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COUNTING FINITE SUBSETS OF AN IMMUNE SET

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ABSTRACT. Let P be a recursive property of finite sets (of nonnegative integers) and ν an immune set of RET (i.e., recursive equivalence type) N. Consider the question, "How many finite subsets of ν have property P?" We shall answer this question if P has the additional property that $P(\alpha)$ if and only if $P(\beta)$, for every two finite sets α and β of the same cardinality.

1. Preliminaries. We use the word *number* for nonnegative integer, *set* for collection of numbers and *class* for collection of sets. The set of all numbers and the empty set are denoted by ε and o respectively. V stands for the class of all sets, Q for the class of all finite sets, $\subset$ for inclusion and $\subset_+$ for proper inclusion. If f is a function, δf and ρf denote its domain and range respectively. Also, f_n means the same as $f(n)$. The cardinality of a collection Γ is denoted by card(Γ) or card Γ. The reader is assumed to be familiar with the basic properties of the collection Λ of all isols. For a survey of basic results see §1 of [**2**]; for a detailed exposition, see [**4**] or [**5**]. We write $\alpha \sim \beta$ for α is equivalent to β, i.e., $\operatorname{card}\alpha = \operatorname{card}\beta$ and $\alpha \simeq \beta$ for α is recursively equivalent to β, i.e., $\operatorname{Req}(\alpha) = \operatorname{Req}(\beta)$. Some properties of combinatorial operators will be used; these are discussed in [**1**], [**4**] and [**5**]. The class Q of all finite sets can be effectively generated without repetitions in an infinite sequence. We shall use a particular sequence of this type, the so-called *canonical enumeration* $\langle \rho_n \rangle$; see [**2**, p. 277]. For each finite set σ there is exactly one number i such that $\sigma = \rho_i$; this number is called the *canonical index* of σ and is denoted by can(σ) or can σ. If $S \subset Q$ we write can S for $\{\operatorname{can}\sigma | \sigma \in S\}$. The function $r_n = \operatorname{card}\rho_n$ is recursive. For $n, k \in \varepsilon$ and $\alpha \in V$,

$$\nu_n = \{x \in \varepsilon \,|\, x < n\}, \qquad 2^\alpha = \{x \in \varepsilon \,|\, \rho_x \subset \alpha\},$$
$$[\alpha; k] = \{x \in 2^\alpha \,|\, r_x = k\}, \qquad \int [n; k] = \operatorname{card}[\nu_n; k].$$

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In displayed formulas $[n; k]$ will also be written in the usual vertical way. Henceforth "*property* P" will only be used if P is a property of finite sets. We call a property P *recursive*, if the property T_P of numbers such that $T_P(x) \iff P(\rho_x)$ is recursive. For a property P and a set ν,

(1) $S_P(\nu) = \{\rho_x \in Q \mid \rho_x \subset \nu \text{ and } P(\rho_x)\}$,

(2) $\Phi_P(\nu) = \{x \in \varepsilon \mid \rho_x \subset \nu \text{ and } P(\rho_x)\}$, i.e., $\Phi_P(\nu) = \text{can}\, S_P(\nu)$,

(3) $\text{RET}\, S_P(\nu) = \text{Req}\, \Phi_P(\nu)$.

For a finite set ν with n elements, $\text{RET}\, S_P(\nu)$ is the number of (finite) subsets of ν with property P. If the property P is recursive and the set ν isolated, we wish to count the finite subsets of ν with property P, i.e., to express $\text{RET}\, S_P(\nu)$ in terms of $N = \text{Req}\, \nu$. We mention two examples. Let $a_1, \dots, a_p,\ b_1, \dots, b_q$ be $p + q$ distinct elements of an isolated set ν with $\text{RET}\, N$. Let a finite set have property P_0, if it contains each of $a_1, \dots, a_p$, but none of $b_1, \dots, b_q$. How many finite subsets of ν have property P_0? Since a subset ρ_x of ν has property P_0 if and only if $\rho_x = (a_1, \dots, a_p) \cup \rho_y$, where $\rho_y \subset \nu - (a_1, \dots, a_p, b_1, \dots, b_q)$, the answer is $2^{N-(p+q)}$. Note that whether ρ_x has property P_0 does not only depend on r_x. Suppose a finite set ρ_x has property E if r_x is even, while ρ_x has property O if r_x is odd. Thus E and O depend only on $r_x = \text{card}\, \rho_x$. For a nonempty isolated set ν with $\text{RET}\, N$ we have $\text{RET}\, S_E(\nu) = \text{RET}\, S_0(\nu) = 2^{N-1}$. This can be proved as follows. Let $p \in \nu$; from now on we keep p fixed. Define the function h by $\delta h = 2^\nu$, $h(x) = x - 2^p$ if $p \in \rho_x$ and $h(x) = x + 2^p$ if $p \notin \rho_x$. Then h maps $\text{can}\, S_E(\nu)$ onto $\text{can}\, S_0(\nu)$ and h has a partial recursive one-to-one extension. Thus the sets $\text{can}\, S_E(\nu)$ and $\text{can}\, S_0(\nu)$ are recursively equivalent. Since they are also separable and their union is 2^ν, we see that

