ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 1, Winter 1990

TRUNCATED ERGODIC THEOREMS FOR NON-SINGULAR AUTOMORPHISMS

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1. Introduction. In this paper we study ergodic theorems for invertible measurable non-singular transformations. We use a skew product construction introduced by Maharam in [**6**] to show the convergence of some new ergodic ratios and the existence of subsequences of integers for which a known ergodic ratio converges. We then look at limits of truncated ergodic ratios, and study these limits as the constants that bound the Radon-Nikodym derivatives increase to infinity. The study of these limits is motivated by an attempt to obtain the Hurewicz-Halmos-Oxtoby ergodic theorem for non-singular transformations (Theorem 1.2) from the Hopf ergodic theorem for measure preserving transformations (Theorem 1.1). We obtain some partial results in this direction. For example, we show (Theorem 3.2) that, for type III_1 automorphisms, a truncated limit of Maharam (Theorem 3.1) is in fact equal to the Hurewicz-Halmos-Oxtoby limit. (Theorem 3.2 has also been obtained independently by D. Maharam (unpublished).) The last section studies other related truncated limits.

Henceforth (X, B, μ) will denote a σ-finite measure space; sometimes we shall simply write X or (X, μ) to denote this space. A *non-singular automorphism* of (X, B, μ) is an invertible transformation T such that A is measurable if and only if TA is measurable and A is null if and only if TA is null. For any integer $n, \mu T^n$ is a measure and there exist Radon-Nikodym derivatives $\omega_n(x) = d\mu T^n/d\mu(x)$. We usually write ω_1 as ω. One can show that the following relation holds a.e.:

$$\omega_{i+j}(x) = \omega_j(x)\ \omega_i(T^j x). \tag{1.1}$$

A non-null set W is said to be *wandering* if $T^{-n}W \cap W = \varnothing$ for $n > 0$. An automorphism T is *conservative* (or *incompressible*) if it

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1980 *Subject Classification*: 28D.
Supported in part by a Williams College Faculty Research Grant.
Received by the editors on August 23, 1985 and in revised form on September 2, 1987.
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admits no wandering sets; it is *measure preserving* if $\mu T^{-1}A = \mu A$, for all measurable sets A, and it is *ergodic* if, whenever A is T-*invariant*, i.e., $T^{-1}A = A$, then $A = \varnothing \pmod 0$ or $A = X \pmod 0$.

Our results are in the spirit of obtaining ergodic theorems for non-singular transformations from the Hopf ergodic theorem for measure preserving transformations, by an appropriate use of the Maharam skew product (cf. [**6**]). For completeness we state the classical results that are used, and sketch a proof of the Hopf ergodic theorem.

We introduce some notation to be used throughout. All functions are measurable by definition or by construction. For a function f and integer n, write

$$f^{(n)}(x) = \sum_{i=0}^{n} f(T^i x)\omega_i(x)$$

(Note that, when T is measure preserving, $f^{(n)}(x) = \sum_{i=0}^{n} f(T^i x)$.)

LEMMA 1.1. (MAXIMAL ERGODIC LEMMA). *Let T be a measure preserving automorphism of a σ-finite measure space X, and f be such that f^+ or f^- is integrable in X. If $E = \cup_{n=0}^{\infty}\{x : f^{(n)}(x) > 0\}$, then*

$$\int_E f\,d\mu \geq 0.$$

PROOF. An elementary proof for $f \in L^1$ can be found in [**2**]. (Jones assumes that $\mu X < \infty$ but one can easily adapt the proof to the infinite σ-finite case. If f^- is integrable choose a non-decreasing sequence of integrable g_k converging to f^+. Let $f_k = g_k - f^-$; then $f_k \in L^1$ and they converge monotonically to f. Let $E_k = \cup_{n=0}^{\infty}\{x : f_k^{(n)}(x) > 0\}$ and $E = \lim E_k$ (the E_k are monotone). Then $\chi_{E_k} \to \chi_E$. By the part already proved we have $\int_{E_k} f_k d\mu \geq 0$, which, by taking limits as $k \to \infty$, gives the desired result. The case when f^+ is integrable is similar. □

COROLLARY 1.1. *Let T be a measure preserving automorphism of a σ-finite measure space X. Let f be integrable and g measurable and non-negative. If $E_\alpha = \cup_{n=0}^{\infty}\{x : f^{(n)}(x) > \alpha g^{(n)}(x)\}$, then*

$\int_{E\alpha} f\,d\mu \geq \alpha \int_{E\alpha} g\,d\mu$. If $E^\beta = \cup_{n=0}^{\infty}\{x : f^{(n)}(x) < \beta g^{(n)}(x)\}$, then $\int_{E^\beta} f\,d\mu \leq \beta \int_{E^\beta} g\,d\mu$.

