|\to\infty} \psi_t(z) = 0.$$

Moreover, if $t > 1$, then the measure $\mu = \mu_t = \psi_t m_t$ is F-invariant, ergodic and equivalent to m_t. In particular, $\mu(J_F^r) = 1$.

Proof. We have first that

$$\|\widehat{\mathcal{L}}_t^n(\mathbf{1})\|_\alpha \le C_1$$

for some $C_1 > 0$ and every $n \ge 0$. Therefore

$$\left\| \frac{1}{n}\sum_{k=1}^{n} \widehat{\mathcal{L}}_t^k(\mathbf{1})\right\|_\alpha = \left\| \widehat{\mathcal{L}}_t\left(\frac{1}{n}\sum_{k=1}^{n-1} \widehat{\mathcal{L}}_t^k(\mathbf{1})\right)\right\|_\alpha \le C_1 \tag{3.6}$$

for every $n \ge 1$. Then it follows from subsection 3.3 that there exists a strictly increasing sequence of positive integers $\{n_j\}_{j\ge 1}$ such that the sequence

$$\left\{\frac{1}{n_j}\sum_{k=1}^{n_j} \widehat{\mathcal{L}}_t^k(\mathbf{1})\right\}_{j\ge 1}$$

converges in $\mathrm{CB}\,(J_F, \mathbf{C})$ to a function $\psi_t : J_F \to \mathbf{R}$. Then, since by (3.6) $\|\psi_t\|_\alpha \le C_1$, we get that $\psi_t \in H_\alpha$. Moreover, by [**4**, Proposition 7.1], m_t is a fixed point of the dual operator $\widehat{\mathcal{L}}_t^*$. Therefore, for every $j \ge 0$,

$$\int \widehat{\mathcal{L}}_t^j(\mathbf{1})\, dm_t = 1,$$

and consequently

$$\int \frac{1}{n}\sum_{j=0}^{n-1} \widehat{\mathcal{L}}_t^j(\mathbf{1})\, dm_t = 1.$$

Applying Lebesgue's dominated convergence theorem together with the fact that the function ϕ from the previous lemma has the Q-property, we obtain the equality

$$\int \psi_t\, dm_t = 1.$$

Moreover, it follows then from [**4**, Proposition 7.1] that $\psi_t > 0$ throughout J_F. Since $\psi_t = \widehat{\mathcal{L}}_t\psi_t$ and since $\mathcal{L}_t\psi_t(z) \le \mathcal{L}_t\mathbf{1}(z)_\infty\|\psi_t\|_\infty$ and

$$\lim_{|\mathrm{Re}\, z|\to\infty} \mathcal{L}_t\mathbf{1}(z) = 0,$$

it follows that

$$\lim_{|\mathrm{Re}\, z|\to\infty} \psi_t(z) = 0.$$

The fact that 1 is an isolated eigenvalue of $\widehat{\mathcal{L}}_t$ follows from Theorem 3.3. The last statement of the corollary follows from above and [**4**, Proposition 4.1].

Therefore, it remains to prove that the isolated eigenvalue 1 is simple, i.e., the eigenspace of 1 is one-dimensional. Let $g = g_1 + ig_2 \in H_\alpha$, where $g_1, g_2 \in H_\alpha$ are real-valued be such that

$$\widehat{\mathcal{L}}_t(g) = g.$$

Since

$$g_1 + ig_2 = g = \widehat{\mathcal{L}}_t(g_1 + ig_2) = \widehat{\mathcal{L}}_t(g_1) + i\widehat{\mathcal{L}}_t(g_2),$$

$$\widehat{\mathcal{L}}_t(g_l) = g_l$$

for $l = 1, 2$. We shall prove that both g_l are equal to $\lambda_l\psi_t$ for some $\lambda_l \in \mathbf{R}$. So let us assume that $g_l \ne \lambda\psi_t$ for all $\lambda \in \mathbf{R}$. Since

$$0 \le \mathbf{1} - \frac{g_l}{||g_l||_\infty},$$

$$0 \le \frac{1}{n_j}\sum_{k=1}^{n_j}\widehat{\mathcal{L}}_t^k\left(\mathbf{1} - \frac{g_l}{||g_l||_\infty}\right) = \frac{1}{n_j}\sum_{k=1}^{n_j}\widehat{\mathcal{L}}_t^k(\mathbf{1}) - \frac{g_l}{||g_l||_\infty}.$$

Therefore, the function

$$h = \frac{\psi_t - (g/||g||_\infty)}{\int \psi_t - (g/||g||_\infty)\, dm_t}$$

is a well-defined nonzero nonnegative function which is a fixed point of $\widehat{\mathcal{L}}_t$ and

$$\int h\, dm_t = 1.$$

But this gives us that $h = \psi_t$ since we know that m_t is ergodic. Then the required contradiction follows. $\square$

3.9. Next we observe that, following [**25**], if we let G be a domain in $\mathbf{C}$ and $\{\Phi_b : J_{F_a} \to \mathbf{C}\}_{b \in G}$ be a family of continuous summable functions such that, for every $z \in J_{F_a}$ the function $G \ni b \mapsto \Phi_b(z) \in \mathbf{C}$ is holomorphic and such that the map $G \ni b \mapsto \mathcal{L}_{\Phi_b} \in L(H_\alpha)$ is

$$b \longmapsto \mathcal{L}_{\Phi_b} \in L(H_\alpha) \tag{3.7}$$

is holomorphic on G.

This important remark on the analyticity of the Perron-Frobenius operators allows us to formulate the next theorem. For a proof, the reader can consult [**25**]. We only mention that the extension from the one-dimensional case to the multi-dimensional case (what we really need, since our parameter space is the $n+1$-dimensional complex space $\mathbf{C}^{n+1}$) is possible due to the well-known Hartogs' theorem.

