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NÖRLUND AND WEIGHTED MEAN MATRICES AS OPERATORS ON l_p

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ABSTRACT. Let $a = \{a_n\}$ be a sequence of positive numbers with $a_n = f(n)$, for n large, where f is a logarithmico-exponential function defined on $[x_0, \infty)$ for some $x_0 > 0$. Let $N_a = \{a_{nk}\}$, where $a_{nk} = a_{n-k}/A_n$ $\left(A_n = \sum_{k=0}^n a_k\right)$ for $0 \le k \le n$ and 0 otherwise; let $W_a = \{a_{nk}\}$, where $a_{nk} = a_k/A_n$ for $0 \le k \le n$ and 0 otherwise. N_a is called a *Nörlund matrix* and W_a is called a *weighted mean matrix*. The principal results in the paper include:

(i) $N_a \in B(l_p)$ $(1 < p < \infty)$ if and only if $\alpha = \lim_{n\to\infty} na_n/A_n < \infty$; and

(ii) $W_a \in B(l_p)$ if and only if $\beta = \lim_{n\to\infty} A_n/na_n < p$. In each case estimates and asymptotic properties of the norms of the operators are obtained.

1. Introduction. Let $a = \{a_n\}$ be a sequence of *positive* numbers and let $A_n = \sum_{k=0}^n a_k$. Define the *Nörlund matrix* $N_a = \{a_{nk}\}$ by

$$a_{nk} = \begin{cases} a_{n-k}/A_n, & \text{for } 0 \le k \le n, \\ 0, & \text{for } k > n. \end{cases}$$

and the *weighted mean matrix* $W_a = \{a_{nk}\}$ by

$$a_{nk} = \begin{cases} a_k/A_n, & \text{for } 0 \le k \le n, \\ 0, & \text{for } k > n. \end{cases}$$

The N_a- or W_a- transform $y = \{y_k\}$ of a sequence $x = \{x_k\}$ is given by $y_n = \sum_{k=0}^n a_{nk}x_k$.

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In this paper we investigate conditions under which $y \in l_p$ whenever $x \in l_p$ $(1 < p < \infty)$, and we provide estimates of the norms of the resulting operators N_a and W_a. We are concerned with the situation in which, for sufficiently large $k, a_k = f(k)$, where f is a logarithmico-exponential function defined and positive for all sufficiently large values of the real variable x. In this case we shall say that *the sequence a is given by the function f.*

Logarithmico-exponential functions are discussed in [**3**]. Let L denote the set of all logarithmico-exponential functions defined on $[x_0, \infty)$ for some $x_0 > 0$. If $f \in L$, then f is eventually monotone, eventually of constant sign and $f' \in L$. We use the notation

$$g \preceq f, \text{ if } g(x) \leq Cf(x), \quad \text{for some } C > 0 \text{ and } x \text{ large},$$

$$g \prec f, \text{ if } \lim_{x \to \infty} g(x)/f(x) = 0.$$

$$g \asymp f, \text{ if } g \preceq f \text{ and } f \preceq g,$$

$$g \sim f, \text{ if } \lim_{x \to \infty} g(x)/f(x) = 1.$$

The symbols $\succeq$ and $\succ$ then have the obvious meaning.

We suppose throughout that $1 < p < \infty,\ 1/p + 1/q = 1$ and define

$$\sigma_1(n) = \sum_{k=0}^{n} a_{nk} \left(\frac{n+1}{k+1} \right)^{1/p},$$

$$\sigma_2(k) = \sum_{n=k}^{\infty} a_{nk} \left(\frac{k+1}{n+1} \right)^{1/q},$$

$$\sigma_1 = \sup_{n \geq 0} \sigma_1(n) \text{ and } \sigma_2 = \sup_{k \geq 0} \sigma_2(k).$$

Let $B(l_p)$ denote the Banach algebra of bounded linear operators on l_p and $||T||_p$ denote the norm of $T \in B(l_p)$.

2. Results. We establish the following results.

THEOREM 1. *Suppose a is given by f. Then* $N_a \in B(l_p)$ *if and only if* $\alpha = \lim_{n\to\infty} na_n/A_n < \infty$, *and then*

$$\frac{\Gamma(\alpha+1)\Gamma(1/q)}{\Gamma(\alpha+1/q)} \le ||N_a||_p \le \sigma_1^{1/q}\sigma_2^{1/p} < \infty \tag{1}$$

and $\lim_{n\to\infty}\sigma_1(n) = \lim_{k\to\infty}\sigma_2(k) = \Gamma(\alpha+1)\Gamma(1/q)/\Gamma(\alpha+1/q)$. *Further, if a is any sequence of positive numbers such that* $\alpha = \lim_{n\to\infty} na_n/A_n = 0$, *then* $N_a \in B(l_p)$ *and* (1) *holds.*

REMARKS. (a) That $\lim_{n\to\infty} na_n/A_n$ exists follows since $f \in L$ (see Lemma 2).