The original form of the following theorem is due to Hopf, but we state Maharam's version of Halmos' improvement of the theorem (cf. [**1**, **6**]). Since, in many of our statements below, we are going to consider a pair of functions f, g such that f is integrable and g is measurable non-negative such that $g(x) = 0$ implies $f(x) = 0$, we decide to call such a pair (f, g) a *Hopf pair.*

THEOREM 1.1. (HOPF ERGODIC THEOREM). *Let T be a measure preserving automorphism of a σ-finite measure space and (f, g) a Hopf pair. If T is conservative, then*

$$h(x) = \lim_{n\to\infty} \sum_{i=0}^{n} f(T^i x) / \sum_{i=0}^{n} g(T^i x)$$

exists and is finite and invariant a.e. Furthermore, for every invariant set A with $\int_A g\,d\mu < \infty$,

$$\int_A f\,d\mu = \int_A h\; g\,d\mu.$$

PROOF. Let Z be the union of all T-invariant sets of positive measure where g vanishes. Clearly Z is invariant (and measurable). The theorem is obviously true for $z \in Z$ (with the convention $0/0 = 0$). Now restrict T to $X - Z$, here g does not vanish on any invariant set of positive measure (it is *invariantly positive*). When g is invariantly positive, the conservativity of T implies that $\lim_n g^{(n)}(x) = \infty$ a.e. [**1**, Theorem 3]. This gives that $\liminf f^{(n)}(x)/g^{(n)}(x)$ and $\limsup f^{(n)}(x)/g^{(n)}(x)$ are invariant and standard arguments deduce the theorem from Corollary 1.1. $\square$

The following theorem is originally due to Hurewicz but it is stated as improved by Halmos and Oxtoby (cf. [**1**]). One can obtain it from Lemma 1.1 and Corollary 1.1 which are also true for non-singular automorphisms.

REMARK 1.1. Lemma 1.1 for the case of non-singular automorphisms (in fact, even for non-singular endomorphisms) can be obtained from the Hopf maximal lemma for positive contractions (see, e.g., [**4**, p. 8]). One could then deduce the following Theorem 1.2 from the version of Corollary 1.1 for non-singular automorphisms in a similar way as Theorem 1.1.

THEOREM 1.2. (HUREWICZ-HALMOS-OXTOBY ERGODIC THEOREM) *Let T be a conservative non-singular automorphism of a σ-finite measure space and (f, g) be a Hopf pair. The following limit exists and is finite and T-invariant a.e.*:

$$h(x) = \lim_{n\to\infty} f^{(n)}(x)/g^{(n)}(x).$$

Furthermore, for every invariant set A with $\int_A g\,d\mu < \infty$,

$$\int_A f\,d\mu = \int_A h\ g\,d\mu.$$

The author is indebted to Dorothy Maharam for several helpful discussions and to the referee for suggesting several improvements. This work is based on part of the author's Ph.D. thesis written under the direction of Professor Maharam.

2. Some ergodic theorems. We begin this section by outlining the skew product introduced in [**6**]. Let T be a non-singular automorphism of X. Let $X^* = X \times \mathbf{R}^+$ and define T^* on X^* by $T^*(x, y) = (Tx, y/\omega(x))$. Let $\mu^* = \mu \times dy$, where dy is Lebesgue measure on $\mathbf{R}$, be the measure on the product X^*. It readily follows that T^* is a measure preserving automorphism of X^*. Also, from (1.1), it follows that

$$T^{*i}(x, y) = (T^i x, y/\omega_i(x)), \tag{2.1}$$

for all integers i. In the proofs below we shall make use of the following theorem.

THEOREM 2.1. [**6**] *The automorphism T^* is measure preserving on (X^*, μ^*). Furthermore, T^* is conservative if and only if T is conservative.*

Now we define some other products and transformations. One can show that they are all isomorphic to T^*, and thus have the same properties as T^*. In the proofs below we shall make use of the product that makes the proof more apparent. The product measure to be considered is indicated in each case. (Skew products isomorphic to T^* were discovered independently after [**6**] and have been used extensively—see, e.g., [**5**, **7**].)

$$\begin{aligned}
&T^+ : X \times \mathbf{R}, \mu \times e^{-y} dy \to X \times \mathbf{R}, \mu \times e^{-y} dy \\
&\qquad (x, y) \mapsto (Tx, y + \ln\, \omega(x)) \\
&T^- : X \times \mathbf{R}, \mu \times e^{y} dy \to X \times \mathbf{R}, \mu \times e^{y} dy \\
&\qquad (x, y) \mapsto (Tx, y - \ln\, \omega(x)) \\
&T^\alpha : X \times \mathbf{R}^+, \mu \times d(y^{1/\alpha}) \to X \times \mathbf{R}^+, \mu \times d(y^{1/\alpha}) \\
&\qquad (x, y) \mapsto (Tx, y/\omega^\alpha(x)) \quad (\alpha > 0)
\end{aligned}$$

THEOREM 2.2. *Let T be a conservative non-singular automorphism of a σ-finite measure space (X, μ), and let (f, g) be a Hopf pair. Then, for every $\alpha > 1$, $0 < \beta < 1$, and almost all $c > 0$, the following limits exist and are finite a.e.*:

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n} f(T^i x)\omega_i^\alpha(x) : \omega_i(x) < c}{\sum_{i=0}^{n} g(T^i x)\omega_i^\alpha(x) : \omega_i(x) < c}$$

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n} f(T^i x)\omega_i^\beta(x) : \omega_i(x) > c}{\sum_{i=0}^{n} g(T^i x)\omega_i^\beta(x) : \omega_i(x) > c}$$

(The sums are to be interpreted so that a term is included only when the condition after the colon is satisfied.)

