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THE ROTATIONS OF $\ell(\phi_{\mathbf{n}})$

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ABSTRACT. A characterization of the linear rotations of a general class of metric linear spaces is given. Sufficient conditions are given for all the rotations of these spaces to be linear.

1. Introduction. Let (p_n) be a sequence of real numbers with $0 < p_n \leq 1$. The linear space $\ell(p)$ is defined [**2**] to be the collection of all real (or complex) sequences (x_n) for which $\sum_{n=1}^{\infty} |x_n|^{p_n}$ is finite. It is a complete metric linear space with the metric given by

$$d(x,y) = \sum_{n=1}^{\infty} |x_n - y_n|^{p_n}. \tag{1.1}$$

A surjective mapping T (not necessarily linear) of a metric linear space (X,d) is a rotation [**1, 2**] if $T(0) = 0$ and $d(Tx,Ty) = d(x,y)$. In [**2**] Maddox asks for a description of the rotations of $\ell(p)$. This question provided the motivation for the present paper.

In this paper we describe a broad class of metric linear spaces, denoted by $\ell(\phi_n)$, which include $\ell(p)$. We characterize, in terms of the action on the space, the linear rotations of $\ell(\phi_n)$. In general it is not known if a rotation of an arbitrary metric linear space is a linear transformation. In fact, in spaces over the complex field this need not be the case. However, for metric linear spaces over the real field, sufficient conditions for rotations to be linear are known [**2**]. In §2 we indicate which of the $\ell(\phi_n)$ spaces satisfies these conditions. Finally, we note that our results answer the question of Maddox except for $\inf p_n = 0$.

2. The spaces $\ell(\phi_{\mathbf{n}})$. Let (ϕ_n) be a sequence of continuous real valued functions defined on $[0,\infty)$ such that $\phi_n(0) = 0, \phi_n(1) = 1, \phi_n(\cdot)$ is increasing, and ${\phi_n}'(\cdot)$ is decreasing.

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REMARK. These conditions are chosen in order to insure that the following lemma will hold. Examples of such functions are easy to construct. Specific examples are: $\phi_n(x) = x^{p_n}, 0 < p_n \leq 1, \phi_n(x) = x/\log(q+x)$, and $\phi_n(x) = (1+n)x/(1+nx)$.

LEMMA 2.1. *Let ϕ be a real valued function defined on $[0,\infty)$ such that $\phi(0) = 0$, $\phi(\cdot)$ is increasing, and $\phi'(\cdot)$ is decreasing. Then, for every pair of complex numbers z and w,*

$$\phi(|z-w|) \leq \phi(|z|) + \phi(|w|), \tag{2.2}$$

and equality holds if and only if $|z|\,|w| = 0$ or both $|z-w| = |z| + |w|$ and ϕ' are constant on the interval from 0 to $|z| + |w|$.

The space $\ell(\phi_n)$ is the linear space of all real (or complex) sequences (x_n) for which $\sum_{n=1}^{\infty} \phi_n(|x_n|)$ is finite. It is straightforward to verify that $\ell(\phi_n)$ is a complete metric linear space with metric given by

$$d(x,y) = \sum_{n=1}^{\infty} \phi_n(|x_n - y_n|). \tag{2.3}$$

The sequences e_n with a 1 in the n^{th} position and zeros elsewhere form a Schauder basis. This fact is important because it implies each continuous linear transformation on $\ell(\phi_n)$ has a matrix representation.

In the terminology of Rolewicz [**3**], $\ell(\phi_n)$ is an F^* space with F norm given by $||x|| = d(x,0)$. Theorem IX 3.1 of Rolewicz [**3**] implies that, whenever an $\ell(\phi_n)$ space is locally bounded, the rotations of the space are linear. The next result gives some conditions under which $\ell(\phi_n)$ is locally bounded

PROPOSITION 2.4. *Let $r_n(t) = \inf\{a > 0 \,|\, \phi_n(at) > 1/2\phi_n(t)\}$, $r_n = \inf_{0<t<\infty} r_n(t)$, and $r = \inf_{n>1} r_n$. If $r > 0$, then $\ell(\phi_n)$ is locally bounded.*

PROOF. By the definition of r and the continuity of the ϕ_n, it follows that

$$\phi_n(rt) \leq \frac{1}{2}\phi_n(t) \quad \text{for } n \geq 1 \text{ and } t > 0. \tag{2.5}$$

Hence, $d(rx, 0) \leq (1/2)\, d(x, 0)$ for each $x \in \ell(\phi_n)$. We claim that if $\delta > 0$, then the set $K_\delta = \{x \,|\, d(x, 0) < \delta\}$ is bounded. For, suppose $\varepsilon > 0$ and $U_\varepsilon = \{x \,|\, d(x, 0) < \varepsilon\}$. Choose n so large that $\delta/2^n < \varepsilon$. Then

$$d(r^n x, 0) \leq \frac{1}{2^n} d(x, 0) < \frac{\delta}{2^n} < \varepsilon. \tag{2.6}$$

Thus, $K_\delta C r^{-n} U_\varepsilon$, and we have shown that K_δ is bounded. Therefore, $\ell(\phi_n)$ is locally bounded.