3.10. Theorem (The main tool). *Suppose that G is an open connected subset of $\mathbf{C}^n$, $n \geq 1$, and $\Phi_b : J_{F_a} \to \mathbf{C}$, $b \in G$, is a family of mappings such that*

(i) *for every $b \in G$, $\Phi_b \in H_\alpha^s$,*

(ii) *for every $b \in G$, Φ_b is dynamically Hölder,*

(iii) *$G \ni b \mapsto \Phi_b \in H_\alpha$ is continuous,*

(iv) *family $\{c_{\Phi_b}\}_{b \in G}$ is bounded,*

(v) *the function $b \mapsto \Phi_b(z) \in \mathbf{C}$, $b \in G$ is holomorphic for every $z \in J_{F_a}$,*

(vi) *for every $d \in G$ there exists $r > 0$ and there exists $c \in G$ such that*

$$\sup\left\{\left|\frac{\Phi_b}{\Phi_c}(z)\right| : b \in \overline{B(d,r)}, z \in \mathbf{C}\right\} < \infty.$$

Then the function $b \mapsto \mathcal{L}_{\Phi_b} \in L(H_\alpha)$, $b \in G$, is holomorphic.

4. The main result. In this section we prove the main result of this paper. For a parameter $b \in \mathbf{C}^{n+1}$ and a map $f = f_b \in \mathcal{H}$, we show that the function $b \mapsto \mathrm{HD}\,(J^r_{f_b})$ is real-analytic. This ultimate goal is established in Theorem 4.17. But first we need to prove that, for $t > 1$, the function $a \mapsto P_a(t)$ is continuous on $\mathcal{H}$. To obtain this we need the following lemma. We separate it because we shall use it one more time later.

4.1. Lemma. *For every $a \in \mathcal{H}$ and for every $\epsilon > 1$, there exists $r > 0$ such that, for $b \in B(a, r)$ and for $z \in J_{f_a}$,*

$$\left| \frac{f_b'(h_b(z))}{f_a'(z)} - 1 \right| < \epsilon.$$

Proof. First note that

(4.1)

$$\begin{aligned} \left| \frac{f_b'(h_b(z))}{f_a'(z)} - 1 \right| &= \left| \frac{f_b'(h_b(z)) - f_a'(z)}{f_a'(z)} \right| \\ &\le \left| \frac{f_b'(h_b(z)) - f_a'(h_b(z))}{f_a'(z)} \right| + \left| \frac{f_a'(h_b(z)) - f_a'(z)}{f_a'(z)} \right|. \end{aligned}$$

Next we split the proof into two cases.

Case 1. Assume that $|\mathrm{Re}\, z| \le M_3 + 1$, where M_3 is the constant from Lemmas 2.2 and 2.3. Observe that there exists $M_5 < \infty$ such that

$$\sup \left\{ \sum_{j=0}^{n} (j-k)^2 e^{2(j-k)\mathrm{Re}\, z} : |\mathrm{Re}\, z| \le M_3 + 2 \right\} \le M_5^2.$$

If b is close to a, then $|\mathrm{Re}\, h_b(z)| \le M_3 + 2$. Therefore, by (1.2), we get

$$\begin{aligned} \left| \frac{f_b'(h_b(z)) - f_a'(h_b(z))}{f_a'(z)} \right| &\le \frac{1}{\delta_a} \sup \left\{ \left\| \frac{\partial f_b'}{\partial b}\,(z) \right\| : |a - b| < r \right\} |b - a| \\ &\le \frac{M_5}{\delta_a}\, |b - a|. \end{aligned}$$

Observe also that there exists $M_6 < \infty$ such that

$$\sup\{ |f_a''(z)| : |\mathrm{Re}\, z| \le M_3 + 2 \} \le M_6.$$

Then

$$\left|\frac{f_a'(h_b(z)) - f_a'(z)}{f_a'(z)}\right| \le \frac{M_6}{\delta_a}\,|h_b(z) - z| \le \frac{M_6}{\delta_a}\left|\frac{\partial h_b}{\partial b}\,(z)\right||b-a|.$$

Since, by Proposition 2.4, $|(\partial h_b/\partial b)(z)|$ is bounded in a small neighborhood of a, we have that (4.1) can be as small as we want.

Case 2. Assume that $|\operatorname{Re} z| > M_3 + 1$. Then, similarly to Case 1 but using Lemma 2.3 instead of estimations by M_5, M_6 and δ_a, we get

$$\left|\frac{f_b'(h_b(z)) - f_a'(h_b(z))}{f_a'(z)}\right| \le \frac{M_2}{M_1}\,|b-a|,$$

and

$$\left|\frac{f_a'(h_b(z)) - f_a'(z)}{f_a'(z)}\right| \le \frac{M_2}{M_1}\left|\frac{\partial h_b}{\partial b}\,(z)\right||b-a|.$$

Therefore, again, if b is close to a, then (4.1) can be as small as we want for b sufficiently close to a. □

4.2. Lemma. *For all $t > 1$ the function $a \mapsto P_a(t)$, $a \in \mathcal{H}$, is continuous.*

Proof. Fix $a \in \mathcal{H}$ and, for this a, $r > 0$ from Lemma 4.1. Next, take any $z \in J_{F_a}$ and $n \ge 1$ and $x \in F_a^{-n}(z)$. First note that, by Proposition 2.4 (iii), h_b, which conjugates f_b and f_a, conjugates also F_b and F_a. Moreover, we have $h_b(F_a^{-n}(z)) = F_b^{-n}(z)$ and for every $i \in \{0, 1, \ldots, n\}$ and every $x \in F_a^{-n}(z)$ we have $h_b \circ f_a^i(x) = f_b^i \circ h_b(x)$. Now we can write

$$\begin{aligned}\frac{|(F_b^n)'(h_b(x))|}{|(F_a^n)'(x)|} &= \frac{|(f_b^n)'(h_b(x))|}{|(f_a^n)'(x)|} = \prod_{i=0}^{n-1}\frac{|f_b'(f_b^i(h_b(x)))|}{|f_a'(f_a^i(x))|}\\ &= \prod_{i=0}^{n-1}\frac{|f_b'(h_b(f_a^i(x)))|}{|f_a'(f_a^i(x))|}.\end{aligned}$$