PROOF. Without loss of generality we may assume that f is non-negative. Below α may be > 1 or < 1. Write

$$f_\alpha(x,y) = (1/y)\, f(x)\chi_{C\alpha}(x,y)$$
$$g_\alpha(x,y) = (1/y) g(x)\chi_{C\alpha}(x,y)$$
$$C_\alpha = X \times D_\alpha, \qquad D_\alpha = \begin{cases} (a,\infty) & \text{if } \alpha > 1 \\ (0,a) & \text{if } \alpha < 1 \text{ (some } a > 0). \end{cases}$$

It follows that f_α, g_α are finite and measurable, g_α is non-negative, and $g_\alpha(x,y) = 0$ implies $f_\alpha(x,y) = 0$. We show that f_α is integrable over $X \times \mathbf{R}^+$ with respect to the product measure $\mu \times d(y^{1/\alpha})$. This follows from Fubini's theorem, since

$$\int_{X\times\mathbf{R}^+} f_\alpha\, d(\mu \times y^{1/\alpha}) = \int_X f(x)\, d\mu(x) \int_{D\alpha} y^{-1}(1/\alpha) y^{1/\alpha-1}\, dy.$$

One notes that, for $\alpha > 1$, the second integral becomes $\int_a^\infty (\frac{1}{\alpha}) y^{1/\alpha-2}\, dy < \infty$. If $0 < \alpha < 1$ one obtains $\int_0^a (1/\alpha) y^{1/\alpha-2} dy < \infty$. The summability of f completes the proof. Now apply Theorem 1.1 to $X \times \mathbf{R}^+, T_\alpha, f_\alpha, g_\alpha$ to obtain that there exists a null set N^* in $X \times \mathbf{R}^+$ such that, for every $(x,y) \in X \times \mathbf{R}^+ - N^*$,

$$\frac{\sum_{i=0}^n f_\alpha(T_\alpha^i(x,y))}{\sum_{i=0}^n g_\alpha(T_\alpha^i(x,y))}$$

converges to a finite limit. One can choose y_0 such that, for almost all x, $(x,y_0) \notin N^*$. Note that $T_\alpha^i(x,y) = (T^i x, y/\omega_i^\alpha(x))$. For $\alpha > 1$, we have

$$f_\alpha^*(T_\alpha^{*^i}(x,y_0)) = \begin{cases} (1/y_0) f(T^i x)\omega_i^\alpha(x), & \text{if } \omega_i(x) < (y_0/a))^{1/\alpha}, \\ 0, & \text{otherwise.} \end{cases}$$

We have a similar expression for $g_\alpha(T_\alpha^i(x,y_0))$. Choose a so that $(y_0/a)^{1/\alpha} = c$. Then the above ratio becomes

$$\frac{(1/y_0)\sum_0^n f(T^i x)\omega_i^\alpha(x) : \omega_i(x) < c}{(1/y_0)\sum_0^n g(T^i x)\omega_i^\alpha(x) : \omega_i(x) < c,}$$

which gives the limit of part (a). The case of $0 < \alpha < 1$ is similar. □

REMARK 2.1. Notice that g in the statement of Theorem 1.1 is not required to be integrable so that g_α in the proof above need not be integrable. Therefore, by letting $g_\alpha(x,y) = (1/y^{1/\alpha})g(x)$, one can obtain the sum $\sum_0^n g(T^i)\omega_i(x)$ (no restriction on the ω_i) in the denominators of the limits above. Likewise, one could obtain $\sum_0^n g(T^i)\omega_i^\alpha(x)$ by taking $g_\alpha(x,y) = (1/y)g(x)$. This remark plays an important role in the theorems of §4.

We observe that the difficulty in obtaining the Hurewicz-Halmos-Oxtoby ergodic theorem from the Hopf ergodic theorem (Theorem 1.1) by means of skew products lies in that the measure of the space X^* is always infinite. One can only obtain expressions where the Radon-Nikodym derivatives are truncated at some constant c; the study of these expressions, which we call c-truncated Hurewicz ratios, as the constants $c \to \infty$ is taken up in §3 and §4. Now we obtain some other results in an attempt to overcome this difficulty.

Before proving the next theorem we recall the construction of the induced transformation of Kakutani [**3**]. Let T be a conservative automorphism of a σ-finite measure space (X,μ), and let A be a subset of X of positive measure. By Poincaré recurrence, for almost all $x \in A$ there exists a least positive integer $l(x)$ such that $T^{l(x)}x \in A$. The induced transformation T_A on A is defined by $T_A x = T^{l(x)}x$. Then T_A is a measurable transformation on A that preserves the induced measure on A whenever T is measure preserving. Furthermore, the function $l(x)$ is measurable.

THEOREM 2.3. *Let T be a conservative non-singular automorphism of (X,μ) and f an integrable function. There exists an integer-valued function $l(i,x)$, measurable in x for each i and with $l(i,x) \to \infty$ as $i \to \infty$ such that the following limit exists and is finite a.e.*:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^{l(i,x)}x)\omega_{l(i,x)}(x).$$

Furthermore, when T^* is ergodic (*i.e., T is type* III$_1$, *cf.* [**8**]) *the limit is* 0 *if $\mu X = \infty$, and if $\mu X < \infty$, then, for any $0 < c < 1$, one can choose $l(i,x)$ so that the limit equals $(c/\mu X)\int f\,d\mu$.*

PROOF. Let $f^*(x,y) = e^{-y}f(x)$ and let T^- be as defined before. Take any subset A of finite positive measure in $\mathbf{R}$ and write $A^* = X \times A$.

