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A REMARK ON DISCRETE QUADRATIC FUNCTIONALS WITH SEPARABLE ENDPOINTS

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ABSTRACT. A characterization of the positivity of a discrete quadratic functional with separable state endpoint constraints is presented in terms of conjugate intervals to 0, various conjoined bases of the associated linear Hamiltonian difference system, and solutions of the implicit and explicit Riccati difference equations. The boundary conditions are in the form of either equalities or (strict) inequalities. Three sets of results are derived under different underlying assumptions.

1. Introduction. Consider the discrete quadratic functional

$$\mathcal{I}(\eta, q) := \eta_0^T \Gamma_0 \eta_0 + \eta_{N+1}^T \Gamma \eta_{N+1} + \sum_{k=0}^{N} \{\eta_{k+1}^T C_k \eta_{k+1} + q_k^T B_k q_k\}$$

subject to $\Delta \eta_k = A_k \eta_{k+1} + B_k q_k$, $k \in [0, N]$, and the boundary conditions

$$\mathcal{M}_0 \eta_0 = 0, \quad \mathcal{M} \eta_{N+1} = 0. \tag{1}$$

The minimization problem for $\mathcal{I}$ will be denoted by (P). This type of functional could be regarded as the second variation of a discrete nonlinear control problem with separated state endpoints. Therefore, studying the positivity of the quadratic form $\mathcal{I}$ would result in sufficiency optimality conditions for nonlinear problems, see [**6, 7, 8**]. The positivity of discrete quadratic functionals has been studied in [**1, 2, 3, 4**]. In [**2, 4**], the positivity of $\mathcal{I}$ was characterized in terms of a specific conjoined basis, that is, the principal solution of the associate linear Hamiltonian difference system. Furthermore, this characterization was also done in terms of the *augmented* implicit Riccati difference equation, see [**2**, Theorem 3] and [**4**, Theorem 2.3].

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In view of [**5**, Proposition 1.3] and [**3**, Theorem 4], the positivity of $\mathcal{I}$ with *fixed endpoints* was characterized in terms of the existence of a solution to the *explicit* Riccati equation. On the other hand, a transformation was developed in [**4**, Lemma 3.1] in order to reduce a separable endpoints problem to a problem with fixed endpoints by extending the time interval to -1 and $N+2$. However, as far as we know, this transformation was not performed on the problem (P) to obtain sufficiency results in terms of the data of (P), which is an important issue in applications.

The positivity of $\mathcal{I}$ when the right endpoint is fixed ($\mathcal{M} = I$) was characterized in terms of the *nonaugmented* implicit Riccati equation [**6**, Theorem 5] and in terms of the explicit Riccati equation with an initial boundary condition of the form of an equality in [**6**, Theorem 6] and inequality in [**8**, Theorem 10]. Note that the result [**6**, Theorem 6] required a certain normality assumption.

In this paper we first extend in Theorems 1 and 2 the results in [**6**] to the case where also the final state endpoint varies as in (1). We provide in Theorem 1 a characterization of the positivity of $\mathcal{I}$ in terms of conjugate intervals to 0, a *natural* conjoined basis, and an implicit Riccati equation. This result extends [**3**, Theorem 3], where an additional condition $\operatorname{Ker}\mathcal{M} \subseteq \operatorname{Im}\overline{X}_{N+1}$ is required, and also completes [**4**, Theorem 3.2] in a sense that the corresponding implicit Riccati equation solution and its boundary conditions are derived. In Theorem 2 under a normality assumption, the positivity of $\mathcal{I}$ is characterized via a conjoined basis (X, U) with X *invertible* and via the *explicit* Riccati equation. This result is based on the perturbation technique from [**6**]. Next we apply the transformation in [**4**] to reduce the problem (P) to a transformed problem (TP) on the time interval $[0, N+2]$. Then we apply to (TP) the results in [**6**] and [**8**]. The translation of these results in terms of the original data for (P) is not a routine exercise. This task requires finding the right form of the boundary conditions associated with each of the conjoined bases and the Riccati equation solutions. The knowledge of the corresponding continuous time results [**9, 10**] is a valuable inspiration in this search. Another important issue that arises during the translation of the results is to figure out the bare minimum conditions that characterizes the positivity of $\mathcal{I}$. The outcome of this method is given mainly in Theorem 3 and also in Theorem 1(ii) where no normality is required.

The difference between the various results of this paper resides in the underlying hypotheses as well as in the form of the initial and final boundary conditions of the conjoined bases and the Riccati equation solutions. These boundary conditions have the form of an equality or a (strict) inequality.

2. Preliminaries. Given $n, N \in \mathbf{N}$ with $N \geq 2$, we denote by $J := [0, N]$ and $J^* := [0, N+1]$ the intervals of integers between the indicated endpoints. We assume that $A_k, B_k, C_k,\ k \in J$ and $\Gamma_0, \Gamma, \mathcal{M}_0, \mathcal{M}$ are $n \times n$-matrices such that $B_k, C_k, \Gamma_0, \mathcal{M}_0, \mathcal{M}$ are symmetric and $\tilde{A}_k := (I - A_k)^{-1}$ exists. Without loss of generality, both $\mathcal{M}_0$ and $\mathcal{M}$ are projections and $\Gamma_0 = (I - \mathcal{M}_0)\Gamma_0(I - \mathcal{M}_0)$, $\Gamma = (I - \mathcal{M})\Gamma(I - \mathcal{M})$. All quantities are supposed to be real valued. The forward difference operator is denoted by Δ, i.e., $\Delta y_k = y_{k+1} - y_k$.

