ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 1, Winter 1990

APPROXIMATION BY SEMI-FREDHOLM OPERATORS WITH FIXED NULLITY

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1. Introduction. Let H be a fixed complex separable Hilbert space. For any (bounded linear) operator T on H, we define the nullity and deficiency, denoted $\operatorname{nul} T$ and $\operatorname{def} T$, to be the dimensions of the kernels of T and T^*, respectively. Of course, the index of T, denoted $\operatorname{ind} T$, is defined to be $(\operatorname{nul} T - \operatorname{def} T)$, with $\infty - \infty$ understood to be 0. We denote the operator norm of T by $\|T\|$ and the spectrum by $\sigma(T)$.

In [**2**] the distance from an arbitrary operator T to the set of invertible operators (and to the Fredholm operators) was determined. This provided a refinement of the classical result in [**5**] that describes the closure of the invertible operators. Subsequently Theorem 12.2 in [**1**] elaborated on [**2**] by showing that the formula given there was actually the distance from an arbitrary operator T to each set of semi-Fredholm operators with an index different from that of T. [**1**] went on to show that the preceding theorem plays a significant role in similarity theory.

In [**7**] the original methods of [**2**] are used to modify Theorem 12.2 to obtain the distance from T to the right invertible operators with a fixed nullity. All of the preceding results and some new methods were used in [**3**] to find the distance from T to the (unrestricted) set of operators with a fixed nullity; the formula obtained in [**3**] is a striking contrast to previously obtained formulas. In this note we determine the distance from T to a natural set which contains the right invertible operators with nullity equal to n and is contained in the set of operators with nullity equal to n. The results have some resemblance to those in [**3**] and some to those in [**7**].

2. Preliminaries. This section contains results that will be used frequently in the subsequent section. These results will be used sometimes without citation.

Received by the editors on August 2, 1986 and in revised form on March 25, 1987.

Recall that the minimum modulus of the operator T, denoted $m(T)$, is defined by

$$m(T) = \inf\{\|Tf\| : \|f\| = 1\}.$$

For the sake of completeness we state the following well-known theorem; a proof can be found in [**2**].

THEOREM 1. (i) $m(T) = \inf\{\lambda : \lambda \in \sigma((T^*T)^{1/2})\}$.

(ii) *There exists an operator B such that $BT = I$ if and only if $m(T) > 0$. In that case, B can be chosen such that $m(T) = 1/\|B\|$.*

(iii) *There exists an operator A such that $TA = I$ if and only if $m(T^*) > 0$. In that case, A can be chosen such that $m(T^*) = 1/\|A\|$.*

(iv) *The operator T is invertible if and only if $m(T)$ and $m(T^*)$ are both positive. In that case, $m(T) = m(T^*)$.*

Recall that the reduced minimum modulus of T, denoted $\gamma(T)$, is defined by

$$\gamma(T) = \inf\{\|Tf\| : \|f\| = 1, f \perp \ker T\}.$$

The next theorem is mostly well known; a nice treatment can be found in [**4**, pp. 364–365].

THEOREM 2. (i) $\gamma(T) > 0$ *if and only if the range of T, denoted TH, is closed.*

(ii) $\gamma(T) = \gamma(T^*)$.

(iii) *If T and A are both operators on H and $\|T - A\| < \gamma(T)$, then* $\operatorname{nul} A \leq \operatorname{nul} T$ *and* $\operatorname{def} A \leq \operatorname{def} T$.

The essential spectrum of an operator T, denoted $\sigma_e(T)$, is the set $\{z : T - zI$ is not a Fredholm operator$\}$. We define the essential minimum modulus $m_e(T)$ by

$$m_e(T) = \inf\{\lambda : \lambda \in \sigma_e((T^*T)^{1/2})\}.$$

The next theorem is a folklore result which is proved in [**6**].