The induced automorphism $S^* = T_A^-$ is measure preserving on the space $(A^*, \mu \times e^y dy)$. Furthermore, it can be readily checked that $f^* \in L^1(A^*)$. Note that

$$S^*(x,y) = (T^-)^{n(x,y)}(x,y)$$

for some integer-valued measurable function $n(x,y)$. The exponent for S^{*2} in T^- is $n(x,y) + n(T^{n(x,y)}(x),\, y - \ln\, \omega_{n(x,y)}(x))$. Let $p(i,x,y)$ be the exponent for S^{*^i}, i.e.,

$$S^{*^i}(x,y) = (T^-)^{p(i,x,y)}(x,y);$$

then $p(i,x,y)$ goes to ∞ with i. Now observe that

$$f^*(S^{*^i}(x,y)) = e^{-y} f(T^{p(i,x,y)}x)\omega_{p(i,x,y)}(x).$$

The Birkhoff ergodic theorem applied to S^*, f^* (restricted to A^*) implies that there exists a set $B^* = A^*(\bmod 0)$ such that, for all $(x,y) \in B^*$, the following limit exists and is finite:

$$\lim_{n->\infty}(1/n)\sum_0^{n-1} f^*(S^{*^i}(x,y)).$$

One can choose y_0 so that, for μ-a.e. $x \in X$, $(x,y_0) \in B^*$. Put $l(i,x) = p(i,x,y_0)$. Then $f^*(S^{*^i}(x,y_0)) = f^*(T^{l(i,x)}x,\ y_0\text{-}\ln \omega_{l(i,x)}(x))$ $= e^{-y_0}\omega_{l(i,x)}(x) f(T^{l(i,x)}x)$. It follows that the limit above gives the desired result.

When T^* is ergodic, then the limit above will equal

$$\left(\lambda A\Big/\int_A e^y dy\right)(1/\mu X)\int f\, d\mu.$$

If $A = (0,a), a > 0$, then $\left(\lambda A/\int_A e^y\, dy\right)$ ranges between 0 and 1. □

Finally we mention the following result whose interest lies in the fact that it can be deduced from Theorem 1.1.

THEOREM 2.4. *Let T be a conservative non-singular automorphism of (X,μ), and let (f,g) be a Hopf pair. Then there exists a positive function $\gamma(x,i)$ such that the following limit exists and is finite a.e.*:

$$\lim_{n\to\infty}\sum_{i=0}^{n} f(T^i x)\omega_i(x)\gamma(x,i)\Big/\sum_{i=0}^{n} g(T^i x)\omega_i(x)\gamma(x,i),$$

PROOF. In order to show that we can obtain this proof from Theorem 1.1 we recall briefly a well-known technique (cf. [**1**]) for deriving the Hurewicz theorem from the Hopf theorem in the case when T admits an invariant measure ν equivalent to μ. Let $h = d\mu/d\nu$; then one can see that $\omega_i(x) = h(T^i x)/h(x)$. Now fh is ν-summable and thus Theorem 1.1 implies the existence of the limit

$$\lim_{n\to\infty}\sum_{i=0}^{n} f(T^i x)h(T^i x)\Big/\sum_{i=0}^{n} g(T^i x)h(T^i x).$$

Dividing both terms by $h(x)$ one obtains the limit of Theorem 1.2.

Now the proof of the theorem. Let $f^*(x,y) = (1/y)f(x), g^*(x,y) = (1/y)g(x)$. We define a measure on X^* equivalent to $\mu\times\lambda$ so that f^* is integrable with respect to this measure. Let ν on $\mathbf{R}^+$ be given by $\nu(A) = \nu_1(A\cap I) + \nu_2(A\cap I^c)$, where I is the unit interval, I^c its complement and $\nu_1(y) = y\,dy$ (so that $\int_0^1 (1/y)d\nu_1 < \infty$), $\nu_2(y) = (1/y)dy$ (so that $\int_1^\infty (1/y)d\nu_2 < \infty$). Let $\nu^* = \mu\times\nu$. Then ν^* is equivalent to μ^* and f^* is ν^*-summable. Now consider $X^*, T^*, f^*, g^*, \nu^*$ and note that even though T^* does not preserve ν^* it does admit an invariant measure (namely μ^*) equivalent to ν^*. Therefore the remark at the beginning of the proof applies and we obtain the existence of the limit

$$\frac{\sum f^*({T^*}^i(x,y))h({T^*}^i(x,y))}{\sum g^*({T^*}^i(x,y))h({T^*}^i(x,y))},$$

where $h = d\nu^*/d\mu^*$. The proof is completed by noting that $f^*({T^*}^i(x,y)) = 1/y\ \omega_i(x)f(T^i x)$ and letting $\gamma(x,i) = h({T^*}^i(x,y_0))/h(x,y_0)$, where y_0 is chosen as in the proof of Theorem 2.2. □

3. The limit of a truncated Hurewicz ratio. In [**6**] Maharam showed the existence of the following limit and asked if one can obtain the Hurewicz-Halmos-Oxtoby ergodic theorem from this limit as m goes to ∞:

$$\lim_{n\to\infty} \frac{\sum_{i=0}^{n} f(T^i x)\omega_i(x) : \omega_i(x) \in C}{\sum_{i=0}^{n} g(T^i x)\omega_i(x) : \omega_i(x) \in C}$$

(where $C = [y_0/m, y_0 m]$, for some constant $y_0 > 0$, and (f, g) is a Hopf pair).