(b) When $||a||_1 = \sum_{k=0}^{\infty} a_k < \infty$, it follows readily from [**1**, Theorem 1] that $1 \le ||N_a||_p \le ||a||_1 \sup_{n\ge 0}(1/A_n)$.

THEOREM 2. *Suppose a is given by f. Then* $W_a \in B(l_p)$ *if and only if* $\beta = \lim_{n\to\infty} A_n/na_n < p$, *and then*

$$\frac{p}{p-\beta} \le ||W_a||_p \le \sigma_1^{1/q}\sigma_2^{1/p} < \infty \tag{2}$$

and $\lim_{n\to\infty}\sigma_1(n) = \lim_{k\to\infty}\sigma_2(k) = p/(p-\beta)$.

Let $l_p(N) = \{x = \{x_k\} \in l_p : x_0 = x_1 = \cdots = x_n = 0\}$, $N \in \mathbf{N}_0$, and let $||N_a||_p^{(N)}$ and $||W_a||_p^{(N)}$ denote the norms of the operators N_a and W_a in $B(l_p(N))$. Further, let

$$\sigma_1^{(N)}(n) = \sum_{k=N+1}^{n} a_{nk}\left[\frac{n+1}{k+1}\right]^{1/p}, \quad \sigma_1^{(N)} = \sup_{n\ge 0}\sigma_1^{(N)}(n),$$

$$\sigma_2^{(N)}(k) = \begin{cases} 0, & \text{for } 0 \le k \le N, \\ \sigma_2(k), & \text{for } k > N, \end{cases}$$

and $\sigma_2^{(N)} = \sup_{k\ge 0}\sigma_2^{(N)}(k)$. Note that $\sigma_1^{(N)} \le \sup_{n>N}\sigma_1(n)$ and $\sigma_2^{(N)} = \sup_{k>N}\sigma_2(k)$.

COROLLARY 1. *Let a be given by f, and suppose* $\lim_{n\to\infty} a_n/A_n = 0$. *Then* $N_a \in B(l_p(N))$ *if and only if* $\alpha = \lim_{n\to\infty} na_n/A_n < \infty$, *and then*

$$\frac{\Gamma(\alpha+1)\Gamma(1/q)}{\Gamma(\alpha+1/q)} \le ||N_a||_p^{(N)} \le (\sigma_1^{(N)})^{1/q}(\sigma_2^{(N)})^{1/p}.$$

Moreover,

$$\lim_{N\to\infty} ||N_a||_p^{(N)} = \Gamma(\alpha+1)\Gamma(1/q)\Gamma(\alpha+1/q). \tag{3}$$

Further, if a is given by f and $\lim_{n\to\infty} a_n/A_n > 0$, *then* $N_a \in B(l_p(N))$ *if and only if* $\sum_{n=0}^{\infty} \exp\{-p(N+1)f'(n)/f(n)\} < \infty$.

COROLLARY 2. *Let a be given by f. Then* $W_a \in B(l_p(N))$ *if and only if* $\beta = \lim_{n\to\infty} A_n/na_n < p$, *and then*

$$p/(p-\beta) \le ||W_a||_p^{(N)} \le (\sigma_1^{(N)})^{1/q}(\sigma_2^{(N)})^{1/p}.$$

Moreover,

$$\lim_{N\to\infty} ||W_a||_p^{(N)} = p/(p-\beta). \tag{4}$$

REMARK. If S is the shift operator that maps $\{x_k\}$ to $\{y_k\}$ where $y_0 = 0$ and $y_k = x_{k-1}$ for $k \ge 1$, then (3) and (4) may be written

$$\lim_{N\to\infty} ||N_a S^N||_p = \Gamma(\alpha+1)\Gamma(1/q)/\Gamma(\alpha+1/q)$$

and

$$\lim_{N\to\infty} ||W_a S^N||_p = p/(p-\beta)$$

respectively.

3. Lemmas and proofs. Our first lemma does not seem to be explicitly stated elsewhere in the literature.

LEMMA 1. *Suppose that* $g \in C_1[x_0, \infty)$, *for some* $x_0 > 0$, *and that* $g > 0$. *Let* $\epsilon = xg'/g$, *suppose that g and* ϵ *are monotonic on*

$[x_0, \infty)$ and that $\lim_{x\to\infty} \epsilon(x) = 0$. Then there is a function μ such that $\lim_{x\to\infty} \mu(x) = \infty$, $\lim_{x\to\infty} \mu(x)/x = 0$ and $\lim_{x\to\infty} g(x)/g(\mu(x)) = 1$. Moreover, if $g \in L$, then we may choose $\mu \in L$.

