ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 1, Winter 1990

THE METRIZATION OF UPPER LIMIT TOPOLOGIES

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1. Introduction. A fruitful source of examples in general topology arises from certain linearly ordered sets endowed with the upper limit topology (e.g., the reals and various spaces derived from the ordinal space $[0, \Omega]$). Their usefulness stems in part from the fact that, although these particular spaces are not metrizable, each possesses many properties necessary for metrization.

In this note, we deduce the non-metrizability of these spaces from a result which characterizes metrizability of the upper limit topology on an arbitrary totally ordered set. The method is both elementary and direct—elementary in that it uses only the definition of the order topology and simple topological properties and direct in that it does not require the use of auxiliary spaces such as the Cartesian square.

The author wishes to thank Art Kruse for his perspicacity in general and for his suggestions regarding Theorem 4 and Corollary 5 in particular.

2. Preliminaries. Let $(X, <)$ be a totally ordered set. The upper limit topology induced by $<$ is the one, a basis for which is

$$\{(a, b] \,|\, a, b \in X, a < b\}.$$

Following Dugundji [**1**, p. 66], we call the resulting space X_u. Suppose d is a metric on the ordered set X; its topology need not be that of X_u.

For $\epsilon > 0$ and $y \in X$, we adopt the notation $N_d(y, \epsilon)$ for

$$\{x \in X \,|\, d(y, x) < \epsilon\}.$$

Now define $f : X \to \mathbf{R} \cup \{\infty\}$ by the rule

$$f(y) = \begin{cases} \text{glb}\{d(x, y) \,|\, y < x\}, & \text{if } (y, \infty) \neq \varnothing, \\ \infty, & \text{otherwise.} \end{cases}$$

Received by the editors on April 1, 1985 and in revised form on March 3, 1986.

Viewing f as a function from X_u to $\mathbf{R} \cup \{\infty\}$, we assert

THEOREM 1. *If d metrizes X_u, then f is positive and upper semicontinuous.*

PROOF. To see that f is positive, choose $y \in X$; since d metrizes X_u, there is $\epsilon > 0$ so that

$$N_d(y, \epsilon) \subset \{x \mid x \leq y\}.$$

Then, for $x > y$, $x \notin N_d(y, \epsilon)$, so $d(x, y) \geq \epsilon$ and

$$\text{glb}\{d(x, y) \mid y < x\} \geq \epsilon.$$

To see that f is upper semicontinuous we appeal to

LEMMA 2. *With the assumption of Theorem* 1, *let $\epsilon > 0$ and $y, w \in X$ so that*

$$(w, y] \subset N_d(y, \epsilon).$$

Then, for $z \in (w, y)$, $f(z) < \epsilon$. (Note that (w, y) might be empty.)

PROOF. For $z \in (w, y)$, in order to show that $f(z) < \epsilon$, it suffices to show there is a point $x > z$ with $d(x, z) < \epsilon$. Take $x = y$. □

Now returning to the proof of Theorem 1 for $b \in \mathbf{R} \cup \{\infty\}$, we set

$$U_b = \{x \in X \mid f(x) < b\};$$

we must show that U_b is open. If $b \leq 0$, U_b is empty. If $b = \infty$, $U_b = X$ or X has a greatest element y_0 and $U_b = \cup_{y<y_0}\{x \mid x \leq y\}$. Thus let $0 < b < \infty$. Choose $y \in U_b$. Since d is a metric for X_u, there is a w so that

$$(w, y] \subset N_d(y, b).$$

Apply Lemma 2 to conclude that

$$(w, y] \subset U_b.$$

□

DEFINITION 3. With $(X, <), d$ and f as above, for $n > 0$, define the closed set

$$F_n = \{y \in X \mid f(y) \geq 1/n\}.$$

The family of sets $\{F_n \mid n > 0\}$ is called the *filtration* of X associated to d, or, more briefly, the filtration of d.