In this section we show that, in the case of type III_1 transformations, this limit equals the limit of the Hurewicz-Halmos-Oxtoby theorem. To this end we first go back to the space X^* where we make use of the Hopf ergodic theorem. We introduce notation:

$$\omega_i^m(x, y) = \omega_i(x)\chi_{[1/m,m]}(y/\omega_i(x)) \qquad (m > 1)$$

$$f_m(x, y) = (1/y)f(x)\chi_{[1/m,m]}(y)$$

It follows that

$$f_m^{(n)}(x, y) = \sum_{i=0}^{n} f_m(T^{*i}(x, y)) = (1/y)\sum_{i=0}^{n} f(T^i x)\omega_i^m(x, y).$$

Remark 3.1. The notation above is in agreement with the fact that, when T is measure preserving and h is any function, $h^{(n)}(x)$ denotes $\sum_{i=0}^{n} h(T^i x)$. It is important to note that in $f_m^{(n)}(x, y)$ one is applying this operation to the function f_m, which perhaps should be denoted by f_m^*. Also, without loss of generality, we may and do assume that all functions in §3 and §4 are non-negative.

From the proof of Theorem 3 (c) in [**6**] one obtains the following result (cf. proof of Theorem 2.2).

Theorem 3.1. [**6**] *Let T be a conservative non-singular automorphism of a σ-finite measure space (X, μ) and let (f, g) be a Hopf pair. The following limit exists and is finite a.e. in X^*:*

$$\lim_{n\to\infty} f_m^{(n)}(x, y)/g_m^{(n)}(x, y).$$

Denote this limit by $\{f/g\}_m(x, y)$.

The following well-known technical result will be used in several of the proofs below.

LEMMA 3.1. *If T is a conservative non-singular automorphism and $g \geq 0$, then $\lim_{n\to\infty} g(x)/g^{(n)}(x) = 0$ and $\lim_{n\to\infty} g(x)/g^{(n-1)}(Tx) = 0$.*

PROOF. Let Z be the union of all invariant sets of positive measure where g vanishes. On Z the limit above is 0 (under the convention $0/0 = 0$). On $X - Z, g$ is invariantly positive and hence

$$\lim_{n\to\infty} g^{(n)}(x) = \infty \quad \text{and} \quad \lim_{n\to\infty} g^{(n-1)}(Tx) = \infty \quad [\mathbf{1}, \text{Theorem 3}].\square$$

COROLLARY 3.1. *If T is conservative and $g \geq 0$, then*

$$\lim_{n\to\infty} g(x)\chi_{[1/m,m]}(y)/g_m^{(n-1)}(T^*(x,y)) = 0.$$

PROOF. Apply Lemma 3.1 to T^* and $g_m(x,y) = (1/y)g(x)\chi_{[1/m,m]}(y)$. $\square$

LEMMA 3.2. *If T^* is conservative then the function $\{f/g\}_m$ is T^*-invariant a.e.*

PROOF. Let χ denote $\chi_{[1/m,m]}$. We have

$$f_m^{(n)}(x,y)/\omega(x) = (1/y\omega(x)) \sum_{i=0}^{n} f(T^i x)\omega_i^m(x,y)$$
$$= (1/y\omega(x)) \left[f(x)\chi(y) + \sum_{i=0}^{n-1} f(T^{i+1}x)\omega_{i+1}^m(x,y) \right].$$

Since

$$\omega_{i+1}^m(x,y)/\omega(x) = \omega_i(Tx)\chi(y/\omega_{i+1}(x)),$$

we have

$$\frac{f_m^{(n)}(x,y)}{g_m^{(n)}(x,y)}$$
$$= \frac{f(x)\chi(y)/\omega(x) + \sum_{i=0}^{n-1} f(T^{i+1}x)\omega_i(Tx)\chi(y/\omega_{i+1}(x))}{g(x)\chi(y)/\omega(x) + \sum_{i=0}^{n-1} g(T^{i+1}x)\omega_i(Tx)\chi(y/\omega_{i+1}(x))}. \tag{3.1}$$

Observe that

$$\omega_i(Tx)\chi(y/\omega_{i+1}(x)) = \omega_i(Tx)\chi(y/\omega(x)\omega_i(Tx)) = \omega_i^m(Tx, y/\omega(x)).$$

Therefore, a general term in one of the sums above has the form

$$f(T^i(Tx))\omega_i^m(Tx, y/\omega(x)).$$

Now we note that

$$\begin{aligned} f_m^{(n-1)}(T^*(x,y)) &= f_m^{(n-1)}(Tx, y/\omega(x)) \\ &= (1/y)\sum_{i=0}^{n-1} f(T^i(Tx))\omega_i^m(Tx, y/\omega(x)). \end{aligned}$$

So, after replacing and simplifying in (3.1), one obtains

$$\begin{aligned} &\frac{f_m^{(n)}(x,y)}{g_m^{(n)}(x,y)} \\ (3.2) \qquad &= \frac{f_m^{(n-1)}(T^*(x,y))(y + f(x)\chi(y)/\omega(x)f_m^{(n-1)}(T^*(x,y)))}{g_m^{(n-1)}(T^*(x,y))(y + g(x)\chi(y)/\omega(x)g_m^{(n-1)}(T^*(x,y)))}. \end{aligned}$$