PROOF. Let $s = \lim_{x\to\infty} g(x)$. Then $0 \le s \le \infty$. If $0 < s < \infty$, we may take $\mu(x) = \sqrt{x}$. (Observe that $g(x)/g(\sqrt{x}) \to s/s = 1$.) The case when $s = 0$ follows from the case when $s = \infty$ with g replaced by $1/g$. If $s = \infty$, then $g(x)$ must increase. Thus $g' \ge 0$ and $\epsilon(x)$ must decrease (since $0 \le \epsilon(x) \to 0$). Define $\mu(x) = xe^{-1/\delta(x)}$, where $\delta(x) = \max\{2/\log x, \sqrt{\epsilon(\sqrt{x})}\} \to 0$. Clearly $\mu \prec x$ and $\mu \ge \sqrt{x} \to \infty$. Finally $0 \le g(x) - g(\mu(x)) = \int_{\mu(x)}^{x} \epsilon(t)g(t)/t\,dt \le g(x)\epsilon(\sqrt{x})\int_{\mu(x)}^{x} \frac{dt}{t} \le g(x)\sqrt{\epsilon(\sqrt{x})} = o(g(x))$ as $x \to \infty$. Hence $\lim_{x\to\infty} g(\mu(x))/g(x) = 1$. □

LEMMA 2. *Suppose $g \in L$, $g(x) > 0$, $xg'/g = o(1)$ as $x \to \infty$ and $0 < \alpha < \infty$, $0 < \nu < \infty$, $\delta < 1$. Then*

$$I(n) = \frac{1}{n+1}\sum_{k=0}^{n} \frac{g(k)}{g(n)}\left(\frac{k+1}{n+1}\right)^{\alpha-1}\left(1 - \frac{k}{n+1}\right)^{-\delta} \longrightarrow \frac{\Gamma(\alpha)\Gamma(1-\delta)}{\Gamma(\alpha+1-\delta)} \quad \text{as } n \to \infty,$$

and

$$J(k) = k^{\nu}\sum_{n=k}^{\infty} \frac{g(n-k)}{g(n)}(n+1)^{-\alpha-\nu}(n+1-k)^{\alpha-1} \longrightarrow \frac{\Gamma(\alpha)\Gamma(\nu)}{\Gamma(\alpha+\nu)} \quad \text{as } k \to \infty.$$

PROOF. Suppose first that $g \equiv 1$. Then

$$I(n) \leq (n+1)^{-\alpha} + \sum_{k=1}^{n-1} \int_k^{k+1} \left(\frac{t}{n+1}\right)^{-1} \left(\frac{t+1}{n+1}\right)^{\alpha} \left(1 - \frac{t}{n+1}\right)^{-1} \left(1 - \frac{t-1}{n+1}\right)^{1-\delta} \frac{dt}{n+1} + (n+1)^{\delta-1}$$

$$= o(1) + \int_{1/(n+1)}^{n/(n+1)} \tau^{\alpha-1} \left(1 + \frac{1}{\tau(n+1)}\right)^{\alpha} (1-\tau)^{-\delta} \left(1 + \frac{1}{(1-\tau)(n+1)}\right)^{1-\delta} d\tau$$

$$\longrightarrow \frac{\Gamma(\alpha)\Gamma(1-\delta)}{\Gamma(\alpha+1-\delta)} \quad \text{as } n \to \infty \quad \left(\text{with } \tau = \frac{t}{n+1}\right)$$

by Lebesgue's dominated convergence theorem. Also,

$$I(n) \geq (n+1)^{-\alpha} + \sum_{k=1}^{n-1} \int_k^{k+1} \left(\frac{t+1}{n+1}\right)^{-1} \left(\frac{t}{n+1}\right)^{\alpha} \left(1 - \frac{t-1}{n+1}\right)^{-1} \left(1 - \frac{t}{n+1}\right)^{1-\delta} \frac{dt}{n+1} + (n+1)^{\delta-1}$$

$$= o(1) + \int_{1/(n+1)}^{n/(n+1)} \tau^{\alpha-1} \left(1 + \frac{1}{\tau(n+1)}\right)^{-1} (1-\tau)^{-\delta} \left(1 + \frac{1}{(1-\tau)(n+1)}\right)^{-1} d\tau$$

$$\to \frac{\Gamma(\alpha\Gamma(1-\delta)}{\Gamma(\alpha+1-\delta)} \qquad \text{as } n \to \infty.$$

The assertion concerning $J(k)$, with $g \equiv 1$, is obtained similarly via the integral

$$\int_k^{\infty} t^{-\alpha-\nu}(t-k)^{\alpha-1}\, dt = k^{-\nu} \int_0^1 \tau^{\alpha-1}(1-\tau)^{\nu-1}\, d\tau = k^{-\nu} \frac{\tau(\alpha)\Gamma(\nu)}{\Gamma(\alpha+\nu)},$$

with $\tau = (t-k)/t$. Now suppose $g \in L$, $g(x) > 0$ and $xg'/g = o(1)$. Then, for all $\eta > 0$, $x^{-\eta} \prec g \prec x^{\eta}$. Hence $x^{\eta} g$ increases and $x^{-\eta} g$

decreases ultimately. Since, for any fixed N,

$$\frac{1}{n+1}\sum_{k=0}^{N}\frac{g(k)}{g(n)}\left(\frac{k+1}{n+1}\right)^{\alpha-1}\left(1-\frac{k}{n+1}\right)^{-\delta}\to 0 \quad \text{as } n\to\infty,$$

we obtain

$$\begin{aligned}\limsup_{n\to\infty} I(n) &\le \lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\left(\frac{k+1}{n+1}\right)^{\alpha-\eta-1}\left(1-\frac{1}{n+1}\right)^{-\delta}\\ &= \frac{\Gamma(\alpha-\eta)\Gamma(1-\delta)}{\Gamma(\alpha-\eta+1-\delta)}\end{aligned}$$

and

$$\liminf_{n\to\infty} I(n) \ge \frac{\Gamma(\alpha+\eta)\Gamma(1-\delta)}{\Gamma(\alpha+\eta+1-\delta)}$$

for all $0<\eta<\alpha$. The same reasoning may be applied to $J(k)$, and then the assertions of our lemma follow. □