Note that if d metrizes X_u, f is positive, the F_n are closed and

$$X = \cup_n F_n.$$

3. A necessary and sufficient condition.

THEOREM 4. *The space X_u is metrizable if and only if it is the union of countably many discrete, closed subspaces.*

PROOF. Suppose that d metrizes X_u. In light of the foregoing discussion we only need to show that each element of the filtration of d is a discrete subspace of X_u. Suppose, for some M, F_M has a limit point, say x_0. Metrizability implies that there is a w so that

$$(w, x_0] \subset N_d(x_0, 1/M).$$

Since x_0 is a limit point of F_M there is a point $y \in (w, x_0) \cap F_M$. Now, by construction, $f(y) \geq 1/M$, but, by Lemma 2, $f(y) < 1/M$.

To demonstrate sufficiency we proceed as follows. Let

$$\{X_n \mid n = 1, 2, \dots\}$$

be a sequence of discrete, closed subspaces of X_u, the union of which is X. Let

$$E = \{\epsilon_n \mid n = 1, 2, \dots\}$$

be a non-increasing sequence of positive, real numbers which converges to 0. Define

$$\varphi : X \to E$$

by the rule

$$\varphi(x) = \epsilon_k,$$

where k is the smallest integer for which $x \in X_k$. Define

$$d : X \times X \to \mathbf{R}$$

by the rule

$$d(x,y) = \begin{cases} \max\{\varphi(z) \,|\, x \leq z < y\}, & x < y, \\ 0, & x = y, \\ d(y,x), & x > y. \end{cases}$$

It is easy to check that d satisfies

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}$$

for x, y and z in X so that d is a metric on X. If $(a,b]$ is given,

$$N_d(b,\epsilon) \subset (a,b],$$

provided that $\epsilon < \min\{d(a,b), \varphi(b)\}$. If $N_d(x,\epsilon)$ is given, let K be such that $\epsilon_k < \epsilon$ for $k > K$. Then there is a point $w < x$ (if x is not the least element of X) so that $(w,x) \cap (X_1 \cup \cdots \cup X_K) = \varnothing$. Then

$$(w,x] \subset N_d(x,\epsilon).$$

Thus d metrizes X_u. □

COROLLARY 5. *If X is uncountable and X_u is separable, then X_u is not metrizable.*

PROOF. Suppose X_u is metrized by d. Then each subspace F_n in the filtration of d is separable metrizable and, by Theorem 4, is discrete. Hence each F_n is countable, and so

$$X = \cup F_n$$

is countable, contrary to hypothesis. □

As a consequence the real line with the upper limit topology (dubbed E^1_{bad} by R.H. Bing) is not metrizable. The fastest classical way to show that E^1_{bad} is not metrizable is, of course, to note that

$$E^1_{\text{bad}} \times E^1_{\text{bad}},$$

the Sorgenfrey plane [**2**], is not normal.

It is of interest to note that (regardless of whether X is uncountable) if X_u is separable, then X_u can be homeomorphically order-embedded in E^1_{bad}. This suggests Corollary 6 below.

COROLLARY 6. *Each uncountable subspace of* E^1_{bad} *is non-metrizable.*

PROOF. Consider any uncountable set Y in $\mathbf{R}$. Let

$$S = \{y \in Y \bigm| \text{there is a } z \in \mathbf{R},\ z < y \text{ and } (z, y] \cap Y \text{ countable}\}.$$

Along standard lines of reasoning, S is countable. Let $X = Y \backslash S$; X is uncountable. Straightforwardly, the relativization to X of the topology of E^1_{bad} is the topology of X_u, which, by Corollary 5, is not metrizable. Thus, Y as a topological subspace of E^1_{bad}, has the non-metrizable subspace X_u and is non-metrizable.

COROLLARY 7. *The spaces* $[0, \Omega]$ *and* $[0, \Omega)$ *in the order topology are not metrizable.*

PROOF. In each case the order topology is the upper limit topology. If either space were metrizable, some filtration element would contain a countable infinity of points and hence a limit point. □

Of course one can dismiss metrizability for $[0, \Omega]$ by simply noting that it is not 1^{st} countable at Ω, but $[0, \Omega)$ is not susceptible to such an elementary argument.

REFERENCES

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