THEOREM 3. *Let T and A be operators such that $\|T - A\| < m_e(T)$. Then*

(i) *A has finite nullity and closed range if and only if T does,*

(ii) *A is Fredholm if and only if T is, and*

(iii) $\operatorname{ind} A = \operatorname{ind} T$.

The following enumeration of the properties of $m_e(T)$ was given in [**2**]. We shall use most of these.

THEOREM 4. (i) *If $E(\cdot)$ is the spectral measure for $R = (T^*T)^{1/2}$, then the smallest nonnegative number α such that $E([\alpha, \alpha + \delta))H$ is infinite dimensional for every positive δ is $\alpha = m_e(T)$.*

(ii) *The range TH is closed and* $\operatorname{nul} T$ *is finite if and only if $m_e(T) >$* 0.

(iii) *The range T^*H is closed and* $\operatorname{def} T$ *is finite if and only if* $m_e(T^*) > 0$.

(iv) *The operator T is Fredholm if and only if $m_e(T)$ and $m_e(T^*)$ are positive. In that case,* $m_e(T) = m_e(T^*)$.

We shall also use the next theorem which is proved in [**7**, Theorem 3.1] for positive n. The case $n = 0$ is dealt with in [**2**].

THEOREM 5. *Let n represent either a nonnegative integer or ∞.*

$$\inf\{\|T - A\| : m(A^*) > 0, \operatorname{nul} A = n\} = \begin{cases} \max\{m_e(T), m_e(T^*)\} \textit{ if } \operatorname{ind} T \neq n, \\ 0 \textit{ otherwise.} \end{cases}$$

3. Distance to the right semi-Fredholm operators. The notation that we are about to define will help us to avoid burdensome repetitions of various conditions.

DEFINITION. Let n denote some nonnegative integer or ∞ and define Φ_n, Ψ_n, and P_n by

$$\Phi_n = \{A : m(A^*) > 0, \operatorname{nul} A = n\},$$
$$\Psi_n = \{A : m_e(A^*) > 0, \operatorname{nul} A = n\},$$
$$P_n = \{A : \operatorname{nul} A = n\}.$$

Define $\psi_n(T), \phi_n(T)$, and $\rho_n(T)$ as follows:

$$\begin{aligned}\phi_n(T) &= \inf\{\|T - A\| : A \in \Phi_n\},\\ \psi_n(T) &= \inf\{\|T - A\| : A \in \Psi_n\},\\ \rho_n(T) &= \inf\{\|T - A\| : A \in \mathrm{P}_n\}.\end{aligned}$$

The set Φ_n consists of the "right invertible" operators with nullity n and Ψ_n consists of the "right semi-Fredholm" operators with nullity n.

LEMMA 6. $\phi_n(T) \geq \psi_n(T) \geq \rho_n(T)$.

PROOF. The inequalities follow from the containments

$$\Phi_n \subset \Psi_n \subset \mathrm{P}_n. \qquad \square$$

The inequalities in the preceding lemma and the simplifications in the next lemma will be used repeatedly.

LEMMA 7. (i) *If* $\operatorname{nul} T < \infty$ *or* $\operatorname{def} T = \infty$, *then* $m_e(T) = \max\{m_e(T), m_e(T^*)\}$.

(ii) *If* $\operatorname{nul} T = \infty$ *or* $\operatorname{def} T < \infty$, *then* $m_e(T^*) = \max\{m_e(T), m_e(T^*)\}$.

PROOF. If $\operatorname{def} T = \infty$, then $m_e(T^*) = 0$ and the conclusion of (i) is obvious from part (iii) of Theorem 4. Assume $\operatorname{nul} T < \infty$. If TH is closed, then $m_e(T) > 0$ according to part (ii) of Theorem 4 and the desired conclusion follows from part (iv) of Theorem 4. If TH is not closed, then T^*H is not closed according to Theorem 2 and both $m_e(T)$ and $m_e(T^*)$ are 0. The conclusion of (i) follows.

To prove part (ii) apply part (i) to T^*. $\square$

The next theorem is our first main result. The distance formulas given here resemble formulas in [**2**] and [**7**].