$$\text{RET}\, S_E(\nu) = \text{RET}\, S_0(\nu) = \frac{1}{2} \cdot 2^N = 2^{N-1}.$$

Definition. A property P is ODC (*only dependent on cardinality*), if $r_x = r_y$ implies $P(\rho_x) \iff P(\rho_y)$, for $x, y \in \varepsilon$, or, equivalently, if there is a property H of numbers such that $P(\rho_x) \iff H(r_x)$, for $x \in \varepsilon$.

If P is an ODC property, we denote the property H of numbers mentioned above by P^*. Clearly, P is recursive if and only if P^* is recursive.

2. Three propositions. For a property P we define

$$\Phi_P^{\varepsilon} = \cup\{\Phi_P(\alpha) \mid \alpha \in V\},$$

$$\Phi_P^{o}(\alpha) = \{x \in \varepsilon \mid \rho_x = \alpha \text{ and } P(\rho_x)\}, \quad \text{for } \alpha \in Q.$$

PROPOSITION P1. *The following conditions are mutually equivalent*:

(a) *the operator* Φ_P *from* V *into* V *is combinatorial*,

(b) *the operator* Φ_P^o *from* Q *into* Q *is dispersive*,

(c) *property* P *is* ODC.

PROOF. Both Φ_P and Φ_P^o map finite sets onto finite sets. Also,

$$x \in \Phi_P(\alpha) \iff \rho_x \subset \alpha, \quad \text{for } x \in \Phi_P^{\varepsilon},\ \alpha \in V,$$

$$\alpha \neq \beta \Rightarrow \Phi_P^o(\alpha) \cap \Phi_P^o(\beta) = o, \quad \text{for } \alpha, \beta \in Q.$$

Using the definitions and basic properties of combinatorial and dispersive operators [**1**, pp. 7–12] we see that, for $\alpha, \beta \in Q$,

$$\begin{aligned} \text{(a)} \quad &\iff (\forall\alpha)(\forall\beta)[\alpha \sim \beta \Rightarrow \{\Phi_P(\alpha) \sim \Phi_P(\beta)\}] \\ &\iff (\forall\alpha)(\forall\beta)[\alpha \sim \beta \Rightarrow \{\Phi_P^o(\alpha) \sim \Phi_P^o(\beta)\}] \iff \quad \text{(b)}. \end{aligned}$$

For $\alpha \in Q$ we have $\operatorname{card} \Phi_P^o(\alpha) \in (0, 1)$, so that, for $\alpha, \beta \in Q$,

$$\begin{aligned} &(\forall\alpha)(\forall\beta)[\alpha \sim \beta \Rightarrow \{\Phi_P^o(\alpha) \sim \Phi_P^o(\beta)\}] \iff \\ &(\forall\alpha)(\forall\beta)[\alpha \sim \beta \Rightarrow \{P(\alpha) \iff P(\beta)\}] \iff \quad \text{(c)}. \end{aligned}$$

For a property P and a combinatorial or dispersive operator Φ we define

$$f_\Phi(n) = \operatorname{card} \Phi(\nu_n), \quad f_P(n) = \operatorname{card} \Phi_P(\nu_n), \quad \text{for } n \in \varepsilon.$$

The functions f_Φ and f_P are the functions *induced* by Φ and P respectively. The combinatorial or dispersive operators Φ and Ψ are *equivalent* (written $\Phi \operatorname{eq} \Psi$) if $f_\Phi = f_\Psi$. With every function f from ε into ε there is associated a unique sequence $\langle c_i \rangle$ of integers such that $f(n) = \sum_{i=0}^{n} c_i[n;i]$; it is called the sequence of *combinatorial coefficients* of f.