Hence, by Lemma 4.1, for every $\gamma > 1$, there exists $0 < r_1 < r$ such that

$$\frac{1}{\gamma^n} < \frac{|(F_b^n)'(h_b(x))|}{|(F_a^n)'(x)|} < \gamma^n.$$

Since $h_b : F_a^{-n}(z) \to F_b^{-n}(h_b(z))$ is a bijection, we obtain

$$\frac{1}{\gamma^{tn}} < \frac{\sum_{x\in F_b^{-n}(h_b(z))} |(F_b^n)'(x)|^{-t}}{\sum_{x\in F_a^{-n}(z)} |(F_a^n)'(x)|^{-t}} < \gamma^{tn}.$$

Hence, taking the logarithm and dividing the last inequality by n, we get

$$-\,t\log\gamma < P_b(t) - P_a(t) < t\log\gamma$$

for all $b \in B(a, r_1)$ and we are done. □

4.3. Let $a \in \mathcal{H}$. Let $r_1 > 0$ be such a real number that, for every $b \in B(a, r_1)$, there exists a quasiconformal map h_b conjugating f_a and f_b, and let

$$\alpha = \alpha(r_1) = \inf\left\{\frac{1}{Q(b)} : b \in B(a, r_1)\right\} > 0. \tag{4.2}$$

The existence of such an r_1 follows from Lemma 2.1 and Proposition 2.4. Therefore, for $b \in B(a, r_1)$ and $t > 1$, we can define a function $\phi_{(\cdot)}(b, t) : J_{F_a} \to \mathbf{R}$ by the formula

$$\phi_z(b, t) = |F_b'(h_b(z))|^{-t}.$$

4.4. Lemma. *If* $\operatorname{Re} t > 1$*, then the functions*

$$\phi_4(z) = -\,t\log|F_b'(h_b(z))| \quad \textit{and} \quad \phi_z(b, t) = |F_b'(h_b(z))|^{-t}$$

are α*-Hölder, where* α *is the constant from* (4.2).

Proof. Since, by Proposition 2.4 (ii), we know that h_b is $(K(Q), (1/Q))$-Hölder, the lemma follows from Lemma 3.5. □

4.5. Then by $\mathcal{L}^0_{b,t}$ we denote the Perron-Frobenius operator associated with $\phi_z(b, t) = |F_b'(h_b(z))|^{-t}$, i.e.,

$$\mathcal{L}^0_{b,t} g(z) = \sum_{x\in F_a^{-1}(z)} \phi_z(b, t) g(x)$$

for $g \in \mathrm{CB}\,(J_F, \mathbf{C})$.

4.6. Embedding of $\mathbf{C}^{n+1}$ into $\mathbf{C}^{2n+3}$. We would like to apply Theorem 3.10, but the direct extension of ϕ_z onto an open subset of $\mathbf{C}^{n+1}\times\mathbf{C}$ is not a holomorphic function of b and t. For this reason we shall embed $\mathbf{C}^{n+1}$ into $\mathbf{C}^{2n+3}$ in some special way described below and then extend ϕ_z.

Let $b=(b_0,b_1,\dots,b_n)\in\mathbf{C}^{n+1}$. Write $b_j=b_j^1+ib_j^2$ for $j=0,1,\dots,n$ where i is the imaginary unit. Then $e_1:\mathbf{C}^{n+1}\hookrightarrow\mathbf{C}^{2n+2}$ is the embedding defined by the following formula

$$e_1(b)=(b_0^1,b_0^2,b_1^1,b_1^2,\dots,b_n^1,b_n^2),$$

and $e:\mathbf{C}^{n+1}\times\mathbf{R}\hookrightarrow\mathbf{C}^{2n+3}$ is the embedding defined as $e(b,t)=(e_1(b),t)$.

4.7. Sketch of the extension. We shall extend the function

$$\phi_z\circ e^{-1}:e(B(a,r_1)\times(1,\infty))\to\mathbf{R},$$

where e^{-1} is the left inverse of e and $r_1>0$ is the constant from subsection 4.3. We cannot do this directly. But observe that

$$\phi_z(b,t)=|F_b'(h_b(z))|^{-t}=\exp\left\{-t\left(\log|\psi_z(b)|+\log|F_a'(z)|\right)\right\},$$

where

$$\psi_z(b)=\frac{F_b'(h_b(z))}{F_a'(z)}.$$

Then, if we have defined Log (a branch of $\exp^{-1}$) on some neighborhood of $\psi_z(a)$, then

$$\operatorname{Re}\operatorname{Log}\psi_z(b)=\log|\psi_z(b)|.$$

Therefore, if we can extend holomorphically $\operatorname{Re}\operatorname{Log}\psi_z\circ e_1^{-1}$ (where e_1^{-1} is the left inverse of e_1), we are done. However, since by Lemma 4.1 there exists $r_2>0$ such that, for $b\in B(a,r_2)$ and for $z\in J_{F_a}$,

$$|\psi_z(b)-1|<\frac{1}{2}, \tag{4.3}$$

the holomorphic function

$$\operatorname{Log}\psi_z:B(a,r_2)\longrightarrow\mathbf{C}$$

is well defined, where Log is that branch of $\exp^{-1}$ which satisfies the condition $\operatorname{Log} 1 = 0$.

Before we start with the extension of $\operatorname{Re}\operatorname{Log}\psi_z \circ e_1^{-1}$, note that by (4.3) there exists $M_7 < \infty$ such that

$$|\operatorname{Log}\psi_z(b)| \leq M_7 \tag{4.4}$$

for all $b \in B(a, r_2)$ and $z \in J_{F_a}$.