THEOREM 8. *Assume* $n \leq \operatorname{nul} T$.

(i) *If* TH *is not closed then* $\psi_n(T) = 0$.

(ii) *If* $n \geq \operatorname{ind} T > -\infty$ *then* $\psi_n(T) = 0$.

(iii) *If* $n < \operatorname{ind} T$ *then* $\psi_n(T) = m_e(T^*)$.

(iv) *If* $\operatorname{ind} T = -\infty$ *then* $\psi_n(T) = m_e(T)$.

PROOF. (i). Since TH is not closed, T^*H is not closed according to Theorem 2 and

$$0 = \max\{m_e(T), m_e(T^*)\}$$

according to parts (ii) and (iii) of Theorem 4. According to Theorem 5 we have $\phi_n(T) = 0$, and Lemma 6 implies that $\psi_n(T) = 0$ as desired.

(ii). First we handle the case that $n = \infty$ and $\operatorname{def} T < \infty$. In view of part (i) we may assume that TH is closed; thus, T^*H is closed and $T \in \Psi_n$ for $n = \infty$. Clearly $\psi_n(T) = 0$ and we are done. In the remaining cases either $n < \infty$ or else each of the quantities $n, \operatorname{nul} T$ and $\operatorname{def} T$ is ∞. In this last case $\max\{m_e(T), m_e(T^*)\} = 0$ and the desired conclusion follows from Theorem 5 and Lemma 6. Henceforth, we assume that n is finite and we note that $\operatorname{def} T \geq \operatorname{nul} T - n$. Let $\{e_1, e_2, \dots\}$ be an orthonormal basis for $\ker T$ and note that this basis could be finite or infinite. Let $\{f_1, f_2, \dots\}$ be an orthonormal basis for $\ker T^*$ and note that it is infinite if the basis for $\ker T$ is infinite. Define A_k by

$$A_k e_j = 0, j = 1, 2, \dots, n, \qquad A_k e_{n+j} = (1/k) f_j, j = 1, 2, \dots,$$

$$A_k|(\ker T)^\perp = T|(\ker T)^\perp.$$

Theorem 2 shows that $A_k H$ is closed if and only if TH is closed. In view of part (i), which is already proved, we may assume that TH is closed. Thus, $A_k H$ is closed. The inequality $\operatorname{ind} T > -\infty$ implies that either $\operatorname{def} T$ is finite or else $\operatorname{nul} T = \infty = \operatorname{def} T$. In either case our construction results in $\operatorname{def} A_k < \infty$. It is routine to verify that $\operatorname{nul} A_k = n$ and $\|T - A_k\| = 1/k$. It follows that A_k belongs to Ψ_n and

$$\psi_n(T) \leq \inf_k \|T - A_k\| = 0.$$

This proves (ii).

(iii). Theorem 5 and Lemma 6 imply that

$$\max\{m_e(T), m_e(T^*)\} \geq \phi_n(T) \geq \psi_n(T).$$

Since $\operatorname{ind} A \leq \operatorname{nul} A$, part (iii) of Theorem 3 implies that

$$\psi_n(T) \geq \max\{m_e(T), m_e(T^*)\},$$

and so we have

$$\psi_n(T) = \max\{m_e(T), m_e(T^*)\}.$$

The inequalities

$$0 \leq n < \operatorname{ind} T$$

imply that $\operatorname{nul} T > \operatorname{def} T$; in particular, $\operatorname{def} T$ is finite. Now Lemma 7 implies that

$$\max\{m_e(T), m_e(T^*)\} = m_e(T^*),$$

and the proof of (iii) is complete.