The sequences $\{\eta_k\}_{k=0}^{N+1}$ and $\{q_k\}_{k=0}^{N}$ of n-vectors form an *admissible pair* (η, q) if they satisfy the equation of motion in (P), i.e., $\Delta\eta_k = A_k\eta_{k+1} + B_k q_k,\ k \in J$. The quadratic functional $\mathcal{I}$ is *nonnegative*, $\mathcal{I} \geq 0$, if $\mathcal{I}(\eta, q) \geq 0$ for all admissible pairs (η, q) satisfying the boundary conditions (1). The functional $\mathcal{I}$ is *positive definite*, $\mathcal{I} > 0$, if $\mathcal{I}(\eta, q) > 0$ for all admissible (η, q) satisfying (1) and $\eta \not\equiv 0$.

The corresponding *linear Hamiltonian difference system* is

$$\text{(H)} \qquad \Delta\eta_k = A_k\eta_{k+1} + B_k q_k, \quad \Delta q_k = C_k\eta_{k+1} - A_k^T q_k.$$

As usual, the vector solutions of (H) will be denoted by small letters and the $n \times n$-matrix solutions by capital ones. Let $(X, U), (\tilde{X}, \tilde{U})$ be solutions of (H). Then $X_k^T\tilde{U}_k - U_k^T\tilde{X}_k \equiv W$, where W is a constant $n \times n$-matrix, sometimes called a *Wronskian* of the solutions (X, U) and $(\tilde{X}, \tilde{U})$. If $W = I$, then these solutions are called *normalized*. A solution (X, U) is said to be a *conjoined basis* if $X^T U$ is symmetric and rank $\begin{pmatrix} X \\ U \end{pmatrix} = n$. Following [**2**], a solution (X, U) of (H) is said to have *no focal points* in $(0, N+1]$, provided

$$\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k \quad \text{and} \quad D_k := X_k X_{k=1}^\dagger \tilde{A}_k B_k \geq 0$$

holds for all $k \in J$, where Ker, † and ≥ 0 denote the kernel, Moore-Penrose inverse and nonnegative definiteness of the given matrix, respectively. We will also use Im, ^{T} and > 0 to denote the image, transpose, and positive definiteness of a matrix. Observe that the matrices D_k are symmetric when the kernel condition holds [**2**, Lemma 2].

A solution (η, q) of (H) has a *generalized zero* in the interval $(m, m+1]$, provided

$$\eta_m \neq 0, \quad \eta_{m+1} \in \operatorname{Im} \tilde{A}_m B_m \quad \text{and} \quad \eta_m^T B_m^\dagger (I - A_m) \eta_{m+1} \leq 0.$$

When the right endpoint is fixed, the generalized zero concept is used to define conjugate intervals to 0. Let $m \in J$. An interval $(m, m+1]$ is said to be *conjugate* to 0 if there exists a solution (η, q) of (H) having $(m, m+1]$ as a generalized zero and, for some $\gamma \in \mathbf{R}^n$ satisfying the initial boundary and transversality conditions

$$\mathcal{M}_0 \eta_0 = 0 \quad \text{and} \quad q_0 = \Gamma_0 \eta_0 + \mathcal{M}_0 \gamma. \tag{2}$$

With (H) the *Riccati matrix difference system*

$$R[W]_k \equiv \Delta W_k - C_k + A_k^T W_k + (W_{k+1} - C_k) \tilde{A}_k (A_k + B_k W_k) = 0 \tag{R}$$

is associated. Implicit Riccati equations, which use the operator $R[W]_k$, will also be considered.

In this paper the following normality concept will be used. A pair (A, B) is called $(\mathcal{M}_0 : I)$-*normal* on J^* if the system

$$-\Delta q_k = A_k^T q_k, \quad B_k q_k = 0, \quad k \in J, \quad q_0 = \mathcal{M}_0 \gamma,$$

$\gamma \in \mathbf{R}^n$, possesses only the zero solution $q_k \equiv 0$ on J^*.

Next, similarly as in [**2**, Remark 3(ii)], we define the transition matrices $\Psi_{k,m}$ and controllability matrices $\tilde{G}_k$ as follows: set $\tilde{G}_{N+1} := 0$, $\Psi_{N+1,N} := I$ and, for $k, m \in J$, $k \leq m$,

$$\begin{aligned} \Psi_{k,m} &:= (I - A_k)(I - A_{k+1}) \dots (I - A_m), \\ \tilde{G}_k &:= \begin{pmatrix} B_k & \Psi_{k,k} B_{k+1} & \dots & \Psi_{k,N-1} B_N \end{pmatrix}. \end{aligned}$$