4.8. Extension of $\operatorname{Re}\operatorname{Log}\psi_z \circ e_1^{-1}$**.** The function $\operatorname{Log}\psi_z : B_{\mathbf{C}^{n+1}}(a, r_2) \to \mathbf{C}$ is analytic. Then

$$\operatorname{Log}\psi_z(b) = \sum_{(i_0,\dots,i_n)\in\mathbf{N}^{n+1}} c_{i_0,\dots,i_n}(z)(a_0 - b_0)^{i_0}\dots,(a_n - b_n)^{i_n}. \tag{4.5}$$

From Cauchy's estimates and (4.4), it follows that

$$|c_{i_0,\dots,i_n}(z)| \leq \frac{M_7}{r_2^{i_0+\dots+i_n}}. \tag{4.6}$$

Recall that $b_j = b_j^1 + ib_j^2$, $a_j = a_j^1 + ia_j^2$ where $j = 0,\dots,n$ and i is the imaginary unit. Note that

$$e_1(b) = (b_0^1, b_0^2, b_1^1, b_1^2, \dots, b_n^1, b_n^2).$$

Since (4.5) can be written as

$$\sum_{(i_0,\dots,i_n)\in\mathbf{N}^{n+1}} c_{i_0,\dots,i_n}(z) \prod_{j=0}^{n} ((a_j^1 - b_j^1) + i(a_j^2 - b_j^2))^{i_j},$$

we have

$$\begin{aligned} &\operatorname{Re}\operatorname{Log}\psi_z \circ e_1^{-1}(b') \\ &\qquad = \sum_{(k_0,\dots,k_{2n+1})\in\mathbf{N}^{2n+1}} c'_{k_0,\dots,k_{2n+1}}(z) \prod_{l=0}^{2n+1} (a'_l - b'_l)^{k_l}, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} c'_{k_0,\dots,k_{2n+1}}(z) = &\operatorname{Re}\left(c_{k_0+k_1,\dots,k_{2n}+k_{2n+1}}(z) i^{k_1+k_3+\dots+k_{2n+1}}\right) \\ &\cdot \binom{k_0+k_1}{k_0}\binom{k_2+k_3}{k_2}\dots,\binom{k_{2n}+k_{2n+1}}{k_{2n}}, \end{aligned} \tag{4.8}$$

and $a' = (a'_0, \dots, a'_{2n+1}) = e_1(a)$, $b' = (b'_0, \dots, b'_{2n+1}) = e_1(b)$. Moreover,

$$\begin{aligned} |c'_{k_0,\dots,k_{2n+1}}(z)| &\le |c_{k_0+k_1,\dots,k_{2n}+k_{2n+1}}(z)| 2^{k_0+k_1+\cdots+k_{2n+1}} \\ &\le M_7 \left(\frac{2}{r_2}\right)^{k_0+k_1+\cdots+k_{2n+1}}. \end{aligned} \tag{4.9}$$

Next, let a' be as before, i.e., $a' = (a'_0, \dots, a'_{2n+1}) = e_1(a)$, and let

$$b' \in \mathbf{D}_{\mathbf{C}^{2n+2}}(a', r_3) = \Big\{ b' = (b'_0, \dots, b'_{2n+1}) \in \mathbf{C}^{2n+2} : \max_{l=0,\dots,2n+1} |a'_l - b'_l| < r_3 \Big\},$$

where $r_3 = r_2/4$. Then

$$\left| c'_{k_0,\dots,k_{2n+1}}(z) \prod_{l=0}^{2n+1} (a'_l - b'_l)^{k_l} \right| \le M_7 \left(\frac{1}{2}\right)^{k_0+k_1+\cdots+k_{2n+1}}. \tag{4.10}$$

Hence $\operatorname{Re} \operatorname{Log} \psi_z \circ e_1^{-1}$ can be holomorphically extended to the functions $\Psi_z : \mathbf{D}_{\mathbf{C}^{2n+2}}(e_1(a), r_3) \to \mathbf{C}$ defined by the

$$\Psi_z(b') = \sum_{(k_0,\dots,k_{2n+1}) \in \mathbf{N}^{2n+1}} c'_{k_0,\dots,k_{2n+1}}(z) \prod_{l=0}^{2n+1} (a'_l - b'_l)^{k_l}. \tag{4.11}$$

Note that

$$|\Psi_z(b')| \le M_7 \sum_{(k_0,\dots,k_{2n+1}) \in \mathbf{N}^{2n+2}} \left(\frac{1}{2}\right)^{k_0+k_1+\cdots+k_{2n+2}} \le M_7 2^{2n+2} \tag{4.12}$$

for $b' \in \mathbf{D}_{\mathbf{C}^{2n+2}}(e_1(a), r_3)$.

4.9. Conclusion of the process. Finally we have the extension of $\phi_z \circ e^{-1} : e(B(a, r_1) \times (1, \infty)) \to \mathbf{R}$. This is the function

$$\Phi_z : \mathbf{D}_{\mathbf{C}^{2n+2}}(e_1(a), r_3) \times B_{\mathbf{C}}(t_0, \rho) \longrightarrow \mathbf{C}$$

(where $\rho = t_0 - 1$) defined by the formula

$$\Phi_z(b', t) = \exp\{-t(\Psi_z(b') + \log|F_a'(z)|)\}. \tag{4.13}$$

4.10. Proposition. *Fix $a \in \mathcal{H}$ and $t_0 > 1$. Then there exist r_4 and ϱ such that, for*

$$(b', t) \in G_3 = \mathbf{D}_{\mathbf{C}^{2n+2}}(e_1(a), r_4) \times B(t_0, \varrho),$$

the Perron-Frobenius operator $\mathcal{L}_{\Phi_{(b',t)}}$ for the function

$$\Phi_{(b',t)}(z) = \Phi_z(b', t),$$

is well defined. Moreover, let $\mathcal{L}$ be the function $G_3 \to L(H_\alpha)$, where $\alpha = \alpha(r_4)$ comes from (4.2), defined by the formula

$$\mathcal{L}((b', t)) = \mathcal{L}_{\Phi_{(b',t)}}.$$

Then $\mathcal{L}$ is holomorphic.