(iv). The equation $\operatorname{ind} T = -\infty$ implies that $\operatorname{nul} T$ and n are finite. By Lemma 7 we have

$$\max\{m_e(T), m_e(T^*)\} = m_e(T).$$

Since any operator in Ψ_n is Fredholm, it has finite index. Part (iii) of Theorem 3 implies that

$$\|T - A\| \geq \max\{m_e(T), m_e(T^*)\},$$

and so $\psi_n(T) \geq \max\{m_e(T), m_e(T^*)\}$. Theorem 5 and Lemma 6 imply that

$$\max\{m_e(T), m_e(T^*)\} \geq \phi_n(T) \geq \psi_n(T),$$

and so

$$\psi_n(T) = \max\{m_e(T), m_e(T^*)\} = m_e(T).$$

This completes the proof. □

The preceding theorem, which computed the distance from T to each set of right semi-Fredholm operators with a fixed n, can be used to obtain the distance to the set of all right semi-Fredholm operators with nullities not exceeding $\operatorname{nul} T$.

THEOREM 9.

$$\inf\{\|T-A\| : A \in \Psi_n, n \leq \operatorname{nul} T\} = \begin{cases} m_e(T), & \textit{if } \operatorname{ind} T = -\infty, \\ 0, & \textit{otherwise.} \end{cases}$$

PROOF. If TH is not closed, then $\psi_n(T) = 0$ according to part (i) of Theorem 8. Henceforth, we assume that TH is closed. Since $\operatorname{ind} T \leq \operatorname{nul} T$ we can take $n = \operatorname{nul} T$ and

$$\operatorname{nul} T \geq n \geq \operatorname{ind} T.$$

According to part (ii) of Theorem 8, $\psi_n(T) = 0$ unless $\operatorname{ind} T = -\infty$. Thus, the only situation where we cannot choose n to get $\psi_n(T) = 0$ is when TH is closed and $\operatorname{ind} T = -\infty$. In that case part (iv) of Theorem 8 implies that $\psi_n(T) = m_e(T)$ for all n. □

The next theorem produces a formula that resembles one in [**3**]. The proof requires the elaboration of an argument found in [**3**] as well as some new arguments.

THEOREM 10. *Assume* $n > \operatorname{nul} T$.

(i) *If* TH *is not closed then* $\psi_n(T) = 0$.

(ii) *If* $\operatorname{def} T = \infty$ *then* $\psi_n(T) = m_e(T)$.

(iii) *If* $\operatorname{def} T < \infty$ *and* TH *is closed, then*

$$\psi_n(T) = \sup\{\lambda : \dim \mathrm{E}([0,\lambda))H < n\},$$

where $\mathrm{E}(\cdot)$ *is the spectral measure for* $(T^*T)^{1/2}$.

PROOF (i). This proof is the same as the proof of part (i) of Theorem 8.

PROOF (ii). The inequality $n > \operatorname{nul} T$ implies that $\operatorname{nul} T$ is finite. Lemma 7 implies that

$$m_e(T) = \max\{m_e(T), m_e(T^*)\}.$$

Since $n > \operatorname{nul} T \geq \operatorname{ind} T$, Theorem 5 implies that $\phi_n(T) = m_e(T)$ and Lemma 6 shows that $m_e(T) \geq \psi_n(T)$. If TH is not closed, then $m_e(T) = 0$ and the proof is complete. Henceforth, we assume that TH is closed. According to Theorem 3 the inequality $\|T - A\| < m_e(T)$ implies that AH is closed, $\operatorname{nul} A < \infty$ and $\operatorname{ind} A = \operatorname{ind} T$. Thus, $A \in \Psi_n$ and $\|T - A\| < m_e(T)$ imply that A is Fredholm and

$$\operatorname{ind} A = \operatorname{ind} T = -\infty.$$

This contradiction proves that

$$\|T - A\| \geq m_e(T) \quad \text{and} \quad \psi_n(T) \geq m_e(T).$$

This completes the proof of (ii)

(iii). Since TH is closed and $\operatorname{nul} T < \infty$, we know that $m_e(T)$ and $\gamma(T)$ are positive.