Then a pair (η, q) with $\mathcal{M} \eta_{N+1} = 0$ is admissible if and only if for all $k \in J$

$$\eta_k = -\tilde{G}_k \begin{pmatrix} q_k \\ \vdots \\ q_N \end{pmatrix} + \Psi_{k,N} \eta_{N+1} = \begin{pmatrix} -\tilde{G}_k \mathcal{P}_k & \Psi_{k,N}(I - \mathcal{M}) \end{pmatrix} \begin{pmatrix} q \\ \alpha \end{pmatrix}, \tag{3}$$

where $q := (q_0^T \dots q_N^T)^T$, $\alpha := \eta_{N+1}$ and $\mathcal{P}_k : \mathbf{R}^{(N+1)n} \to \mathbf{R}^{(N-k+1)n}$ is the restriction operator onto the last $N-k+1$ entries of q, i.e., cutting the first k entries. Note that $\mathcal{P}_0$ is the identity (matrix). Moreover, if (X, U) is a conjoined basis of (H) with $\operatorname{Ker} X_{k=1} \subseteq \operatorname{Ker} X_k$ on J, then

$$\eta_k \in \operatorname{Im} X_k \quad \text{implies} \quad \eta_{k+1} \in \operatorname{Im} X_{k+1}. \tag{4}$$

In order to transform the variable endpoint at $N+1$ to a fixed endpoint at $N+2$, we use the result of [**4**]. We define the matrices $A_{N+1} := 0$, $\tilde{A}_{N+1} := I$, $B_{N+1} := I - \mathcal{M}$, $\mathcal{C}_k := C_k$, $k \in [0, N-1]$, $\mathcal{C}_N := C_N + \Gamma - (I - \mathcal{M})$ and $\mathcal{C}_{N+1} := 0$. Then consider the discrete quadratic functional

$$\mathcal{J}(\eta, q) := \eta_0^T \Gamma_0 \eta_0 + \sum_{k=0}^{N+1} \{\eta_{k+1}^T \mathcal{C}_k \eta_{k+1} + q_k^T B_k q_k\} \tag{TP}$$

subject to $\mathcal{J}$-*admissible pairs* (η, q), i.e., $\Delta\eta_k = A_k \eta_{k+1} + B_k q_k$, $k \in \mathcal{J}^*$, satisfying the boundary conditions

$$\mathcal{M}_0 \eta_0 = 0, \quad \eta_{N+2} = 0. \tag{5}$$

The relation between the definiteness of $\mathcal{I}$ and $\mathcal{J}$ is stated next.

Proposition 1 [**4**, Lemma 3.1]. $\mathcal{I} > 0$ ($\mathcal{I} \geq 0$) *over admissible pairs* (η, q) *satisfying* (1) *if and only if* $\mathcal{J} > 0$ ($\mathcal{J} \geq 0$) *over* $\mathcal{J}$*-admissible pairs* (η, q) *satisfying* (5).

Naturally we need to describe also the relation between the solutions (X, U) of the Hamiltonian system (H) corresponding to $\mathcal{I}$ and the solutions (Y, V) of the transformed Hamiltonian system $(\mathcal{H})$ corresponding to $\mathcal{J}$, i.e.,

$$\Delta\eta_k = A_k \eta_{k+1} + B_k q_k, \quad \Delta q_k = \mathcal{C}_k \eta_{k+1} - A_k^T q_k, \quad k \in J^*. \tag{$\mathcal{H}$}$$

Lemma 1. *Let* (X, U) *be a solution of* (H) *on* J. *Then for a solution* (Y, V) *of the transformed Hamiltonian system* $(\mathcal{H})$ *with*

$(Y_0, V_0) = (X_0, U_0)$ *we have that* $Y_k = X_k$ *for all* $k \in J^*$, $V_k = U_k$ *for all* $k \in J$ *and*

$$\begin{aligned} Y_{N+2} &= (\Gamma + \mathcal{M})X_{N+1} + (I - \mathcal{M})U_{N+1}, \\ V_{N+1} &= U_{N+1} + [\Gamma - (I - \mathcal{M})]X_{N+1}. \end{aligned}$$

Proof. For $k \in [0, N]$ the first equations of (H) and $(\mathcal{H})$ are the same. Also, for $k \in [0, N-1]$ the second equations of (H) and $(\mathcal{H})$ are the same. Thus, $Y_k = X_k$ on J^* and $V_k = U_k$ on J. Finally the second equation of $(\mathcal{H})$ at $k = N$ yields the expression for V_{N+1} and then the first equation of $(\mathcal{H})$ at $k = N+1$ yields Y_{N+2}. $\square$

3. Main results. The following result is obtained via a direct approach with the exception of the conjugate intervals condition (ii), which will require using the transformation in Proposition 1. This conjugate intervals condition can also be derived by applying [**4**, Theorem 2.3 (ii)] to a transformed problem on the interval $[-1, N+1]$. Note that such transformations do not produce a *coupled intervals* condition as is known in [**7**] for the discrete calculus of variations case. Such a condition must be derived independently in a future work.

Theorem 1 (Characterization of $\mathcal{I} > 0$). *The following are equivalent.*

(i) $\mathcal{I} > 0$, *i.e.,* $\mathcal{I}(\eta, q) > 0$ *for all admissible* (η, q) *with* $\mathcal{M}_0\eta_0 = 0$, $\mathcal{M}\eta_{N+1} = 0$ *and* $\eta \not\equiv 0$.