Proof. To prove the proposition it is enough to check the conditions from Theorem 3.10.

Condition (v). This is satisfied from the construction (4.6)–(4.8).

Condition (i). First, we prove that the function $\Phi_{(b',t)}(z)$ is summable. From (4.12) it follows that

$$|\Phi_{(b',t)}(z)| = \exp\{\mathrm{Re}\,(-t\Psi_z(b'))\}|F_a'(z)|^{-\mathrm{Re}\,t} \le e^{M_7 2^{2n+2}|t|}|F_a'(z)|^{-\mathrm{Re}\,t}.$$

Since in Lemma 3.7 we proved that the function $\phi_2(z) = |F_a'(z)|^{-\mathrm{Re}\,t}$ is summable, it follows that $\Phi_{(b',t)}$ is summable.

Next, we shall prove that the function $\Phi_{(b',t)}$ is Hölder. First we shall show that for the function $z \mapsto \Psi_{(z)}(b')$. To do this we start with $b \in B(a, r_2)$. Using the proof of Lemma 3.5 (i) and Proposition (2.4) (ii), we can get that

$$|\Psi_z(e_1(b)) - \Psi_w(e_1(b))| = |\mathrm{Re}\,\mathrm{Log}\,\psi_z(b) - \mathrm{Re}\,\mathrm{Log}\,\psi_w(b)| \le M_{100}|z-w|^\alpha$$

for some constant M_{100}. Next we prove that the extension is Hölder by showing that the coefficients of the extension are Hölder. To get this we have to take a look at the function $\operatorname{Log}\psi_{(\cdot)}$. From estimation of the arg in Koebe's theorem, we can find a constant M_{101} such that

$$\begin{aligned}|\arg\psi_z(b)-\arg\psi_w(b)| &= \left|\arg\frac{\psi_z(b)}{\psi_w(b)}\right| \le \left|\arg\frac{F_b'(h_b(z))}{F_b'(h_b(w))}\right| + \left|\arg\frac{F_a'(z)}{F_a'(w)}\right| \\ &\le \frac{6}{2\delta}\,|z-w| + \frac{6}{2\delta}\,|h_b(z)-h_b(w)| \\ &\le M_{101}|z-w|^\alpha.\end{aligned}$$

Therefore,

$$\begin{aligned}&|\operatorname{Log}\psi_z(b)-\operatorname{Log}\psi_w(b)| \\ &\qquad \le |\operatorname{Re}\operatorname{Log}\psi_z(b)-\operatorname{Re}\operatorname{Log}\psi_w(b)| + |\arg\psi_z(b)-\arg\psi_w(b)| \\ &\qquad \le (M_{100}+M_{101})|z-w|^\alpha.\end{aligned}$$

Then, by (4.5) and Cauchy's estimation,

$$|c_{i_0,\dots,i_n}(z)-c_{i_0,\dots,i_n}(w)| \le \frac{M_{100}+M_{101}}{r_2^{i_0+\cdots+i_n}}|z-w|^\alpha.$$

Hence, by (4.8),

$$\begin{aligned}(4.14)\quad &|c'_{k_0,\dots,k_{2n+1}}(z)-c'_{k_0,\dots,k_{2n+1}}(w)| \\ &\qquad \le (M_{100}+M_{101})\left(\frac{2}{r_2}\right)^{k_0+\cdots+k_{2n+1}}|z-w|^\alpha.\end{aligned}$$

Then, for $b'\in\mathbf{D}_{\mathbf{C}^{2n+2}}(a',r_3)$, $r_3=r_2/4$,

$$|\Psi_z(b')-\Psi_w(b')| \le (M_{100}+M_{101})2^{2n+2}|z-w|^\alpha.$$

Next, as in the proof of Lemma 3.5 (ii), observe that M_9 exists such that, if

$$|x| \le \left((M_{23}+M_{24})2^{2n+2}+2\log\frac{1}{\Delta_a}\right)|t|,$$

then

$$|e^x-1| \le M_9|x|.$$

Therefore

$$
\begin{aligned}
(4.15)\quad & |\Phi_z(b',t)-\Phi_w(b',t)| \\
&\le |e^{-t(\Psi_z(b')+\log|F_a'(z)|)}| \\
&\cdot M_9(t)|t|\left((M_{23}+M_{24})2^{2n+2}|z-w|^\alpha+\frac{9}{2\delta_a}\,|z-w|\right) \\
&\le M_{11}(t)\cdot|z-w|^\alpha
\end{aligned}
$$

for come constant M_{11}.

Conditions (ii) *and* (iv). Now we check conditions (ii) and (iv), i.e., we show that $\Phi_{b,t}$ is dynamically Hölder, with exponent α, for (b,t) in some neighborhood $G\subset \mathbf{C}^{2n+3}$ of $(e_1(a),t_0)$ and with constants $c_{\phi_{(b,t)}}$ uniformly bounded (in G). Denote

$$\varphi(z) = -\,t(\Psi_z(b)+\log|F_a'(z)|)$$

and

$$\phi(z)=\Phi_z(b,t)=e^{\varphi(z)}$$

where $(b,t)\in G=\mathbf{D}_{\mathbf{C}^{2n+2}}(a,r_3)\times B(t_0,\varrho)$ for some t_0 with $\mathrm{Re}\,t_0>1$ and $\rho=t_0-1$. Then we need to show that, for every $n\ge 1$, every $x,y\in J_{F_a}$ such that $|x-y|\le\delta$, every $v\in F^{-n}(x)$ and

$$\phi_n(F_v^{-n}(z))=\prod_{k=0}^{n-1}\phi(F^k(F_v^{-n}(z)))=e^{\sum_{k=0}^{n-1}\varphi(F^k(F_v^{-n}(z)))}$$

we have

$$|\phi_n(F_v^{-n}(y))-\phi_n(F_v^{-n}(x))|\le c_\phi|\phi_n(F_v^{-n}(x))||y-x|^\alpha$$

where c_ϕ is uniformly bounded in G.