If UR is the usual polar factorization for T, then $\ker T = \ker R$ and $\mathrm{E}(\cdot)$ is the spectral measure for R. Choose $A \in \Psi_n$ and $\lambda > 0$ such that $\dim \mathrm{E}([0, \lambda))H < n$; let P denote the projection $\mathrm{E}([0, \lambda))$. Note that

$$\begin{aligned}
\|T - A\| &\geq \|(T - A)|\ker A\| \\
&= \|T|\ker A\| \\
&= \|R|\ker A\| \\
&\geq \|(I - P)R|\ker A\| \\
&= \|R(I - P)|\ker A\| \\
&= \|R|(I - P)\ker A\| \\
&\geq \lambda,
\end{aligned}$$

provided that $(I-P)\ker A$ is nontrivial since $(I-P)\ker A \subset (I-P)H = \mathrm{E}([\lambda, \infty))H$. Clearly

$$\operatorname{nul}(I - P) = \dim PH < n = \operatorname{nul} A,$$

and thus $(I-P)\ker A$ must be nontrivial. From the first displayed inequality it follows that

$$\psi_n(T) \geq \lambda \quad \text{and} \quad \psi_n(T) \geq \mu,$$

where μ is $\sup\{\lambda : \dim \mathrm{E}\,([0,\lambda))H < n\}$.

Now we prove the inequality

$$\psi_n(T) \leq \mu.$$

From the usual properties of the spectral measure it follows that

$$\dim \mathrm{E}([0,\mu+1/k))H \geq n \geq \dim \mathrm{E}([0,\mu))H.$$

First we deal with the case that n is a nonnegative integer. Choose $G(k) = \{g_1^{(k)}, g_2^{(k)}, \ldots, g_{l(k)}^{(k)}\}$ to be an orthonormal set from $\mathrm{E}\,((\mu,\mu+1/k))H$ such that $M_k = \text{closed span}\,(\mathrm{E}\,([0,\mu])H \cup G(k))$ has dimension n. Let P_k be the orthogonal projection onto M_k and note that $\|T(I-P_k)f\| = \|R(I-P_k)f\|$ is zero if and only if f belongs to $P_kH = M_k$. Thus,

$$\operatorname{nul} T(I-P_k) = \dim M_k = n.$$

Let A_k be $T(I-P_k)$ and note that $\|T-A_k\| = \|TP_k\| = \|RP_k\| \leq \mu + 1/k$. Since $\mathrm{E}\,([\mu+1/k,\infty)) \leq I - P_k$ it follows that A_kH is closed, and it suffices to show that $\operatorname{def} A_k < \infty$. Observe that $\operatorname{def} A_k = \operatorname{nul} A_k^* = \operatorname{nul}\,(I-P_k)T^*$ and $\ker(I-P_k)T^*$ is the set theoretic inverse of $P_kH = M_k$ under T^*. Because $\operatorname{nul} T^* = \operatorname{def} T$ and $\dim M_k = n$ are both finite, we see that

$$\operatorname{def} A_k = \operatorname{nul}(I-P_k)T^* < \infty.$$

Thus, A_k belongs to Ψ_N and

$$\psi_n(T) \leq \inf_k \|T - A_k\| \leq \mu.$$

Now we consider the case $n = \infty$. Note that $\dim \mathrm{E}\,([0,\mu+1/k))H = \infty > \dim \mathrm{E}\,([0,\mu-1/k))H$ for all positive integers k sufficiently large that $\mu - 1/k > 0$. Define A_k by

$$A_k = (1/k)UV\mathrm{E}\,([\mu-1/k,\mu+1/k)) + T\mathrm{E}\,([\mu+1/k,\infty)),$$

where V is the adjoint of a unilateral shift with infinite multiplicity defined on $\mathrm{E}([\mu - 1/k, \mu + 1/k))H$. Since $\ker A_k$ contains $\ker V$, we know that $\mathrm{nul}\, A_k = \infty$. It is routine to verify that

$$T - A_k = T\mathrm{E}([0, \mu + 1/k)) - (1/k)UV\mathrm{E}([\mu - 1/k, \mu + 1/k)),$$

and it follows that

$$\begin{aligned} \|T - A_k\| &\le \|T\mathrm{E}([0, \mu + 1/k))\| + 1/k \\ &\le \mu + 2/k. \end{aligned}$$