Lemma 3 elucidates the relationship between $\lim_{x\to\infty} xf'(x)/f(x)$ and $\lim_{n\to\infty} na_n/A_n$.

LEMMA 3. *Suppose a is given by f and that*

$$\lim_{x\to\infty} xf'(x)/f(x) = \alpha - 1 \qquad (-\infty \le \alpha \le +\infty).$$

Then

$$\lim_{n\to\infty} na_n/A_n = \begin{cases}\alpha & \text{if } \alpha \ge 0\\ 0 & \text{if } \alpha < 0.\end{cases}$$

Conversely, if $\lim_{n\to\infty} na_n/A_n = \alpha > 0$, *then* $\lim_{x\to\infty} xf'(x)/f(x) = \alpha - 1$. (*Observe that* $\lim_{x\to\infty} xf'(x)/f(x)$ *always exists by monotonicity.*)

PROOF. If $\alpha=\infty$, then $f \succ x^{\Delta}$ for all $\Delta>0$ by [**3**, Theorem 23 with $v(x)=x$]. Moreover, $A_n \sim a_n$ or $A_n \asymp \int_1^n f(t)\,dt = F(n)$ by [**3**, Theorem 33]. Since $F(x) \le f(x)x^{-\Delta}\int_1^x t^{\Delta}\,dt + x_0(\Delta f(x) \le (xf(x))/(\Delta+1)+x_0 f(x)$, for all $\Delta>0$, we obtain that $na_n/A_n\to\infty$.

If $0 < \alpha < \infty$, then $f \sim x^{\alpha-1}g$ with $x^{-\delta} \prec g \prec x^{\delta}$ for all $\delta > 0$ by [**3**, Theorem 23] and $A_n \sim \int_1^n f(t)\,dt$ by [**3**, Theorem 33]. Hence, $A_n \sim na_n/\alpha$ by [**3**, Theorem 25]. If Σa_n is convergent, which happens if $\alpha < 0$ and sometimes when $\alpha = 0$, then $na_n \to 0$. Hence $na_n/A_n \to 0$. Finally, we may assume $\alpha = 0$ and $\Sigma a_n = \infty$. Then $f = g/x$ with $x^{-\delta} \prec g \prec x^{\delta}$ for all $\delta > 0$ by [**3**, Theorem 23] and $A_n \sim \int_l^n f(t)\,dt$ by [**3**, Theorem 33]. Now, Lemma 2 implies that

$$A_n \geq \int_{\mu(n)}^{n} f(t)\,dt \sim g(n)\log\left(\frac{n}{\mu(n)}\right) = na_n \log\left(\frac{n}{\mu(n)}\right) \succ na_n,$$

which completes the proof. □

LEMMA 4. *Suppose $f \in L$, $f(x) > 0$ and $\lim_{x\to\infty} xf'(x)/f(x) = \infty$. Then, for γ real,*

$$\int_x^\infty \frac{dt}{t^\gamma F(t)} \sim \frac{1}{x^\gamma f(x)} \quad \text{as } x \to \infty,$$

where $F(x) = \int_c^x f(t)\,dt$ for any sufficiently large constant c.

PROOF. By [**3**, Theorem 23], our hypothesis implies that $f \succ x^{\Delta}$ for all $\Delta > 0$, so that the integral exists. Integrating by parts and using [**3**, Theorem 25], we have

$$\begin{aligned}\int_x^\infty \frac{dt}{t^\gamma F(t)} &= \int_x^\infty \left\{-\frac{F}{t^\gamma F'}\right\}\left\{-\frac{F'}{F^2}\right\} dt \\ &= \frac{1}{x^\gamma f(x)} + \int_x^\infty \frac{1}{F(t)} o(t^{-\gamma})\,dt \\ &= \frac{1}{x^\gamma f(x)} + o(1)\int_x^\infty \frac{dt}{F(t)t^\gamma}. \quad \square\end{aligned}$$

Using Lemma 4 with [**3**, Theorems 25 and 33], we obtain

LEMMA 5. *Suppose $f \in L$, $f(x) > 0$ and $\lim_{x\to\infty} xf'/f = \infty$. Let γ be real, $a_n = f(n)$ and $A_n = \sum_{k=0}^n a_k$. Then*

$$\sum_{n=k}^\infty \frac{1}{A_n n^\gamma} \sim \frac{1}{a_k k^\gamma}.$$

PROOF. We may assume without loss of generality that f is increasing on $[0,\infty)$. Let

$$F(x)=\int_0^x f(t)\,dt$$

and

$$A_n=\sum_{k=0}^{n}a_k=\sum_{k=0}^{n}f(k).$$

Then, for all sufficiently large k, we have

$$\int_{k+1}^{\infty}\frac{dt}{F(t)(t-1)^{\gamma}}\le\sum_{n=k}^{\infty}\frac{1}{A_n n^{\gamma}}\le\int_{k-1}^{\infty}\frac{dt}{F(t)t^{\gamma}}.$$

We now distinguish three cases.