By (3.5) and (4.15) we get that

$$
\begin{aligned}
(4.16)\quad & |\varphi(z)-\varphi(w)| \\
&\le (|t_0|+\rho)(|\Psi_z(b)-\Psi_w(b)|+|\log|F_a'(z)|-\log|F_a'(w)||) \\
&\le (|t_0|+\rho)\Big((M_{23}+M_{24})2^{2n+2}|z-w|^\alpha+\frac{9}{2\delta}\,|z-w|\Big) \\
&\le M_{15}(t_0+\rho)|z-w|^\alpha
\end{aligned}
$$

for all $z, w \in J_{F_a}$ such that $|z-w| \leq \delta$ and some constant M_{15}. Then it follows from (3.1) that

$$\begin{aligned}\Big|\sum_{k=0}^{n-1} \varphi(F^k(F_v^{-n}(z))) - \sum_{k=0}^{n-1} \varphi(F^k(F_v^{-n}(w)))\Big| \\ \leq M_{102} \sum_{k=0}^{n-1} |F^k(F_v^{-n}(z)) - F^k(F_v^{-n}(z))|^\alpha \\ \leq M_{13}|z-w|^\alpha\end{aligned}$$

for some constants M_{102} and M_{13}. Therefore putting

$$M_{14} = \sup\left\{\left|\frac{e^z-1}{z}\right| : |z| \leq M_{13}\right\} < \infty,$$

we get

$$\text{(4.17)} \quad \left|\phi_n(F_v^{-n})(z) - \phi_n(F_v^{-n}(w))\right| \leq M_{13}M_{14}|\phi_n(F_v^{-n}(z))|\,|z-w|^\alpha.$$

Consequently, the functions ϕ are dynamically Hölder and we can see that the constants c_ϕ are uniformly bounded. So the assumptions (ii) and (iv) of the Main Tool are verified.

Condition (iii). Now let $G = \mathbf{D}_{\mathbf{C}^{2n+2}}(e_1(a), r_4) \times B(t_0, \rho) \subset \mathbf{C}^{2n+3}$ where $r_4 = r_2/16$. We have to show that

$$G \ni (b,t) \longmapsto \Phi_z(b,t) \in H_\alpha(J_{F_a})$$

is a continuous function. Since

$$\Phi_z(b,t) = e^{-t\Psi_z(b)}|F_a'(z)|^{-t},$$

we can do this by proving that the following two functions are continuous in each of the variables b and t,

$$(b,t) \longmapsto e^{-t\Psi_z(b)},$$

$$(b,t) \longmapsto |F_a'(z)|^{-t}$$

because both these functions are in H_α which is a Banach algebra.

The function $(b,t) \mapsto e^{-t\Psi_z(b)}$ is continuous in the variable t as a function $\mathbf{R} \mapsto H_\alpha$ where b is fixed since $||tg||_\alpha = |t|||g||_\alpha$, and then

$$||-t_1\Psi_{(\cdot)}(b) + t_2\Psi_{(\cdot)}(b)||_\alpha = |t_1 - t_2|\,||\Psi_{(\cdot)}(b)||_\alpha < M_{103}|t_1 - t_2|$$

for some constant M_{103}. For continuity with respect with the variable b, we recall that

$$||\Psi_{(\cdot)}(b) - \Psi_{(\cdot)}(c)||_\alpha = v_\alpha(\Psi_{(\cdot)}(b) - \Psi_{(\cdot)}(c)) + ||\Psi_{(\cdot)} \cdot (b) - \Psi_{(\cdot)}(c)||_\infty$$

where $b = (b_0, \dots, b_{2n+1})$, $c = (c_0, \dots, b_{2n+1})$. Then, by (4.14), we get

$$\begin{aligned}(4.18)\quad & |(\Psi_w(b) - \Psi_w(c)) - (\Psi_z(b) - \Psi_z(c))| \\ &= \Bigg| \sum_{k_0,\dots,k_{2n+1}} (c'_{k_0,\dots,k_{2n+1}}(w) - c'_{k_0,\dots,k_{2n+1}}(z)) \\ &\qquad \cdot \Bigg(\prod_{l=0}^{2n+1} (b_l - a'_l)^{k_l} - \prod_{l=0}^{2n+1} (c_l - a'_l)^{k_l} \Bigg) \Bigg| \\ &\leq \sum_{k_0,\dots,k_{2n+1}} (M_{100} + M_{101}) \left(\frac{2}{r_2}\right)^{k_0+\cdots+k_{2n+1}} |z-w|^\alpha \\ &\qquad \cdot \left| \prod_{l=0}^{2n+1} (b_l - a'_l)^{k_l} - \prod_{l=0}^{2n+1} (c_l - a'_l)^{k_l} \right|.\end{aligned}$$

where $a' = (a'_0, \dots, a'_{2n+1}) = e_1(a)$. But

$$\begin{aligned}&\left| \prod_{l=0}^{2n+1} (b_l - a'_l)^{k_l} - \prod_{l=0}^{2n+1} (c_l - a'_l)^{k_l} \right| \\ &= \Bigg| \sum_{l=0}^{2n+1} (b_0 - a'_0)^{k_0} \cdots (b_{l-1} - a'_{l-1})^{k_{l-1}} \big((b_l - a'_l)^{k_l} - (c_l - a'_l)^{k_l} \big) \\ &\qquad (c_{l+1} - a'_{l+1})^{k_{l+1}} \cdots (c_{2n+1} - a'_{2n+1})^{k_{2n+1}} \Bigg|.\end{aligned}$$