(ii) *There is no interval* $(m, m+1] \subseteq (0, N+1]$ *conjugate to* 0, *i.e., the Jacobi sufficient condition holds and any solution* (η, q) *of* (H) *with*

$$\mathcal{M}_0\eta_0 = 0, \quad q_0 = \Gamma_0\eta_0 + \mathcal{M}_0\gamma, \quad \mathcal{M}\eta_{N+1} = 0, \quad \eta_{N+1} \neq 0$$

satisfies

$$\eta_{N+1}^T(\Gamma\eta_{N+1} + q_{N+1}) > 0. \tag{6}$$

(iii) *The conjoined basis* $(\overline{X}, \overline{U})$ *of* (H) *given by the initial conditions*

$$\overline{X}_0 = I - \mathcal{M}_0, \quad \overline{U}_0 = \Gamma_0 + \mathcal{M}_0 \tag{7}$$

has no focal points in $(0, N+1]$ *and satisfies*

$$\overline{X}^T_{N+1}(\Gamma\overline{X}_{N+1} + \overline{U}_{N+1}) \geq 0 \quad \textit{on } \operatorname{Ker}\mathcal{M}\overline{X}_{N+1}, \tag{8}$$

$$\operatorname{Ker}(I-\mathcal{M})(\Gamma\overline{X}_{N+1} + \overline{U}_{N+1}) \cap \operatorname{Ker}\mathcal{M}\overline{X}_{N+1} \subseteq \operatorname{Ker}\overline{X}_{N+1}. \tag{9}$$

(iv) *The implicit Riccati matrix equation*

(10)

$$R[\overline{W}]_k(-\tilde{G}_k\mathcal{P}_k \quad \Psi_{k,N}(I-\mathcal{M})) = 0 \quad \textit{on } \operatorname{Ker}\mathcal{M}_0(-\tilde{G}_0 \quad \Psi_{0,N}(I-\mathcal{M})),$$

$k \in J$, *has a symmetric solution* $\overline{W}_k$ *on* J^* *such that*

$$\overline{\mathcal{D}}_k = B_k - B_k - B_k\tilde{A}^T_k(\overline{W}_{k+1} - C_k)\tilde{A}_kB_k \geq 0$$

holds for all $k \in J$, *and*

$$\overline{W}_0 = \Gamma_0, \tag{11}$$

$$\Gamma + \overline{W}_{N+1} > 0 \quad \textit{on } \operatorname{Ker}\mathcal{M} \cap \operatorname{Im}\overline{X}_{N+1}. \tag{12}$$

Remark 1. When we attempted to derive the conjoined basis and Riccati equation conditions (iii) and (iv) via the transformation in Proposition 1, we obtained conditions that are *stronger* and *more complicated* than (iii) and (iv).

Lemma 2. *Suppose that* (iv) *of Theorem 1 is true without* (12). *Then*

$$\operatorname{Ker}\overline{X}_{k+1} \subseteq \operatorname{Ker}\overline{X}_k \quad \textit{for all } k \in J,$$

where $(\overline{X}, \overline{U})$ *is the conjoined basis of* (H) *given by the initial conditions* (7).

Proof. If there exists $m \in J$ such that $\operatorname{Ker}\overline{X}_{m+1} \not\subseteq \operatorname{Ker}\overline{X}_m$, then there is a $d \in \mathbf{R}^n$, $d \neq 0$, such that $\overline{X}_{m+1}d = 0$ and $\overline{X}_m d \neq 0$. Define the pair (η, q) as $(\overline{X}_k d, \overline{U}_k d)$ for $k \in [0, m]$ and $(0, 0)$ for $k \in [m+1, N+1]$. Then it follows that (η, q) is admissible and satisfies the boundary conditions $\mathcal{M}_0\eta_0 = 0$, $\eta_{N+1} = 0$. Hence, by (3),

$$\eta_k = (-\tilde{G}_k\mathcal{P}_k \quad \Psi_{k,N}(I-\mathcal{M}))\begin{pmatrix} q \\ 0 \end{pmatrix} \quad \text{for all } k \in J.$$

Then $\mathcal{M}_0\eta_0 = 0$ implies that $\binom{q}{0} \in \operatorname{Ker}(-\tilde{G}_0 \quad \Psi_{0,N}(I-\mathcal{M}))$ which by (10) yields $R[\overline{W}]_k\eta_k = 0$ for all $k \in J$. Using [**2**, Lemma 2(i)], $\eta_{N+1} = 0$ and (11) we get

$$\begin{aligned}\mathcal{I}(\eta, q) &= \eta_0^T(\Gamma_0 - \overline{W}_0)\eta_0 + \eta_{N+1}^T(\Gamma + \overline{W}_{N+1})\eta_{N+1} + \sum_{k=0}^{N} z_k^T \overline{\mathcal{D}}_k z_k \\ &= \sum_{k=0}^{N} z_k^T \overline{\mathcal{D}}_k z_k.\end{aligned}$$

On the other hand, by the definition of (η, q) we have $\mathcal{I}(\eta, q) = 0$. Hence, for all $k \in J$, $\overline{\mathcal{D}}_k z_k = 0$. By [**2**, Lemma 2(i)] again we get that η satisfies the identity of the form $Z_k\eta_{k+1} = \eta_k$ for all $k \in J$. Since $\eta_{m+1} = 0$, we obtain that $\eta_m = \overline{X}_m d = 0$ that yields a contradiction. $\square$

Proof of Theorem 1. (i) $\Leftrightarrow$ (ii). By Proposition 1 we know that (i) is equivalent to

(i)$'$ $\mathcal{J} > 0$ over $\mathcal{M}_0\eta_0 = 0$ and $\eta_{N+2} = 0$, $\eta \not\equiv 0$.