Case 1. $f(x)\succ e^{\Delta x}$ for every $\Delta>0$. Referring to [**3**, Theorem 33], we have $A_{n-1}\prec a_n$ and $A_n\sim a_n$. Now

$$\sum_{n=k}^{\infty}\frac{1}{A_n n^{\gamma}}\le\frac{1}{A_k k^{\gamma}}+\frac{1}{A_{k+1}(k+1)^{\gamma}}+\int_{k+1}^{\infty}\frac{dt}{F(t)t^{\gamma}}.$$

Thus, using Lemma 4, we see

$$\limsup_{k\to\infty}a_k k^{\gamma}\sum_{n=k}^{\infty}\frac{1}{A_n n^{\gamma}}\le 1.$$

On the other hand,

$$\sum_{n=k}^{\infty}\frac{1}{A_n n^{\gamma}}\ge\frac{1}{A_k k^{\gamma}},$$

so

$$\liminf_{k\to\infty}a_k k^{\gamma}\sum_{n=k}^{\infty}\frac{1}{A_n n^{\gamma}}\ge 1,$$

and the result follows.

Case 2. $f(x)\prec e^{\delta x}$ for every $\delta>0$. In this case we have, using [**3**, Theorems 33 and 25], $A_n\sim F(n)$ and $a_n/A_n=o(1)$. Now, for all sufficiently large k,

$$\int_k^\infty \frac{dt}{t^\gamma F(t)} \le \sum_{n=k}^\infty \frac{1}{n^\gamma F(n)} \le \int_{k-1}^\infty \frac{dt}{t^\gamma F(t)}.$$

Since, for large k, we have

$$\int_{k-1}^k \frac{dt}{t^\gamma F(t)} \le \frac{1}{(k-1)^\gamma F(k-1)} \sim \frac{1}{k^\gamma A_k} \prec \frac{1}{k^\gamma a_k},$$

and $\sum_{n=k}^\infty 1/A_n n^\gamma \sim \sum_{n=k}^\infty 1/F(n)n^\gamma$, the result follows from Lemma 4.

Case 3. $f(x) = e^{ax}b(x)$, where $a > 0$ and $e^{-\delta x} \prec b(x) \prec e^{\delta x}$ for all $\delta > 0$.

Here, from [**3**, Theorem 33], $A_n \sim \{a/(1-e^{-a})\}F(n)$ and $a_n \sim \{a/(1-e^{-a})\}c_n$ where $c_n = \int_{n-1}^n f(t)\,dt$. So $a_k/A_k \sim c_k/F(k) \sim 1-e^{-a}$ and $A_k/A_{k+r} \sim e^{-ar}$ as $k \to \infty$ with r fixed. Now, for r fixed, we have

$$a_k k^\gamma \sum_{n=k}^\infty \frac{1}{A_n n^\gamma} \le \frac{a_k}{A_k} + \cdots + \frac{a_k}{A_{k+r}} + a_k k^\gamma \int_{k+r}^\infty \frac{dt}{F(t)t^\gamma}.$$

Thus

$$\limsup_{k\to\infty} a_k k^\gamma \sum_{n=k}^\infty \frac{1}{A_n n^\gamma} \le (1-e^{-a})(1+e^{-a}+\cdots+e^{-ar}) + e^{-ar},$$

and so

$$\limsup_{k\to\infty} a_k k^\gamma \sum_{n=k}^\infty \frac{1}{A_n n^\gamma} \le 1.$$

On the other hand, for fixed r,

$$a_k k^\gamma \sum_{n=k}^\infty \frac{1}{A_n n^\gamma} \ge \sum_{\nu=0}^r \frac{a_k k^\gamma}{A_{k+\nu}(k+\nu)^\gamma} \quad \longrightarrow \quad (1-e^{-a}) \sum_{\nu=0}^r e^{-\nu a}$$

so that

$$\liminf_{k\to\infty} a_k k^\gamma \sum_{n=k}^\infty \frac{1}{A_n n^\gamma} \ge 1,$$

and the result follows. □

4. Proof of Theorem 1 and Corollary 1.

Case 1. $\lim_{n\to\infty} a_n/A_n > 0$. Note that $\alpha = \lim_{n\to\infty} na_n/A_n = \infty$.

Now $\sum_{n=0}^{\infty} |a_{n0}|^p = \sum_{n=0}^{\infty} |a_n/A_n|^p = \infty$. Hence $N_a e^0 \notin l_p$, where $e^0 = (1, 0, 0, \dots)$, so that $N_a \notin B(l_p)$. For Corollary 1 a more delicate argument is required.