PROPOSITION P2. *Let Φ be a combinatorial operator, $f = f_\Phi$, and $\langle c_i \rangle$ the sequence of combinatorial coefficients of f (hence $0 \le c_i$, for $i \in \varepsilon$). Then the following conditions are mutually equivalent*:

(a) $\Phi \operatorname{eq} \Phi_P$, *for some* ODC *property* P,

(b) $f = f_P$, *for some* ODC *property* P,

(c) $0 \le c_i \le 1$, for $i \in \varepsilon$.

PROOF. (a) $\Longleftrightarrow$ (b) is true since $\Phi \operatorname{eq} \Phi_P$ means the same as $f = f_P$. Now assume (b). Then the operator

$$\begin{aligned}\Phi_P^o(\alpha) &= \{x \in \varepsilon \mid \rho_x = \alpha \text{ and } P(\rho_x)\}\\ &= \{x \in \varepsilon \mid \rho_x = \alpha \text{ and } P^*(r_x)\}, \quad \text{for } \alpha \in Q,\end{aligned}$$

is dispersive by Proposition P1 and induces c_i by [**1**; P14, P18], so that

$$c_i = \operatorname{card} \Phi_P^o(\nu_i) = \begin{cases} 1, & \text{if } P^*(i) \text{ is true,} \\ 0, & \text{if } P^*(i) \text{ is false.} \end{cases}$$

This implies (c). Now assume (c). Define a property P by $P(\rho_x) \Longleftrightarrow [c_{r(x)} = 1]$; then P is ODC and

$$\begin{aligned}&\Phi_P(\alpha) = \{x \in \varepsilon | \rho_x \subset \alpha \text{ and } P(\rho_x)\} = \{x \in \varepsilon | \rho_x \subset \alpha \text{ and } c_{r(x)} = 1\},\\ &f_P(n) = \operatorname{card} \Phi_P(\nu) = \sum_{i=0}^{n} c_i \binom{n}{i} = f(n).\end{aligned}$$

Hence (b) is true. □

REMARK. Assume the hypothesis of Proposition P2. Then the combinatorial operator Φ is not uniquely determined by f; see [**1**, P16]. On the other hand, it follows from our proof of (b) $\Longleftrightarrow$ (c) that if $f = f_P$ (or equivalently $\Phi \operatorname{eq} \Phi_P$) for some ODC property P, then

$$P(\rho_x) \iff P^*(r_x) \iff [c_{r(x)} = 1], \quad \text{for } x \in \varepsilon,$$

so that P is uniquely determined by c_i, hence also by f (and Φ).

Let f be a combinatorial function and $\langle c_i \rangle$ the sequence of its combinatorial coefficients. Then we define

$$\Psi(\alpha) = \{j(x,y) \in \varepsilon | \rho_x \subset \alpha \text{ and } y < c_{r(x)}\}, \quad \text{for } \alpha \in V. \tag{4}$$

We are only interested in the case where the following mutually equivalent conditions are satisfied: f is a recursive function, c_i is a recursive function, Ψ is a recursive combinatorial operator. It follows by [**1**, Remark on p. 51] that, for every recursive combinatorial operator Φ which induces f, we have, for $\alpha, \beta \in V$,

$$\begin{gathered} \Phi(\alpha) \simeq \Psi(\alpha), \qquad \alpha \simeq \beta \iff \Phi(\alpha) \simeq \Phi(\beta), \\ \alpha \text{ isolated } \Rightarrow \Phi(\alpha) \text{ isolated.} \end{gathered} \tag{5}$$

We can now define

$$f_\Lambda(N) = \operatorname{Req} \Phi(\nu), \quad \text{for } \nu \in N,\ N \in \Lambda. \tag{6}$$

This is *Myhill's canonical extension* of the function f from ε into ε to a function from Λ into Λ.

DEFINITION. For a recursive ODC property P,

$$F_P(N) = \operatorname{Req} \Phi_P(\nu), \quad \text{for } \nu \in N,\ N \in \Lambda. \tag{7}$$

If we count finite subsets of an isolated set ν with RET N using isols rather than cardinals $\leq \aleph_0$ we obtain that there are $F_P(N)$ finite subsets of ν with the recursive ODC property P.