Apply to (i)$'$ the results of [**6**, Theorem 5] to obtain that (i)$'$ is equivalent to

(ii)$'$ there is no interval $(m, m+1] \subseteq (0, N+1]$ conjugate to 0 and $(N+1, N+2]$ is not conjugate to 0.

Condition $(N+1, N+2]$ being not conjugate to 0 is equivalent to the fact that every solution (η, q) of $(\mathcal{H})$ with the initial conditions (2), $\eta_{N+1} \neq 0$ and $\eta_{N+2} \in \operatorname{Im} \tilde{A}_{N+1}B_{N+1} = \operatorname{Im}(I-\mathcal{M})$ satisfies

$$\eta_{N+1}^T B_{N+1}^\dagger (I - A_{N+1})\eta_{N+2} > 0. \tag{13}$$

Since η_{N+2} must also be a multiple of

$$\overline{X}_{N+2} = (\Gamma + \mathcal{M})\overline{X}_{N+1} + (I-\mathcal{M})\overline{U}_{N+1}$$

from Lemma 1, it follows that $\mathcal{M}\eta_{N+1} = \mathcal{M}\eta_{N+2} = 0$. Since $B_{N+1}^\dagger = I - \mathcal{M}$, condition (13) is equivalent to (6).

(i) $\Rightarrow$ (iii). Since (i) implies that $\mathcal{I} > 0$ over all admissible pairs (η, q) with $\mathcal{M}_0\eta_0 = 0$ and $\eta_{N+1} = 0$, we have from [**6**, Theorem 5] that the conjoined basis $(\overline{X},\overline{U})$ has no focal points in $(0,N+1]$. Suppose now that $\beta\in\operatorname{Ker}\mathcal{M}\overline{X}_{N+1}$ and define an admissible pair $(\eta,q):=(\overline{X}\beta,\overline{U}\beta)$. Then η satisfies the boundary conditions (1) and it follows from (i) that $\mathcal{I}(\eta,q)\geq 0$. On the other hand, as (η,q) is a solution of (H), $\mathcal{I}(\eta,q)=\beta^T\overline{X}_{N+1}^T(\Gamma\overline{X}_{N+1}+\overline{U}_{N+1})\beta$, so that (8) is shown. Finally, to prove (9) let $\beta\in\mathbf{R}^n$ be such that $(I-\mathcal{M})(\Gamma\overline{X}_{N+1}+\overline{U}_{N+1})\beta=0$ and $\mathcal{M}\overline{X}_{N+1}\beta=0$. It follows that the admissible pair (η,q) defined as above satisfies $\mathcal{I}(\eta,q)=0$. If $\overline{X}_{N+1}\beta\neq 0$, then $\eta_{N+1}\neq 0$, i.e., $\eta\not\equiv 0$ and thus (i) would imply that $\mathcal{I}(\eta,q)>0$, which is a contradiction. Therefore, $\overline{X}_{N+1}\beta=0$ and (9) is shown.

(iii) $\Rightarrow$ (iv). Let $(\overline{X},\overline{U})$ be the conjoined basis from (iii) and let $(\tilde{X},\tilde{U})$ be the conjoined basis of (H) completing $(\overline{X},\overline{U})$ to normalized conjoined bases of (H), i.e.,

$$\tilde{X}_0=-(\Gamma_0+\mathcal{M}_0)(I+\Gamma_0^2)^{-1},\quad \tilde{U}_0=(I-\mathcal{M}_0)(I+\Gamma_0^2)^{-1}.$$

For $k\in J^*$ define the $n\times n$-matrices

$$\overline{W}_k=\overline{U}_k\overline{X}_k^\dagger+(\overline{U}_k\overline{X}_k^\dagger\tilde{X}_k-\tilde{U}_k)(I-\overline{X}_k^\dagger\overline{X}_k)\overline{U}_k^T.$$

Then $R[\overline{W}]_k\overline{X}_k=0$, $\overline{W}_k\overline{X}_k=\overline{U}_k\overline{X}_k^\dagger\overline{X}_k$ and $\overline{\mathcal{D}}_k\geq 0$ for all $k\in J$, by [**2**, Lemma 2(ii)]. Let $\binom{q}{\alpha}$ be arbitrary in $\operatorname{Ker}\mathcal{M}_0(-\tilde{G}_0\quad\Psi_{0,N}(I-\mathcal{M}))$. Define $\eta_{N+1}:=(I-\mathcal{M})\alpha$ and $\{\eta_k\}_{k=0}^N$ by (3). Then (η,q) is admissible and satisfies the boundary conditions (1). From (4) we obtain that $\eta_k=\overline{X}_kc_k$ for some $c_k\in\mathbf{R}^n$, $k\in J^*$. Therefore,