Suppose first that

$$\sum_{n=0}^{\infty} \exp\{-p(N+1)f'(n)/f(n)\} < \infty, \tag{5}$$

so $\lim_{x\to\infty} f'(x)/f(x) = \infty$. Then, by [**3**, Theorem 33], $f(x) \succ e^{\Delta x}$ for all $\Delta > 0$, and $A_n \sim a_n$. Let $x \in l_p(N)$ and $y = N_a x$. Using Hölder's inequality we obtain

$$\begin{aligned}\sum_{n=N+1}^{\infty} |y_n|^p &\le \sum_{n=N+1}^{\infty} \left(\sum_{k=N+1}^{n} |x_k|^p \right) \left(\sum_{k=N+1}^{n} (a_{n-k}/A_n)^q \right)^{p/q} \\ &\le ||x||_p^p \sum_{n=N+1}^{\infty} A_n^{-p} \left(\sum_{k=0}^{n-N-1} a_k^q \right)^{p/q} \\ &\sim ||x||_p^p \sum_{n=N+1}^{\infty} (f(n-N-1)/f(n))^p.\end{aligned}$$

Since $f'(t)/f(t)$ is eventually increasing, we have $\log(f(n)/f(n-N-1)) = \int_{n-N-1}^{n} f'(t)/f(t)\, dt \ge (N+1)f'(n-N-1)/f(n-N-1)$ for large n, so that $\sum_{n=N+1}^{\infty}(f(n-N-1)/f(n))^p < \infty$ by comparison with (5).

Suppose now that

$$\sum_{n=0}^{\infty} \exp\{-p(N+1)f'(n)/f(n)\} = \infty. \tag{6}$$

Let $x = (0, 0, \dots, 0, 1, 0, \dots)$ with the 1 in the $(N+1)^{\text{st}}$ place. Then, if $y = N_a x$, $y_n = a_{nN+1}$. Since, for some c, we have $0 < c \le a_n/A_n \le 1$, it follows that

$$\sum_{n=N+1}^{\infty} |y_n|^p \succeq \sum_{n=N+1}^{\infty} (f(n-N-1)/f(n))^p.$$

Since f'/f is eventually monotonic we have, for large n,

$$\int_{n-N-1}^{n} f'(t)/f(t)\,dt \leq (N+1)\max\{f'(n)/f(n), f'(n-N-1)/f(n-N-1)\}$$

and $\sum_{n=N+1}^{\infty}(f(n-N-1)/f(n))^p = \infty$ by comparison with (6). This completes the proof of Corollary 1 in Case 1.

Case 2. $\lim_{n\to\infty} a_n/A_n = 0$ and $\alpha = \lim_{n\to\infty} na_n/A_n = \infty$. It follows, by Lemma 3, that $\lim_{x\to\infty} xf'(x)/f(x) = \infty$. Since $A_n \to \infty$, we have $A_n \sim \int_c^n f(t)\,dt$ where c is constant. From [**3**, Theorem 25] we have $\int_c^x f(t)\,dt \sim f^2/f'$ (since $f \succ x^\Delta$ for all $\Delta > 0$); therefore, $h(n) \sim A_n/a_n \to \infty$ with $h = f/f'$. Hence, from [**4**, Lemma 2 with $\lambda_3(x) = x$ and $\lambda = f$], we obtain

$$\sum_{n-h(n)\leq k\leq n} a_k \asymp a_n h(n).$$

Putting $\delta = 1/p$, we have

$$\sum_{k=0}^{n} a_{n-k}(k+1)^{-\delta} \geq \sum_{n-h(n)\leq k\leq n} a_k(n+1-k)^{-\delta} \succeq (h(n))^{1-\delta}a_n.$$

Thus $\sigma_1(n) \succeq (1/A_n)(n+1)^\delta a_n(h(n))^{1-\delta} \asymp (na_n/A_n)^\delta \to \infty$. From [**2**, Theorem 4 with $b = d_n = 1$], it follows that $N_a \notin B(l_p)$. Since $\sum_{k=n-N}^{n} a_k \prec A_n$ we have $\sum_{n-h(n)\leq k\leq n-N} a_k \asymp a_n h(n) \sim A_n$, and the arguments above imply that $N_a \notin B(l_p(N))$.

Case 3. $\lim_{n\to\infty} na_n/A_n = \alpha$ with $0 < \alpha < \infty$. Hence, by Lemma 3, $\lim_{x\to\infty} xf'(x)/f(x) = \alpha - 1$. Now, $f = (x+1)^{\alpha-1}g$, where $x^{-\epsilon} \prec g \prec x^\epsilon$ for all $\epsilon > 0$.