COROLLARY 2.7. *$\ell(p)$ is locally bounded whenever* $\inf p_n > 0$.

REMARK. It is known [**3**] that, for $\inf p_n = 0$, $\ell(p)$ is not locally bounded.

3. The rotations of $\ell(\phi_{\mathbf{n}})$.

LEMMA 3.1. *Let T be a rotation of $\ell(\phi_n)$ and let (x_n) and (y_n) be sequences in $\ell(\phi_n)$ such that*

$$x_n \cdot y_n = 0. \tag{3.2}$$

Then

$$(Tx)_n (Ty)_n = 0. \tag{3.3}$$

PROOF. Let (x_n) and (y_n) satisfy (3.2). Then, by definition of the metric, we have

$$d(x, y) = d(x, 0) + d(y, 0). \tag{3.4}$$

Since T is a rotation, $T(0) = 0$ and T preserves the metric. Hence,

$$d(Tx, Ty) = d(Tx, 0) + d(Ty, 0). \tag{3.5}$$

Thus

$$0 = \sum_{n=1}^{\infty} [\phi_n(|(Tx)_n|) + \phi_n(|(Ty)_n|) - \phi_n(|(Tx)_n - (Ty)_n|)]. \tag{3.6}$$

By 2.1, the summands are all nonnegative and

$$\phi_n(|(Tx)_n - (Ty)_n|) = \phi_n(|(Tx)_n|) + \phi_n(|(Ty)_n|) \tag{3.7}$$

for each n. This remains valid on replacing y by $-y$. By 2.1,

$$(Tx)_n \cdot (Ty)_n = 0 \quad \text{for} \quad n = 1, 2 \dots . \tag{3.8}$$

This completes the proof of the lemma. □

THEOREM 3.9. *T is a surjective linear isometry of $\ell(\phi_n)$ if and only if there is a permutation σ of the positive integers, and a sequence of scalars (t_n) such that*

$$(Tx)_n = t_n x_{\sigma(n)} \quad \textit{and} \quad \phi_n(|t_n|) = 1 \quad \textit{and} \quad \phi_{\sigma(n)} = \phi_n \tag{3.10}$$

for each n.

PROOF. Suppose T is a surjective linear isometry of $\ell(\phi_n)$. Let (t_{ij}) be the matrix representation of T relative to the basis e_n. For $k \neq j$, the vectors e_k and e_j satisfy the hypothesis of Lemma 3.1. Hence,

$$(Te_k)_i \cdot (Te_j)_i = t_{ik} t_{ij} = 0 \quad \text{for } i = 1, 2, 3 \dots . \tag{3.11}$$

From (3.11) it follows that each row of (t_{ij}) has at most one non zero element. Since T is surjective, each row must have at least one non zero element and, therefore, has precisely one. We define the mapping σ as follows. Since, for each n, there is a unique n' for which $t_{nn'} \neq 0$, we define $\sigma(n) = n'$. If we let $t_n = t_{n\sigma(n)}$, then the action of T on the space is given by

$$(Tx)_n = t_n x_{\sigma(n)}. \tag{3.12}$$

We claim σ is one to one. For suppose there exist distinct positive integers n and m such that $\sigma(n) = \sigma(m)$. This implies that, for every sequence in the range of T,

$$y_n = \frac{t_n y_m}{t_m}. \tag{3.13}$$

Since the t_n's depend only on the basis, this relation is impossible if T is to be surjective. Therefore, σ is one to one. The surjectivity of σ follows from (3.12) and the injectivity of T.

To see that $\phi_n(|t_n|) = 1$ for every n, note that $d(Te_n, 0) = d(e_n, 0) = 1$ and use the fact that $\phi_n(1) = 1$. Finally, since $d(Tx, Ty) = d(x, y)$ for every x and y, it follows that $\phi_{\sigma(n)}(\cdot) = \phi_n(\cdot)$.

We omit the proof of the sufficiency. □

COROLLARY 3.14. *Let T be a rotation of a real $\ell(\phi_n)$ space. If the hypotheses of Proposition* (2.4) *are satisfied, then every rotation of the space is given by* (3.10). *In particular, every rotation of $\ell(p)$, when* $\inf p_n > 0$ *and the scalars are real, is given by* (3.10).

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