PROPOSITION P3. *Let P be a recursive* ODC *property and $f = f_P$. Then $F_P(N) = f_\Lambda(N)$ for $N \in \Lambda$.*

PROOF. Under the hypothesis, Φ_P is a recursive combinatorial operator. The desired relation now follows from (6) and (7).

COROLLARY. *Let P be a recursive* ODC *property. Then $F_P(N) \leq 2^N$, for $N \in \Lambda$.*

PROOF. Let $\nu \in N$ and $N \in \Lambda$. Then the sets $\Phi_P(\nu)$ and $2^\nu - \Phi_P(\nu)$ are separable, hence $\mathrm{Req}\,\Phi_P(\nu) \leq \mathrm{Req}\,2^\nu$, i.e., $F_P(N) \leq 2^N$.

3. Miscellaneous remarks.

(A) Proposition P3 can be generalized to k-ary relations. The case $k = 2$ is as follows. Call a relation R between finite sets *recursive* if the relation $T_R(x, y)$ between numbers such that $R(\rho_x, \rho_y) \iff T_R(x, y)$ is recursive. Relation R is ODC, if

$$[r_x = r_u \text{ and } r_y = r_v] \Rightarrow [R(\rho_x, \rho_y) \iff R(\rho_u, \rho_v)], \quad \text{for} \quad x, y, u, v \in \varepsilon.$$

Define Φ_R, f_R and F_R by

$$\begin{aligned} &\Phi_R(\mu, \nu) = \{j(x, y) \in \varepsilon \mid \rho_x \subset \mu \text{ and } \rho_y \subset \nu \text{ and } R(\rho_x, \rho_y)\}, \mu, \nu \in V, \\ &f_R(m, n) = \mathrm{card}\,\Phi_R(\nu_m, \nu_n), \\ &F_R(M, N) = \mathrm{Req}\,\Phi_R(\mu, \nu), \text{ for } \mu \in M, \nu \in N, \end{aligned}$$

If the ODC relation R is recursive and $f = f_R$, we have $F_R(M, N) = f_\Lambda(M, N)$, for $M, N \in \Lambda$. The proof will be deleted since it is similar to that for $k = 1$. Here is a simple example. How many ordered pairs $\langle \rho_x, \rho_y \rangle$ of finite sets are there with $\rho_x \subset \mu$, $\rho_y \subset \nu$, $r_x = r_y$, $\mu \in M$, $\nu \in N$ and $M, N \in \Lambda$? In this case

$$f(m, n) = \mathrm{card}\,\Phi_R(\nu_m, \nu_n) = \sum_{i=0}^{m} \sum_{k=0}^{n} c_{ik} \binom{m}{i} \binom{n}{k}.$$

We know from combinatorics that, for $s = \min(m, n)$,

$$\binom{m}{o}\binom{n}{o} + \cdots + \binom{m}{s}\binom{n}{s} = \frac{(m+n)!}{m!\,n!}.$$

Since the function c_{ik} is uniquely determined by f, we see that c_{ik} is Kronecker's delta function. Hence the "number" of ordered pairs satisfying the requirements is

$$F_R(M, N) = \frac{(M+N)!}{M!\,N!}, \quad \text{for } M, N \in \Lambda.$$

The analogue of the corollary of P3 is *for every recursive* ODC *relation* R *we have*

$$F_R(M,N) \le 2^{M+N}, \quad \text{for } M,N \in \Lambda.$$

(B) The following question deals with a recursive relation between finite sets which is not ODC. Let ν be an isolated set with RET N. How many ordered pairs are there of disjoint finite subsets of ν? We refer to [**2**, p. 292] for the definition of the recursive function $r_n(x) = r(n,x)$ and the sets $\delta_e r_n$ and $\rho_e r_n$. For RET 's A and B with $A > 0$, A^B can be defined as $\mathrm{Req}\, \alpha^\beta$, where

$$\alpha^\beta = \{n \in \varepsilon \,|\, \delta_e r_n \subset \beta \text{ and } \rho_e r_n \subset \alpha\}, \quad \text{for } 0 \in \alpha, \alpha \in A, \beta \in B.$$

Let $\gamma = (0,1,2)^\nu$. Then $\gamma \simeq \{j(x,y) | \rho_x, \rho_y \subset \nu \text{ and } \rho_x \rho_y = o\}$. Hence the desired RET is $\mathrm{Req}\, \gamma = 3^N$.