Now it suffices to show that $\mathrm{def}\, A_k < \infty$. Note that $A_k H$ contains $U\mathrm{E}([\mu - 1/k, \mu + 1/k))H$ and $UR\mathrm{E}([\mu + 1/k, \infty))H = U\mathrm{E}([\mu + 1/k, \infty))H$. Thus,

$$A_k H \supset U\mathrm{E}([\mu - 1/k, \infty))H,$$

and, because U is isometric on $RH = \mathrm{E}((0, \infty))H$ and $\ker T^* = \ker R = \mathrm{E}(\{0\})H$, we see that

$$\begin{aligned} \mathrm{def}\, A_k &\le \dim \mathrm{E}(\{0\})H + \dim \mathrm{E}((0, \mu - 1/k))H \\ &= \dim \mathrm{E}([0, \mu - 1/k))H \\ &< \infty. \end{aligned}$$

We may conclude that

$$\psi_\infty(T) \le \inf_k \|T - A_k\| \le \mu.$$

This completes the proof. □

Of course the preceding theorem allows us to determine the distance from T to the set of right semi-Fredholm operators with nullities exceeding $\mathrm{nul}\, T$. In [**3**] it was proved that

$$m_e(T) \ge \sup\{\lambda : \dim \mathrm{E}([0, \lambda))H < n\} \ge \gamma(T).$$

These inequalities exhibit the least upper bound and the greatest lower bound for the supremum appearing in part (iii) of Theorem 10 and in the next theorem.

THEOREM 11.

$$\inf\{\|T-A\| : A \in \Psi_n, n > \mathrm{nul} T\} = \begin{cases} m_e(T), & \text{if } \mathrm{def}\, T = \infty, \\ \sup\{\lambda : \dim \mathrm{E}([0,\lambda))H < 1 + \mathrm{nul} T\}, & \text{if } m_e(T^*) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. If $\mathrm{def}\, T = \infty$, then $\psi_n(T) = m_e(T)$ for all n according to part (ii) of Theorem 10. Henceforth, we assume $\mathrm{def}\, T < \infty$. If TH is closed, then part (iii) of Theorem 10 implies the second part of the above formula. In the only remaining case TH is not closed, and the formula follows from part (i) of Theorem 10. □

Using Theorems 9 and 11 it is possible to describe the distance from T to the right semi-Fredholm operators. Theorems 12 and 13 can also be deduced from Theorem 12.2 of [**1**].

THEOREM 12.

$$\inf\{\|T-A\| : AH \textit{ is closed}, \mathrm{def}\, A < \infty\} = \begin{cases} m_e(T), & \textit{if } \mathrm{def}\, T = \infty, \\ 0, & \textit{otherwise.} \end{cases}$$

PROOF. Note that

$$\{A : AH \text{ is closed}, \mathrm{def}\, A < \infty\} = \cup\{\Psi_n : n = \infty, 0, 1, \ldots\}. \qquad \square$$

4. Distance to the left semi-Fredholm operators. Each of the theorems in the preceding section can be restated to give a conclusion about left semi-Fredholm operators by taking adjoints throughout. We shall only restate Theorem 12.

THEOREM 13.

$$\inf\{\|T-A\| : AH \textit{ is closed}, \mathrm{nul}\, A < \infty\} = \begin{cases} m_e(T^*), & \textit{if } \mathrm{nul}\, T = \infty, \\ 0, & \textit{otherwise.} \end{cases}$$

It follows from Theorems 12 and 13 that any operator T has 0 distance to either the right semi-Fredholm operators or the left semi-Fredholm operators. Thus, the closure of the union of these two sets consists of all operators.

The author is grateful to the referee for several helpful suggestions.

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