Then, since

$$(b_l - a'_l)^{k_l} - (c_l - a'_l)^{k_l} = \sum_{j=0}^{k_l - 1} (b_l - a'_l)^j (b_l - c_l)(c_l - a'_l)^{k_l - j - 2},$$

$$\Big| \prod_{l=0}^{2n+1} (b_l - a'_l)^{k_l} - \prod_{l=0}^{2n+1} (c_l - a'_l)^{k_l} \Big| \\ \leq (k_0 + \ldots + k_{2n+1}) \frac{16}{r_2} \left(\frac{r_2}{16}\right)^{k_0+\cdots+k_{2n+1}} |b-c|.$$

Therefore, by (4.18),

$$|(\Psi_w(b) - \Psi_w(c)) - (\Psi_z(b) - \Psi_z(c))| \\ \leq \frac{16}{r_2} (M_{100} + M_{101}) |b-c| \cdot |z-w|^\alpha \\ \cdot \sum_{k_0,\ldots,k_{2n+1}} \left(\frac{1}{8}\right)^{k_0+\cdots+k_{2n+1}} (k_0 + \cdots + k_{2n+1}),$$

So, this shows that

$$v_\alpha(\mathrm{Re}\,\mathrm{Log}\,\psi_z(b) - \mathrm{Re}\,\mathrm{Log}\,\psi_z(c)) \leq M_{15}|b-c|$$

for some constant $M_{15} > 0$. Similarly,

$$||\Psi_z(b) - \Psi_z(c)||_\infty \leq M_{16}|a-b|$$

for some $M_{16} > 0$. Therefore,

$$||\Psi_z(b) - \Psi_z(c)||_\alpha \leq (M_{14} + M_{15})|a-b|,$$

and the continuity is proven because using the same arguments which led to (4.17), we can get

$$||e^{-t\Psi_{(\cdot)}(b)} - e^{-t\Psi_{(\cdot)}(c)}||_\alpha \leq ||e^{-t\Psi_{(\cdot)}(b)+t\Psi_{(\cdot)}(c)} - 1||_\alpha ||e^{-t\Psi_{(\cdot)}(c)}||_\alpha \\ \leq M_{17}|b-c|$$

for some constant M_{17}.

To prove the continuity of

$$(b,t) \longmapsto |F'_a(z)|^{-t} \in H_\alpha,$$

we first observe that continuity in the variable b is clear simply because the function is constant as a function of b. To get the continuity in the variable t, we can for example proceed like before i.e., writing

$$|F'_a(z)|^{-t} = e^{-t\log|F'_a(z)|}$$

and first estimating

$$||-\ t_1 \log|F_a'(\cdot)| + t_2 \log|F_a'(\cdot)||\,|_\alpha.$$

Condition (vi). Let $(d, t_d) \in B_{\mathbf{C}^{2n+3}}((e_1(a), t_0), \min\{r_3/4, \rho\})$. Let γ be such a real number that

$$B_{\mathbf{C}^{2n+3}}((d, t_d), 2\gamma) \subset B_{\mathbf{C}^{2n+3}}((e_1(a), t_0), \min\{r_3/4, \rho\}),$$

and let t_c be an arbitrary point from the interval $(1, \mathrm{Re}\,(t_b - \gamma))$. Now, let $(b, t_b) \in B_{\mathbf{C}^{2n+3}}((d, t_d), \gamma)$. Then, by (4.12) we get that

$$\begin{aligned} |e^{t_c(\Psi_z(e_1(a)) - \Psi_z(b))}| &\le e^{t_c 2^{2n+3} M_7}, \\ |e^{(t_c - t_b)\Psi_z(b)}| &\le e^{\rho M_7 2^{2n+2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \left|\frac{\Phi_{b,t_b}(z)}{\Phi_{e_1(a),t_c}(z)}\right| &\le \frac{e^{-t_b\Psi_z(b)}}{e^{-t_c\Psi_z(e_1(a))}}\,|F_a'(z)|^{-(t_b - t_c)} \\ &\le |e^{t_c(\Psi_z(e_1(a)) - \Psi_z(b))}|\,|e^{(t_c - t_b)\Psi_z(b)}|\,|F_a'(z)|^{\mathrm{Re}\,(t_c - t_b)} \\ &\le e^{(t_c+\rho)2^{2n+3}M_{22}}\Delta_a^{-\rho} \end{aligned}$$

where Δ_a is defined by the formula (3.4) and is a strictly positive real number. □

4.11. Now we shall invoke the following perturbation theorem, see [**22**, Theorem XII.8 and Problem 8, p. 70].

4.12. Kato-Rellich theorem. *Let H be a complex Banach space, $L(H)$ the Banach space of bounded operators on H, G an open region in $\mathbf{C}^m$ for some $m \ge 1$ and*

$$G \ni b \longmapsto \mathcal{L}_b \in L(H)$$

a holomorphic function. If $\mathcal{L}_a$ with $a \in G$ has a simple eigenvalue α_a which is an isolated point of the spectrum of $\mathcal{L}_a$ with the associated *eigenvector g_a, then for every $\varepsilon > 0$ small enough there exists $\delta > 0$ such that, if $|b - a| < \delta$, then the operator $\mathcal{L}_b$ has a simple eigenvalue α_b and eigenvector g_b with the properties*

(i) *the functions $b \mapsto \alpha_b$ and $b \mapsto g_b$ are holomorphic on $B_{\mathbf{C}^m}(a, \delta)$,*

(ii) *if $|b - a| < \delta$, then*

$$\text{spectrum}\,(\mathcal{L}_b) \bigcap B(\alpha_a, \varepsilon) = \{\alpha_b\}. \tag{4.19}$$