Putting $\delta = 1/p$ and applying [**3**, Theorem 25] together with Lemma 2, we find

$$\begin{aligned}\sigma_1(n) &= \frac{(n+1)^\delta}{A_n}\sum_{k=0}^{n} a_k(n+1-k)^{-\delta}\\ &\sim \frac{\alpha}{n}\sum_{k=0}^{n}\frac{g(k)}{g(n)}\left(\frac{k+1}{n+1}\right)^{\alpha-1}\left(1-\frac{k}{n+1}\right)^{-\delta}\\ &\sim \frac{\alpha\Gamma(\alpha)\Gamma(1/q)}{\Gamma(\alpha+1/q)}.\end{aligned}$$

Thus, from [**2**, Theorem 4], $||N_a||_p \geq \Gamma(\alpha+1)\Gamma(1/q)\Gamma(\alpha+1/q)$. Also, putting $\nu = 1/q$, by Lemma 2,

$$\begin{aligned}\sigma_2(k) &\sim \alpha k^\nu \sum_{n=k}^{\infty}\frac{g(n-k)}{g(k)}(n+1)^{-\alpha-\nu}(n+1-k)^{\alpha-1}\\ &\sim \alpha\frac{\Gamma(\alpha)\Gamma(1/q)}{\Gamma(\alpha+1/q)}.\end{aligned}$$

Thus, from [**2**, Theorem 1 with $b_{nk} = (n+1)/(k+1)$], (1) holds. It follows similarly that

$$\lim_{n\to\infty}\sigma_1^{(N)}(n) = \lim_{k\to\infty}\sigma_2^{(N)}(k) = \Gamma(\alpha+1)\Gamma(1/q)/\Gamma(\alpha+1/q),$$

and Corollary 1 is established for $0 < \alpha \leq \infty$.

Case 4. Here, $a = \{a_n\}$ is any sequence of positive numbers such that $\alpha = \lim_{n\to\infty} na_n/A_n = 0$. Let $\delta = 1/p$ and $0 < \theta < 1$. Now, for θ fixed,

$$1 \leq \sigma_1(n) = \frac{(n+1)^\delta}{A_n}\left(\sum_{0\leq k\leq\theta n} + \sum_{\theta n<k\leq n}\right)\frac{a_{n-k}}{(k+1)^\delta}$$

and

$$\frac{(n+1)^\delta}{A_n}\sum_{0\leq k\leq\theta n}\frac{a_{n-k}}{(k+1)^\delta} = o(1) \quad \text{as } n\to\infty,$$

while

$$\frac{(n+1)^\delta}{A_n}\sum_{\theta n<k\leq n}\frac{a_{n-k}}{(k+1)^\delta} \leq \left(\frac{n+1}{\theta n+1}\right)^\delta \quad \longrightarrow \quad \theta^{-\delta} \quad \text{as } n\to\infty.$$

Consequently, $1 \le \liminf \sigma_1(n) \le \limsup \sigma_1(n) \le 1/\theta^\delta$, and since θ is arbitrary, we have $\lim_{n\to\infty} \sigma_1(n) = 1$. Since $\sigma_1^{(N)}(n) \le \sigma_1(n)$ and $\sigma_1^{(N)}(n) \ge (1/A_n) \sum_{k=0}^{n-N-1} a_k \to 1$, we also have $\lim_{n\to\infty} \sigma_1^{(N)}(n) = 1$. Again, [**2**, Theorem 4] yields $||N_a||_p \ge ||N_a||_p^{(N)} \ge 1$. For $0 < \theta < 1$ and k large enough that $(n+1)a_n/A_n < \theta$ for all $n < k$, we have, with $\nu = 1/q$,

$$(k+1)^\nu \sum_{n=2k+1}^{\infty} \frac{a_{n-k}}{A_n(n+1)^\nu} \le (k+1)^\nu \sum_{n=k+1}^{\infty} \frac{a_n}{A_n(n+1)^\nu} \le (k+1)^\nu \theta \int_{k+1}^{\infty} \frac{dt}{t^{\nu+1}} = q\theta$$

and

$$(k+1)^\nu \sum_{n=k}^{2k} \frac{a_{n-k}}{A_n(n+1)^\nu} \le 1.$$

Thus $\limsup \sigma_2(k) \le 1 + q\theta$, and, since θ is arbitrary, $\limsup \sigma_2(k) \le 1$. Now [**2**, Theorem 1] yields (1), and Corollary 1 is established for $\alpha = 0$. □

REMARK. When $\lim_{n\to\infty} na_n/A_n = \alpha$ with $0 \le \alpha < \infty$, then [**1**, Theorem 2] shows that $N_a \in B(l_p)$ and the proof of [**1**, Theorem 2] yields $||N_a||_p \le \sigma_1^{1/q}\sigma_2^{1/p}$.

5. Proof of Theorem 2 and Corollary 2.

Case 1. $\beta = \lim_{n\to\infty} A_n/na_n = 0$, so that, by Lemma 3, $\lim_{x\to\infty} xf'(x)/f(x) = \infty$.