Since T^* is conservative, Corollary 3.1 gives

$$\begin{aligned} &\lim_{n\to\infty} f(x)\chi(y)/f_m^{(n-1)}(T^*(x,y)) \\ &\quad = \lim_{n\to\infty} g(x)\chi(y)/g_m^{(n-1)}(T^*(x,y)) = 0. \end{aligned}$$

It follows from (3.2) that $\{f/g\}_m$ is T^*-invariant. □

We now give an application of Lemma 3.2 to the case when T^* is ergodic. Before stating the next theorem we introduce some additional notation. Recall that

$$\omega_i^m(x,y) = \omega_i(x)\chi_{[1/m,m]}(y/\omega_i(x)) = \omega_i(x)\chi_C(\omega_i(x)),$$

where $C = [y/m, ym]$. When it is clear from context that y has been fixed, we omit writing the dependence on y and write simply $\omega_i^m(x)$ for $\omega_i^m(x,y)$. For a given function f defined on X, write

$$f_m^{(n)}(x) = \sum_{i=0}^{n} f(T^i x)\omega_i^m(x).$$

Note that $f_m^{(n)}(x,y) = (1/y)\ f_m^{(n)}(x)$ and that, for fixed y_0,

$$f_m^{(n)}(x,y_0)/g_m^{(n)}(x,y_0) = f_m^{(n)}(x)/g_m^{(n)}(x).$$

As mentioned in the introduction, Theorem 3.2 has been obtained independently by D. Maharam.

THEOREM 3.2. *Let T be a conservative non-singular automorphism of non-atomic σ-finite measure space (X,μ), and let (f,g) be a Hopf pair such that g is integrable. If T is type* III_1, *then*

$$\lim_{n\to\infty} f_m^{(n)}(x)/g_m^{(n)}(x) = \int_X f(x)d\mu(x) \Big/ \int_X g(x)\, d\mu(x).$$

PROOF. Since T^* is conservative ergodic, $\{f/g\}_m(x,y)$ is constant for each fixed m (Lemma 3.2). Then, since $\int_{X^*} g_m^* < \infty$, the Hopf ergodic theorem applied to T^* obtains

$$\int_{X^*} f_m^* d\mu^* = \int_{X^*} \{f/g\}_m\ g_m^*\ d\mu^*.$$

This gives $\{f/g\}_m(x,y) = \int_X f(x)d\mu(x)/\int_X g(x)d\mu(x)$, for all m and y; then one chooses y_0 as in Theorem 2.2. □

4. The limit of another truncated Hurewicz ratio. In this section we investigate another truncated ratio. We use this new ratio to define a class of transformations which we call of Hurewicz type. This class includes those transformations admiting an invariant measure (type II), but does not include type III_1 transformations. If it turns out Hurewicz type is the same as type II we would have a new characterization of transformations admiting an invariant measure. Finally, for Hurewicz type transformations, we show that the Hurewicz-Halmos-Oxtoby ergodic theorem can be deduced from the Hopf ergodic theorem.

Since we are not able to prove Theorem 4.2 below in the context of X^* (and hence in the context of measure preserving automorphisms), we are forced to use the version of Corollary 1.1 for non-singular automorphisms. However, to use notation already introduced in §3, we prove Theorem 4.1 and Lemma 4.2 in X^* although they are used in the proof of Theorem 4.2 only when restricted to X.

Our starting point is a slight modification of Theorem 3(c) in [**6**]. The modification is obtained by redefining g^* in the proof of Theorem 3(c) of [**6**] by $g^*(x,y) = (1/y)g(x)$ (cf. proof of Theorem 2.2 and Remark 2.1).

Thus, for any function g, write $g_*(x,y) = (1/y)g(x)$ and

$$g_*^{(n)}(x,y) = \sum_{i=0}^{n} g_*(T^{*^i}(x,y)) = (1/y)\sum_{i=0}^{n} g(T^i x)\omega_i(x).$$

Hence $g_*^{(n)}(x,y) = (1/y)g^{(n)}(x)$.

THEOREM 4.1. *Let T be a conservative automorphism of a σ-finite measure space and (f,g) a Hopf pair. The following limit exists and is finite for all (x,y) outside a null set N:*

$$\lim_{n\to\infty} f_m^{(n)}(x,y)/g_*^{(n)}(x,y).$$

Furthermore, we can assume the set NT^*-invariant and fixed from now on. Denote this limit by $[f/g]_m(x,y)$.

LEMMA 4.1. *For every $(x,y) \notin N$ and $m > 1$,*

$$[f/g]_m(x,y) = [f/g]_m(T^*(x,y)) \text{ a.e.}$$

PROOF. Let χ denote $\chi_{[1/m,m]}$. As in the proof of Lemma 3.2 we have

$$\frac{f_m^{(n)}(x,y)}{g_*^{(n)}(x,y)} = \frac{f(x)\chi(y)/\omega(x) + \sum_{i=0}^{n-1} f(T^{i+1}x)\omega_i(Tx)\chi(y/\omega_{i+1}(x))}{g(x)\chi(y)/\omega(x) + \sum_{i=0}^{n-1} g(T^{i+1}x)\omega_i(Tx)}$$

and, therefore,

$$\frac{f_m^{(n)}(x,y)}{g_*^{(n)}(x,y)}$$

$$(4.1) \qquad = \frac{f_m^{(n-1)}(T^*(x,y))(y + f(x)\chi(y)/\omega(x) f_m^{(n-1)}(T^*(x,y)))}{g_*^{(n-1)}(T^*(x,y))(y + g(x)\chi(y)/\omega(x) g_*^{(n-1)}(T^*(x,y)))}.$$