$$R[\overline{W}]_k(-\tilde{G}_k\mathcal{P}_k\quad\Psi_{k,N}(I-\mathcal{M}))\begin{pmatrix}q\\ \alpha\end{pmatrix}=R[\overline{W}]_k\eta_k=R[\overline{W}]_k\overline{X}_kc_k=0,$$

and since $\binom{q}{\alpha}$ was arbitrary, (10) holds true. Initial condition (11) follows from (iii) $\Rightarrow$ (iv) in [**6**, Theorem 5]. To show (12), let $\gamma\in\operatorname{Ker}\mathcal{M}$, $\gamma=\overline{X}_{N+1}\delta$ for some $\delta\in\mathbf{R}^n$. Then $\mathcal{M}\overline{X}_{N+1}\delta=0$ and (8) with the equality $\overline{X}^T\overline{WX}=\overline{X}^T\overline{U}$ yield

$$\begin{aligned}\gamma^T(\Gamma+\overline{W}_{N+1})\gamma&=\delta^T\overline{X}_{N+1}^T(\Gamma+\overline{W}_{N+1})\overline{X}_{N+1}\delta\\&=\delta^T\overline{X}_{N+1}^T(\Gamma\overline{X}_{N+1}+\overline{U}_{N+1})\delta\geq 0.\end{aligned}$$

If $\gamma^T(\Gamma + \overline{W}_{N+1})\gamma = 0$, then $(I - \mathcal{M})(\Gamma + \overline{W}_{N+1})\gamma = 0$. For a vector $\beta := \overline{X}^{\dagger}_{N+1}\gamma$ we have that $\overline{X}_{N+1}\beta = \overline{X}_{N+1}\delta = \gamma$, $\mathcal{M}\overline{X}_{N+1}\beta = 0$, and

$$\begin{aligned}(I - \mathcal{M})(\Gamma\overline{X}_{N+1} + \overline{U}_{N+1})\beta &= (I - \mathcal{M})(\Gamma\overline{X}_{N+1} + \overline{U}_{N+1})\overline{X}^{\dagger}_{N+1}\overline{X}_{N+1}\delta \\ &= (I - \mathcal{M})(\Gamma + \overline{W}_{N+1})\gamma = 0.\end{aligned}$$

Thus, (9) implies that $\gamma = \overline{X}_{N+1}\beta = 0$ and so condition (12) is proven.

(iv) $\Rightarrow$ (i). By Lemma 2, $\operatorname{Ker}\overline{X}_{k+1} \subseteq \operatorname{Ker}\overline{X}_k$ holds for all $k \in J$. Let (η, q) be an admissible pair satisfying the boundary conditions (1). Then $\eta_0 \in \operatorname{Im}\overline{X}_0$ and (4) yield $\eta_{N+1} \in \operatorname{Im}\overline{X}_{N+1}$. Also, by (3) with $\alpha := \eta_{N+1}$, $\binom{q}{\alpha} \in \operatorname{Ker}\mathcal{M}_0(-\tilde{G}_0 \quad \Psi_{0,N}(I - \mathcal{M}))$ and, for all $k \in J$,

$$R[\overline{W}]_k\eta_k = R[\overline{W}]_k(-\tilde{G}_k\mathcal{P}_k \quad \Psi_{k,N}(I - \mathcal{M}))\begin{pmatrix} q \\ \alpha \end{pmatrix} = 0.$$

Whence, by [**2**, Lemma 2(i)] with $z_k := q_k - \overline{W}_k\eta_k$ and by using (12) we get

$$\mathcal{I}(\eta, q) = \eta_0^T(\Gamma_0 - \overline{W}_0)\eta_0 + \eta_{N+1}^T(\Gamma + \overline{W}_{N+1})\eta_{N+1} + \sum_{k=0}^{N} z_k^T\overline{\mathcal{D}}_k z_k \geq 0,$$

so that we showed that $\mathcal{I} \geq 0$. If now $\mathcal{I}(\eta, q) = 0$ for some admissible (η, q) satisfying (1), then $\eta_{N+1} \in \operatorname{Im}\overline{X}_{N+1}$, $\overline{\mathcal{D}}_k z_k = 0$ for all $k \in J$, and

$$\eta_{N+1}^T(\Gamma + \overline{W}_{N+1})\eta_{N+1} = 0.$$

Hence, condition (12) implies that $\eta_{N+1} = 0$. Via the identity of the form $Z_k\eta_{k+1} = \eta_k$, $k \in J$, from [**2**, Lemma 2] we then get that $\eta_k \equiv 0$ on J^*. Hence $\mathcal{I} > 0$ and the proof is complete. □

Remark 2. If $\overline{X}_{N+1}$ is invertible, then (8)–(9) and (12) are rephrased, respectively, as

$$(14) \qquad \begin{aligned}\overline{X}^T_{N+1}(\Gamma\overline{X}_{N+1} + \overline{U}_{N+1}) > 0 &\quad \text{on } \operatorname{Ker}\mathcal{M}\overline{X}_{N+1}, \\ \Gamma + \overline{W}_{N+1} > 0 &\quad \text{on } \operatorname{Ker}\mathcal{M}.\end{aligned}$$

Remark 3. In (iv) of Theorem 1, the implicit Riccati equation (10) may take the equivalent form

(15)

$$R[\overline{W}]_k(\Phi_{k,0}(I - \mathcal{M}_0) \;\; G_k\tilde{\mathcal{P}}_k) = 0 \quad \text{on } \operatorname{Ker}\mathcal{M}(\Phi_{N+1,0}(I - \mathcal{M}_0) \;\; G_{N+1}),$$

where $\Phi_{k,m}$ and G_k are the transition and controllability matrices, see [**2**, Remark 3].