Now, using [**3**, Theorem 25] $(\delta = 1/p)$,

$$\sigma_1(n) = \frac{(n+1)^\delta}{A_n} \sum_{k=0}^{n} a_k(k+1)^{-\delta} \quad \longrightarrow \quad 1 \quad \text{as } n \to \infty$$

and $\sigma_1^{(N)}(n) \to 1$ as $n \to \infty$. Thus, by [**2**, Theorem 4],

$$||W_a||_p \ge ||W_a||_p^{(N)} \ge 1.$$

Also, since, by Lemma 5, $\sum_{n=k}^{\infty} 1/A_n n^{\nu} \sim 1/a_k k^{\nu}$ where $\nu = 1/q$, we obtain $\lim_{k\to\infty} \sigma_2(k) = 1$. This, together with [**2**, Theorems 2 and 4], establishes Theorem 2 and Corollary 2 in the case $\beta = 0$.

Case 2. $\beta = \lim_{n\to\infty} A_n/na_n$ with $0 < \beta < \infty$ so that, by Lemma 3, $\lim_{x\to\infty} xf'(x)/f(x) = \alpha - 1$, where $\alpha = 1/\beta$.

We have $f = (x+1)^{\alpha-1}g$, where $x^{-\epsilon} \prec g \prec x^{\epsilon}$ for all $\epsilon > 0$. Then, using [**3**, Theorem 25] and putting $\delta = 1/p$,

$$\begin{aligned}\sigma_1(n) &= \frac{(n+1)^{\delta}}{A_n}\sum_{k=0}^{n} a_k(k+1)^{-\delta} \\ &\sim \alpha(n+1)^{\delta-\alpha}\frac{1}{g(n)}\sum_{k=0}^{n} g(k)(k+1)^{\alpha-\delta-1}.\end{aligned}$$

(i) If $\alpha < \delta$ (i.e., $\beta > p$), then $\sum_{k=0}^{\infty} g(k)(k+1)^{\alpha-\delta-1} < \infty$ so that $\sigma_1(n) \to \infty$ and [**2**, Theorem 4] shows $W_a \notin B(l_p)$. Also $\sigma_1^{(N)}(n) \to \infty$, so $W_a \notin B(l_p(N))$.

(ii) Suppose $\alpha = \delta$, i.e., $\beta = p$. Then, with the notation of Lemma 1,

$$\begin{aligned}\sigma_1(n) &= \frac{(n+1)^{\delta}}{A_n}\sum_{k=0}^{n} a_k(k+1)^{-\delta} \\ &\sim \frac{\alpha}{g(n)}\sum_{k=0}^{n} g(k)/(k+1) \\ &\geq \frac{\alpha}{g(n)}\sum_{\mu(n)\leq k\leq n} g(k)/(k+1) \\ &\sim \alpha\log(n/\mu(n)) \quad \longrightarrow \quad \infty\end{aligned}$$

and $W_a \notin B(l_p)$. Also $W_a \notin B(l_p(N))$.

(iii) Suppose $\alpha > \delta$, i.e., $\beta < p$. Then, from [**3**, Theorem 25 with $a = \alpha - \delta - 1$] and from [**3**, Theorem 33],

$$\sigma_1(n) \sim \alpha/(\alpha-\delta) = p/(p-\beta).$$

Also $\sigma_2(k) = a_k(k+1)^\nu \cdot \sum_{n=k}^{\infty} 1/A_n(n+1)^\nu$ where $\nu = 1/q$. So

$$\sigma_2(k) \sim g(k)k^{\alpha-\delta}\alpha \sum_{n=k}^{\infty} n^{\delta-\alpha-1}/g(n) \quad \longrightarrow \quad p/(p-\beta) \quad \text{as } k \to \infty.$$

With [**2**, Theorems 2 and 4] we get Theorem 2 and Corollary 2 for the case $0 < \alpha < \infty$.

Case 3. Now suppose $\beta = \lim_{n\to\infty} A_n/na_n = \infty$. Then, by Lemma 3, $\lim_{x\to\infty} xf'/f = \rho \leq -1$. Hence, $\lim_{n\to\infty} A_n = s < \infty$ if $\rho < -1$, and $f(x) = g(x)/(x+1)$ with $x^{-\epsilon} \prec g \prec x^{\epsilon}$ for all $\epsilon > 0$, if $\rho = -1$. If $A_n \to s$ we obtain (with $\delta = 1/p$),

$$\sigma_1(n) = \frac{(n+1)^\delta}{A_n} \sum_{k=0}^{n} a_k(k+1)^{-\delta} \sim \frac{(n+1)^\delta}{s} \sum_{k=0}^{n} a_k(k+1)^{-\delta} \to \infty,$$

and if $\rho = -1$ we also get $\sigma_1(n) \to \infty$, because, in that case,

$$\sum_{k=0}^{\infty} a_k(k+1)^{-\delta} = \sum_{k=0}^{\infty} g(k)(k+1)^{-\delta-1} < \infty$$

and $A_n = \sum_{k=0}^{n} g(k)/(k+1) \prec n^\epsilon$ for all $\epsilon > 0$. Now [**2**, Theorem 4] shows that $W_a \notin B(l_p)$ and $W_a \notin B(l_p(N))$. This completes the proofs of Theorem 2 and Corollary 2. □

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