From Lemma 3.1 and Corollary 3.1,

$$\lim_{n\to\infty} f(x)\chi(y)/\omega(x)f_m^{(n-1)}(T^*(x,y))$$

$$= \lim_{n\to\infty} g(x)\chi(y)/\omega(x)g_*^{(n-1)}(T^*(x,y)) = 0,$$

which gives T^*-invariance. □

Remark 4.1. As mentioned before, the results above are used when restricted to X. Now we restate them in this new context. We can choose y_0 almost arbitrarily so that Theorem 4.1 and Lemmas 4.1 hold for all x outside a null set N_0 in X. Note that, according to our notation,

$$f_m^{(n)}(x) = y_0 f_m^{(n)}(x, y_0)$$
$$g^{(n)}(x) = y_0 g_*^{(n)}(x, y_0).$$

Therefore, for fixed y_0,

$$f_m^{(n)}(x, y_0)/g_*^{(n)}(x, y_0) = f_m^{(n)}(x)/g^{(n)}(x).$$

It follows that the following limit exists and is finite a.e.:

$$[f/g]_m(x) = \lim_{n\to\infty} f_m^{(n)}(x)/g^{(n)}(x)$$

(for some fixed y_0).

We now need a technical lemma.

Lemma 4.2.

$$\lim_{m\to\infty}[f/g]_m(Tx, y_0/\omega(x)) = \lim_{m\to\infty}[f/g]_m(Tx, y_0).$$

Proof. Recall that

$$f_m^{(n)}(x,y) = (1/y)\sum_{i=0}^{n} f(T^i x)\omega_i(x)\chi(y/\omega_i(x)),$$

(where $\chi = \chi_{[1/m,m]}$). Define

$$\begin{aligned} f^{(n)}_{\langle m\rangle}(Tx,y) &= (\omega(x)/y)\sum_{i=0}^{n} f(T^{i+1}x)\omega_i(Tx)\chi(y/\omega(x)\omega_i(Tx)) \\ &= f^{(n)}_m(T_x, y/\omega(x)) \end{aligned}$$

It suffices to show

$$\begin{aligned} &\lim_{m\to\infty}\lim_{n\to\infty} f^{(n)}_{\langle m\rangle}(Tx,y_0)/g^{(n)}_*(Tx,y_0) \\ &\quad = \lim_{m\to\infty}\lim_{n\to\infty} \omega(x) f^{(n)}_m(Tx,y_0)/g^{(n)}_*(Tx,y_0). \end{aligned}$$

This equality is a consequence of the following inequalities (Note that the denominators in the two limits above are the same.):

(a) If $\omega(x) \geq 1$, then

$$\omega(x) f^{(n)}_{m/\omega(x)}(Tx,y) \leq f^{(n)}_{\langle m\rangle}(Tx,y) \leq \omega(x) f^{(n)}_{m\omega(x)}(Tx,y).$$

(b) If $\omega(x) < 1$, then

$$\omega(x) f^{(n)}_{m\omega(x)}(Tx,y) \leq f^{(n)}_{\langle m\rangle}(Tx,y) \leq \omega(x) f^{(n)}_{m/\omega(x)}(Tx,y).$$

The inequalities can be established directly from the definitions. We prove one inequality as an illustration. Assume $\omega(x) \geq 1$. Suppose $y/m\omega(x) < \omega_i(Tx) < ym/\omega(x)$. Then

$$y/m\omega(x) < \omega_i(Tx) < ym\omega(x),$$

which means $f^{(n)}_{\langle m\rangle}(Tx,y) \leq \omega(x) f^{(n)}_{m\omega(x)}(Tx,y)$. □

Lemma 4.3. *Write* $[f/g](x) = \lim_{m\to\infty}[f/g]_m(x,y_0)$. *Then* $[f/g](x)$ *is* T*-invariant.*

Proof. We have

$$\begin{aligned} \lim_{m\to\infty}[f/g]_m(x) &= \lim_{m\to\infty}[f/g]_m(x,y_0) \\ &= \lim_{m\to\infty}[f/g]_m(T^*(x,y_0)) = \lim_{m\to\infty}[f/g]_m(Tx,y_0/\omega(x)) \\ &= \lim_{m\to\infty}[f/g]_m(Tx,y_0) = \lim_{m\to\infty}[f/g]_m(Tx). \quad \square \end{aligned}$$

A transformation T is said to be of *Hurewicz type* if, for every integrable $f > 0$,

$$h_T(x) = \lim_{m\to\infty} \lim_{n\to\infty} f_m^{(n)}(x)/f^{(n)}(x) > 0 \text{ a.e.}$$

Since $\lim_{n\to\infty} f_m^{(n)}(x)/f^{(n)}(x)$ is non-decreasing in m and $f_m^{(n)}(x)/f^{(n)}(x) \le 1$, the double limit above always exists.) It is immediate that when T is measure preserving then $h_T = 1$.