Proof. Similarly as in (3), a pair (η, q) with $\mathcal{M}_0\eta_0 = 0$ is admissible if and only if for all $k \in [1, N+1]$

$$\eta_k = \Phi_{k,0}\eta_0 + G_k \begin{pmatrix} q_0 \\ \vdots \\ q_{k-1} \end{pmatrix} = (\Phi_{k,0}(I - \mathcal{M}_0) \; G_k\tilde{\mathcal{P}}_k) \begin{pmatrix} \alpha \\ q \end{pmatrix},$$

where $\alpha = \eta_0$. The matrix $\tilde{\mathcal{P}}_k$ is the restriction operator onto the first k entries of q, i.e., $\tilde{\mathcal{P}}_k q = (q_0^T, \dots, q_{k-1}^T)^T$. Note that $\tilde{\mathcal{P}}_{N+1}$ is the identity (matrix). Thus, condition $R[\overline{W}]_k\eta_k = 0$ is equivalent to (15), which is what we needed to show. $\square$

The following result represents an extension of [**6**, Theorem 6] to the case where also the right endpoint varies. As in Theorem 1 the initial conditions of the conjoined basis (X, U) and the Riccati equation solution W are in the form of equalities. However, X is now invertible and W solves the *explicit* Riccati equation. The price for this richer result is the assumption of $(\mathcal{M}_0 : I)$-normality, which incidentally does not yield that $\overline{X}_k$ in Theorem 1 is invertible for all k. However, when $\mathcal{I} > 0$, the $(\mathcal{M}_0 : I)$-normality implies that $\overline{X}_{N+1}$ is invertible, as it is shown in [**6**, Lemma 4].

Let the two conjoined bases $(\overline{X}, \overline{U})$ and $(\hat{X}, \hat{U})$ of (H) be given by the initial values (7) and

$$\hat{X}_{N+1} = 0, \quad \hat{U}_{N+1} = -I,$$

respectively.

Theorem 2 (Characterization of $\mathcal{I} > 0$). *Assume that (A, B) is $(\mathcal{M}_0 : I)$-normal on J^*. Then the following are equivalent.*

(i) $\mathcal{I} > 0$*, i.e.,* $\mathcal{I}(\eta, q) > 0$ *for all admissible pairs* (η, q) *satisfying* $\mathcal{M}_0\eta_0 = 0$*,* $\mathcal{M}\eta_{N+1} = 0$*,* $\eta \not\equiv 0$*.*

(ii) *There exists a conjoined basis* (X,U) *of* (H) *with no focal points in* $(0,N+1]$, X_k *invertible for all* $k \in J^*$, *and satisfying*

$$
\begin{aligned}
&(I-\mathcal{M}_0)(\Gamma_0 X_0 - U_0) = 0,\\
(16) \qquad & X_{N+1}^T(\Gamma X_{N+1} + U_{N+1}) > 0 \quad \textit{on } \operatorname{Ker} \mathcal{M} X_{N+1}.
\end{aligned}
$$

This conjoined basis is given explicitly for all $k \in J^*$ *by*

$$
(17) \qquad X_k := \varepsilon \hat{X}_k \overline{X}_{N+1}^{T-1} \mathcal{M}_0 + \overline{X}_k, \quad U_k := \varepsilon \hat{U}_k \overline{X}_{N+1}^{T-1} \mathcal{M}_0 + \overline{U}_k,
$$

where ε *is small enough.*

(iii) *There exists a symmetric solution* W_k *on* J^* *of the explicit Riccati matrix equation* (R) *with* $I+B_kW_k$ *nonsingular and* $(I+B_kW_k)^{-1}B_k \geq 0$ *for all* $k \in J$, *and satisfying*

$$
\begin{aligned}
&(I-\mathcal{M}_0)W_0 - \Gamma_0 = 0,\\
(18) \qquad & \Gamma + W_{N+1} > 0 \quad \textit{on } \operatorname{Ker} \mathcal{M}.
\end{aligned}
$$

Proof. (i) $\Rightarrow$ (ii). We know from [**6**, Theorem 6] that (ii) holds except of (16). Since $\overline{X}_{N+1}$ is invertible and $\mathcal{I} > 0$, condition (14) holds, see Remark 2. On the other hand, condition (16) where $\varepsilon = 0$ reduces to (14). Thus, by perturbing $(\overline{X}, \overline{U})$ as in (17) we obtain that, for ε small enough, (16) is valid as well.

(ii) $\Rightarrow$ (iii). This is automatic by $W_k := U_k X_k^{-1}$, $k \in J^*$.