4.13. Kato-Rellich theorem works. Fix $b = (b_0, \dots, b_n) \in B_{\mathbf{C}^{n+1}}(a, r_3/4)$ and $t \in (t_0 - \varrho, t_0 + \varrho)$. It follows from Corollary 3.8 that $e^{P_b(t)}$ is a simple isolated eigenvalue of $\mathcal{L}_{b,t}$. Note also that

$$\mathcal{L}^0_{b,t} = \mathcal{L}(e_1(b), t)$$

and, if $b \neq a$, then $\mathcal{L}_{b,t} \neq \mathcal{L}^0_{b,t}$. But

$$\mathcal{L}_{a,t} = \mathcal{L}^0_{a,t} = \mathcal{L}(e_1(a), t).$$

Therefore, $e^{P_a(t_0)}$ is a simple isolated eigenvalue of $\mathcal{L}(e_1(a), t_0)$. Then it follows from Kato-Rellich that, for every $\varepsilon > 0$ small enough, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that, for $(b', t) \in \mathbf{C}^{2n+3}$ such that $|b' - e_1(a)| < \delta_1$ and $|t - t_0| < \delta_2$, $\mathcal{L}(b', t)$ has a simple eigenvalue $\alpha_{(b',t)}$ with the properties

(i) the function $(b', t) \mapsto \alpha_{(b',t)}$ is holomorphic on $B_{\mathbf{C}^{2n+2}}(e_1(a), \delta_1) \times B_{\mathbf{C}}(t_0, \delta_2)$,

(ii) if $|b' - e_1(a)| < \delta_1$ and $|t - t_0| < \delta_2$, then

$$\text{spectrum}\,(\mathcal{L}(b', t)) \bigcap B(e^{P_a(t_0)}, \varepsilon) = \{\alpha_{(b',t)}\}.$$

So next we shall prove that $\alpha_{(e_1(b),t)} = e^{P_b(t)}$ for $b \in B(a, \delta_1)$.

4.14. Diagram. We claim that $b \in B_{\mathbf{C}^{n+1}}(a, r_3/4)$; the following diagram is commutative

$$\begin{array}{ccc} H_1(J_{F_b}) & \xrightarrow{\mathcal{L}_{b,t}} & H_1(J_{F_b}) \\ {\scriptstyle T_b}\big\downarrow & & \big\downarrow{\scriptstyle T_b} \\ H_\alpha(J_{F_a}) & \xrightarrow{\mathcal{L}(e_1(b),t)} & H_\alpha(J_{F_a}), \end{array}$$

where $T_b(g) = g \circ h_b$. Since T_b is linear and continuous, T_b is bounded. Moreover, since h_b is Hölder, $T_b(H_1(J_{F_b})) \subset H_\alpha(J_{F_a})$. To prove the claim, observe

$$(\mathcal{L}(e_1(b), t) \circ T_b)(g)(z) = \sum_{x \in F_a^{-1}(z)} |(F_b' \circ h_b)(x)|^{-t} g(h_b(z)),$$

$$(T_b \circ \mathcal{L}_{b,t})(g)(z) = \mathcal{L}_{b,t}(g(h_b(z))) = \sum_{x \in F_b^{-1}(h_b(z))} |(F_b)'(x)|^{-t} g(x).$$

Since F_a and F_b are conjugated by the homeomorphism h_b, see Proposition 2.4 (iii),

$$F_b^{-1}(h_b(z)) = \{h_b(y) : y \in F_a^{-1}(z)\}.$$

This finishes the proof of the claim. $\square$

4.15. We prove that $\alpha_{(e_1(b),t)} = e^{P_b(t)}$. Let $g_{b,t} \in H_1(J_{F_b})$ be an eigenvector that is associated to the eigenvalue $e^{P_b(t)}$ of the operator $\mathcal{L}_{b,t}$, see Corollary 3.8. From diagram 4.14 it follows that

$$\mathcal{L}(e_1(b), t)(g_{b,t} \circ h_b) = e^{P_b(t)}(g_{b,t} \circ h_b).$$

Therefore, $\alpha_{(e_1(b),t)}$ and $e^{P_b(t)}$ are eigenvalues of $\mathcal{L}(e_1(b), t)$. Since we have $\alpha_{(e_1(a),t_0)} = e^{P_a(t_0)}$ and since $e^{P_b(t)}$ is continuous for (b, t) close to (a, t_0) (see Lemma 4.2 and [**4**, Proposition 3.1]), we have that

$$e^{P_b(t)} \in B(e^{P_a(t_0)}, \varepsilon).$$

Then from (4.19) we have the equality $\alpha_{(e_1(b),t)} = e^{P_b(t)}$ and the following corollary.

4.16. Corollary. *The function* $(b, t) \mapsto P_b(t)$ *is real-analytic in some neighborhood of* (a, t_0) *in* $\mathbf{C}^{n+1} \times (1, \infty)$.

4.17. Theorem. *The Hausdorff dimension of* $\mathrm{HD}\,(J_{f_a}^r)$ *is real-analytic with respect to* $a \in \mathcal{H} \subset \mathbf{C}^{n+1}$.

Proof. To prove that

$$a \longmapsto \mathrm{HD}\,(J^r_{F_b})$$

with $a \in \mathcal{H}$ real analytic, it suffices to show that the solution of the equation

$$P_b(t) = 0$$

(which by [**4**, Theorem C] is some $t > 1$ equal to $\mathrm{HD}\,(J^r_{F_b})$) is real-analytic for every b in some neighborhood of a. Since the function $P_b : (1, \infty) \to \mathbf{R}$ is real-analytic (Corollary 4.16), convex, strictly decreasing [**4**, Proposition 3.1], and

$$\frac{\partial P_b(t)}{\partial t} \neq 0,$$

the theorem follows from the implicit function theorem. □

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