LEMMA 4.4. *Suppose T is ergodic. If T admits an invariant measure, then it is of Hurewicz type.*

PROOF. Let ν be a T-invariant measure equivalent to μ and write $k(x) = d\nu/d\mu(x)$. Let $A^* = \{(x,y) : k(x) < y < 2k(x)\}$. Then A^* is T^*-invariant since $k(Tx)\omega(x) = k(x)$. Let $f_*(x,y) = (1/y)f(x)$, $f_m(x,y) = (1/y)f(x)\chi_{[1/m,m]}(y)$. One finds that $\int_{A^*} f_* d\mu \times \lambda = \ln 2(\int_X f\, d\mu) < \infty$. Hence Theorem 1.1 applied to T^*, f_m, f_* when restricted to A^* gives that

$$h_T(x,y,m) = \lim_{n\to\infty} \sum_{i=0}^{n} f(T^i x)\omega_i^m(x) \Big/ \sum_{i=0}^{n} f(T^i x)\omega_i(x)$$

exists and is finite a.e. and

$$0 < \int_{A^*} f_m d\mu \times \lambda = \int_{A^*} h_T f_*\, d\mu \times \lambda.$$

It follows that $h_T(x,y,m) > 0$ on a set of positive measure in X^* and for all $m > 1$. Hence there exists y_0 so that $h_T(x, y_0, m) > 0$ on a set of positive measure in X. By Lemma 4.3 $h_T(x)$ is T-invariant, hence constant. Since $h_T(x) = \lim_{m\to\infty} h_T(x, y_0, m)$, where the limit is non-decreasing in m and h_T constant, then $h_T > 0$ a.e. □

LEMMA 4.5. *If T is type III_1, then $h_T = 0$ and so is not of Hurewicz type.*

Proof. We have seen that $\lim_{n\to\infty} f_m^{(n)}(x,y)/f_*^{(n)}(x,y)$ is T^*-invariant. Thus, in a way similar to that in the proof of Theorem 3.2, since T^* is ergodic and $\int_{X^*} f_*\, d\mu\times\lambda = \infty$, the Hopf ergodic theorem implies $\lim_{n\to\infty} f_m^{(n)}(x,y)/f_*^{(n)}(x,y) = 0$. □

The following Theorem 4.2 could be deduced from Theorem 1.2; however we use it below (Theorem 4.3) to prove Theorem 1.2. One can also obtain Theorem 4.2 if one assumes that $\lim_{m\to\infty}\lim_{n\to\infty} f_m^{(n)}(x)/g_m^{(n)}(x)$ exists. This follows from the equality

$$\left(g_m^{(n)}(x)\Big/g^{(n)}(x)\right)\left(f_m^{(n)}(x)\Big/g_m^{(n)}(x)\right) = f_m^{(n)}(x)\Big/g^{(n)}(x).$$

If the converse of Lemma 4.4 is true, then Theorem 4.3 below is trivial. However, one would then have a new characterization of transformations admiting an invariant measure. In view of these results it would be of interest to investigate further the properties of Hurewicz type transformations.

Theorem 4.2. *Let T be a conservative automorphism of a σ-finite measure space (X,μ) and (f,g) be a Hopf pair. The following limit exists and is finite and T-invariant a.e.*:

$$[f/g](x) = \lim_{m\to\infty}\lim_{n\to\infty} f_m^{(n)}(x)/g^{(n)}(x).$$

Proof. It suffices to prove this for g invariantly positive and $f \geq 0$. Recall that $[f/g](x) = \lim_{m\to\infty}[f/g]_m(x,y_0)$ and $[f/g]_m(x,y)$ is finite for almost all (x,y) and a non-decreasing function of m. Thus it suffices to show it is bounded. Write $A = \{x : [f/g](x) = \infty\}$. Note that $A = \{x : \sup_m \lim_n f_m^{(n)}(x)/g^{(n)}(x) = \infty\}$. Since $f^{(n)}(x)/g^{(n)}(x) \geq f_m^{(n)}(x)/g^{(n)}(x)$ for all m, n, if $B = \{x : \lim_n f^{(n)}(x)/g^{(n)}(x) = \infty\}$ then $A \subset B$. Let $E_\alpha = \{x : \sup_n f^{(n)}(x)/g^{(n)}(x) > \alpha\} = \cup_{n=0}^{\infty}\{x : f^{(n)}(x) > \alpha g^{(n)}(x)\}$. Then $A \subset B \subset E_\alpha$ for all $\alpha > 0$. By Lemma 4.3, A is T-invariant. By applying the version of Corollary 1.1 for non-singular transformations (cf. Remark 1.1) to T restricted to A, one obtains

$$\int_{E_\alpha\cap A} f\, d\mu \geq \alpha\int_{E_\alpha\cap A} g\, d\mu,$$

which gives $\int_A f\, d\mu \geq \alpha \int_A g\, d\mu$, for all $\alpha > 0$. But since f is integrable and g invariantly positive, $\mu A = 0$. □

THEOREM 4.3. *If T is of Hurewicz type, then the Hurewicz-Halmos-Oxtoby theorem can be deduced from the Hopf ergodic theorem.*

PROOF. This follows from the following equality and Theorem 4.2.

$$\frac{f_m^{(n)}(x)}{f^{(n)}(x)}\frac{f^{(n)}(x)}{g^{(n)}(x)} = \frac{f_m^{(n)}(x)}{g^{(n)}(x)}. \quad \square$$

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