(iii) $\Rightarrow$ (i). Via the Picone identity [**2**, Theorem 1] we obtain $\mathcal{I} \geq 0$. Now, if $\mathcal{I}(\eta, q) = 0$ for some admissible pair (η, q) satisfying (1), then $(\Gamma + W_{N+1})\eta_{N+1} = 0$. Hence, by (18), $\eta_{N+1} = 0$. Via the identity of the form $Z_k \eta_{k+1} = \eta_k$, $k \in J$, from [**2**, Lemma 2], we get $\eta_k \equiv 0$ on J^*. □

Remark 4. When the left endpoint is free ($\mathcal{M}_0 = 0$), then (A,B) is automatically $(\mathcal{M}_0 : I)$-normal, and since in this case (17) implies $(X,U) \equiv (\overline{X}, \overline{U})$, the corresponding conditions of Theorem 2 and Theorem 1 coincide.

Next a characterization of the positivity of $\mathcal{I}$ in terms of the *explicit* Riccati equation is given without any normality assumption. Note that the initial conditions are now in the form of strict inequalities, as opposed to the equalities in Theorems 1 and 2. The proof of this result is via the transformation technique in Proposition 1.

Theorem 3 (Characterization of $\mathcal{I} > 0$). *The following are equivalent.*

(i) $\mathcal{I} > 0$, *i.e.*, $\mathcal{I}(\eta, q) > 0$ *for all admissible pairs* (η, q) *with* $\mathcal{M}_0\eta_0 = 0$, $\mathcal{M}\eta_{N+1} = 0$, $\eta \not\equiv 0$.

(ii) *There exists a conjoined basis* (X, U) *of* (H) *with no focal points in* $(0, N+1]$, X_k *invertible for all* $k \in J^*$ *and satisfying*

$$\begin{aligned} &X_0^T(\Gamma_0 X_0 - U_0) > 0 \quad \textit{on } \operatorname{Ker}\mathcal{M}_0 X_0, \\ &X_{N+1}^T(\Gamma X_{N+1} + U_{N+1}) > 0 \quad \textit{on } \operatorname{Ker}\mathcal{M} X_{N+1}. \end{aligned} \tag{19}$$

(iii) *There exists a symmetric solution* W_k *on* J^* *of the explicit Riccati matrix equation* (R) *with* $I + B_k W_k$ *invertible and* $(I + B_k W_k)^{-1} B_k \geq 0$ *for all* $k \in J$, *and satisfying*

$$\begin{aligned} \Gamma_0 - W_0 > 0 \quad &\textit{on } \operatorname{Ker}\mathcal{M}_0, \\ \Gamma + W_{N+1} > 0 \quad &\textit{on } \operatorname{Ker}\mathcal{M}. \end{aligned}$$

Proof. (i) $\Rightarrow$ (ii). By Proposition 1, $\mathcal{I} > 0$ is equivalent to $\mathcal{J} > 0$. Our results in [**8**, Theorem 10] then yields that there exists a conjoined basis (X, U), $k \in [0, N+2]$ of (H) satisfying all the conditions in (ii) except of (19), but instead $D_{N+1} := X_{N+1} X_{N+2}^{-1} \tilde{A}_{N+1} B_{N+1} \geq 0$. We will show that this implies (19). By Lemma 1, it follows that $X_{N+2} = (\Gamma + \mathcal{M}) X_{N+1} + (I - \mathcal{M}) U_{N+1}$. Thus, $D_{N+1} \geq 0$ is equivalent to

$$(I - \mathcal{M}) X_{N+1}^{T-1} [(\Gamma + \mathcal{M}) X_{N+1} + (I - \mathcal{M}) U_{N+1}]^T \geq 0,$$

which in turn, by transposing, is equivalent to

$$(I - \mathcal{M})(\Gamma + U_{N+1} X_{N+1}^{-1})(I - \mathcal{M}) \geq 0.$$

Multiplying from the left by X_{N+1}^T and from the right by X_{N+1}, we obtain the condition

$$X_{N+1}^T(\Gamma X_{N+1} + U_{N+1}) \geq 0 \quad \text{on } \operatorname{Ker}\mathcal{M} X_{N+1}.$$

If now $\mathcal{M}X_{N+1}\alpha = 0$ with $\alpha^T X_{N+1}^T(\Gamma X_{N+1} + U_{N+1})\alpha = 0$, then $X_{N+1}^T(\Gamma X_{N+1} + U_{N+1})\alpha = 0$, so that $(\Gamma X_{N+1} + U_{N+1})\alpha = 0$ by the invertibility of X_{N+1}. It follows that

$$\begin{aligned} X_{N+2}\alpha &= [(\Gamma + \mathcal{M})X_{N+1} + (I - \mathcal{M})U_{N+1}]\alpha \\ &= [\Gamma X_{N+1} + (I - \mathcal{M})U_{N+1}]\alpha \\ &= (I - \mathcal{M})(\Gamma X_{N+1} + U_{N+1})\alpha = 0. \end{aligned}$$

The invertibility of X_{N+2} now yields that $\alpha = 0$. Thus (19) holds true.

(ii) $\Rightarrow$ (iii). This is straightforward by $W_k = U_k X_k^{-1}$ for all $k \in J^*$.

(iii) $\Rightarrow$ (i). This follows by the Picone identity as in the proof of Theorem 2. $\square$

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