(C) Let us take ordered k-tuples $\langle \alpha_1, \ldots, \alpha_k \rangle$ with $k \ge 2$ of finite subsets of an isolated set ν with RET N, instead of ordered pairs. Then the question discussed in (B) can be generalized in (at least) two ways: (a) by requiring that $\alpha_1, \ldots, \alpha_k$ be mutually disjoint and (b) by requiring that the intersection of *all* k sets $\alpha_1, \ldots, \alpha_k$ be empty. Case (a) can be dealt with as its special case $k = 2$. The answer is $(k+1)^N$. Now consider case (b). Then the answer is $(2^k - 1)^N$. To prove this we generalize Stanley's proof of the case where ν is a finite set of cardinality n; see [**6**; Example 1.1.16, p. 12]. Let $T_{k\nu}$ denote the family of all functions r_m from ν into the class of all proper subsets of $(1, \ldots, k)$ and $S_{k\nu}$ the family of all ordered k-tuples $\langle \alpha_1, \ldots, \alpha_k \rangle$ of finite subsets of ν with $\alpha_1 \ldots \alpha_k = o$. Define

$$\alpha = \{x \in \varepsilon \,|\, \rho_x \subset_+ (1, \ldots, k)\}, \qquad \tau_{k\nu} = \alpha^\nu,$$

$$\sigma_{k\nu} = \{j_k \langle a_1, \ldots, a_k \rangle \in \varepsilon \,|\, a_1, \ldots, a_k \in 2^\nu \text{ and } \rho_{a(1)} \ldots \rho_{a(k)} = o\},$$

where j_k is a one-to-one recursive from ε^k onto ε. We use $\tau_{k\nu}$ and $\sigma_{k\nu}$ as the sets of G-numbers of the members of the families $T_{k\nu}$ and $S_{k\nu}$ respectively. Since $0 \in \alpha$ and $\mathrm{card}\, \alpha = 2^k - 1$, we have $\mathrm{Req}\, \tau_{k\nu} = (2^k - 1)^N$. Define the mapping ψ by $\delta\psi = \tau_{k\nu}$; if $m \in \delta\psi$, then $\psi(m) = j_k \langle y_1, \ldots, y_k \rangle$, where

$$i \in \rho_{y(j)} \iff j \in \rho_{r(m,i)}, \quad \text{for } i \in \nu,\ 1 \le j \le k.$$

It can now be proved that ψ maps $\tau_{k\nu}$ one-to-one onto $\sigma_{k\nu}$ and that both ψ and ψ^{-1} have partial recursive extensions. Thus $\tau_{k\nu} \simeq \sigma_{k\nu}$ and $\mathrm{Req}\,\sigma_{k\nu} = (2^k - 1)^N$.

(D) Let $P_1, \ldots, P_k$ be recursive properties of finite sets and ν an isolated set of RET N. Assume that the k sets $\{x \in 2^\nu \mid P_1(\rho_x)\}, \ldots, \{x \in 2^\nu \mid P_k(\rho_x)\}$ are *recursively distinct*, i.e., that they are distinct and we can, given any number $x \in 2^\nu$, find all numbers $i \in (1, \ldots, k)$ such that ρ_x has property P_i. Let $\overline{P}_i$ denote the negation of P_i and define

$$P'(\rho_x) = P_1(\rho_x) \vee \cdots \vee P_k(\rho_x), \quad P''(\rho_x) = \overline{P}_1(\rho_x) \& \cdots \& \overline{P}_k(\rho_x).$$

Let ν be an isolated set with RET N. Write $F'(N)$, $F''(N), F_i(N)$ for $F_{P'}(N), F_{P''}(N), F_{P(i)}(N)$ respectively. Using the inclusion-exclusion principle [**3**] we can express $F'(N)$ and $F''(N)$ in terms of $F_1(N), \ldots, F_k(N)$:

$$\begin{aligned} F'(N) &= \sum_{i=1}^{k} F_i(N) - \sum_{1 \le i < j \le k} F_i(N) F_j(N) + \cdots \\ &\qquad + (-1)^{k-1} F_1(N) \cdots F_k(N), \\ F''(N) &= 2^N - F'(N). \end{aligned}$$

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