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# ZERO-ONE SET FUNCTIONS AND ABSOLUTE CONTINUITY

WILLIAM D.L. APPLING

ABSTRACT. Suppose that $U$ is a set and $\mathbf{F}$ is a field of subsets of $U$. By means of functions from $\mathbf{F}$ into $\{0,1\}$, there are obtained a finitely additive analogue of a classical absolute continuity partitioning theorem, a characterization of uniform absolute continuity, and an elementary argument for a theorem equivalent to a uniform absolute continuity theorem of Brooks and Dinculeanu (J. Math. Anal. Appl. **15** (1974), 156-175.)

**1. Introduction.** Let us begin by stating a well-known absolute continuity partitioning theorem.

THEOREM. *Suppose that $U$ is a set, $S$ is a $\sigma$-field of subsets of $U$, each of $\mu$ and $\xi$ is a real nonnegative-valued countably additive measure on $S$, and $\xi$ is absolutely continuous with respect to $\mu$. Then there is a sequence $\{V_k\}_{k=1}^{\infty}$ of mutually exclusive sets of $S$ such that $\cup_{k=1}^{\infty} V_k = U$ and such that if $n$ is a positive integer, $W$ is in $S$ and $W \subseteq V_n$, then $(n-1)\mu(W) \leq \xi(W) \leq n\mu(W)$.*

The heuristic notion that underlies much of this paper is that, for the finitely additive case, zero-one set functions bear strong similarity to the elements of a $\sigma$-field.

We give the basic setting of this paper and then state the first of our three main theorems. It is a finitely additive analogue of the above theorem.

Suppose that $U$ is a set, $\mathbf{F}$ is a field of subsets of $U$, $ba(\mathbf{F})$ is the set of all real-valued bounded finitely additive functions defined on $\mathbf{F}$, and, for each $\mu$ in $ba(\mathbf{F})$, $A_\mu$ denotes the set of all elements of $ba(\mathbf{F})$

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absolutely continuous with respect to $\mu$. Let $ba(\mathbf{F})^+$ denote the set of all nonnegative-valued elements of $ba(\mathbf{F})$.

Now, in the theorem stated immediately below, certain of the notions will be zero-one set function analogues of notions of the theorem given above. We shall indicate, by appropriate parenthetical remarks, the comparison to be made.

THEOREM 3.2. (see §3). *Suppose that each of $\mu$ and $\xi$ is in $ba(\mathbf{F})^+$ and $\xi$ is in $A_\mu$. Then there is a sequence $\{\beta_k\}_{k=1}^{\infty}$ of functions from $\mathbf{F}$ into $\{0,1\}$ such that*:

(i) *if $n$ is a positive integer, then the integral* (see §2) $\int_U \beta_n(I)\mu(I)$ *exists*,

(ii) *for each $I$ in $\mathbf{F}$ and pair of distinct positive integers $k'$ and $k''$, $\beta_{k'}(I)\beta_{k''}(I) = 0$ ("mutually exclusive")*,

(iii) *if $n$ is a positive integer and $\gamma$ is a function from $\mathbf{F}$ into $\{0,1\}$ such that $\int_U \gamma(I)\mu(I)$ exists and $0 = \int_U \gamma(I)[1-\beta_n(I)]\mu(I)$ ("inclusion"), then $(n-1)\int_U \gamma(I)\mu(I) \leq \int_U \gamma(I)\xi(I) \leq n\int_U \gamma(I)\mu(I)$, and*

(iv) $\sum_{k=1}^{\infty}\int_U \beta_k(I)\mu(I) = \mu(U)$ *("union $U$")*.

In various papers [**2**, **4**, **6**, **8**, **9**, **10**, **11**, **12**, **13**], functions from $\mathbf{F}$ into $\{0,1\}$ figure nontrivially in the analysis of such things as absolute continuity, the existence of various set function integrals (see §2), and the representation of certain bounded finitely additive set functions.

With our analogues somewhat more specifically in mind, we state and discuss the second and third of our main theorems.

THEOREM 4.1. (see §4). *Suppose that $\mu$ is in $ba(\mathbf{F})^+$ and $G \subseteq A_\mu \cap ba(\mathbf{F})^+$* (we shall, without loss of generality, confine our attention to $ba(\mathbf{F})^+$). *Then the following two statements are equivalent*:

(1) *$G$ is uniformly absolutely continuous with respect to $\mu$.*

(2) *Suppose that $\{\beta_k\}_{k=1}^{\infty}$ is a sequence of functions from $\mathbf{F}$ into $\{0,1\}$ such that if $r$ and $s$ are distinct positive integers, then the integrals $\int_U \beta_r(I)\mu(I)$ and $\int_U \beta_s(I)\mu(I)$ exist and $\int_U \min\{\beta_r(I),\beta_s(I)\}\mu(I) = 0$. Then $\int_U \beta_n(I)\xi(I) \to 0$ as $n \to \infty$, for $\xi$ in $G$, uniformly on $G$.*

Certain parts of the above theorem are consequences of theorems of Bell [**12**]; however, in the interests of self-containment, we give our own arguments.

THEOREM 5.1. (see §5). *Suppose that* $\eta$ *is in* $ba(\mathbf{F})^+$, $G \subseteq ba(\mathbf{F})^+$, $G$ *is uniformly absolutely continuous with respect to* $\eta$, $\mu$ *is in* $ba(\mathbf{F})^+, G \subseteq A_\mu$ *and* $\lambda$ *is the element of* $A_\mu$ *given by* $\lambda(V) = \sup\{\int_V \min\{\eta(I), K\mu(I)\} : 0 \leq K\}$ (see §2). *Then* $G$ *is uniformly absolutely continuous with respect to* $\lambda$ (and so, trivially, with respect to $\mu$; in the cited theorem of [**14**] the hypothesis is equivalent to that of Theorem 5.1 and the conclusion is the uniform absolute continuity with respect to $\mu$).

Clearly, Theorem 5.1 is a consequence of the theorem of Brooks and Dinculeanu cited. However, the point that we wish to make is that Theorem 5.1 follows in an immediate way from very simple considerations involving zero-one set functions. A fact central to the matter, which we shall state again in §2 for reference in proving this result, is that if each of $\xi, \zeta$ and $\xi - \zeta$ is in $ba(\mathbf{F})^+$, then $\int_U \min\{\xi(I) -\zeta(I), \zeta(I)\} = 0$ if and only if, for some function $\beta$ from $\mathbf{F}$ into $\{0,1\}$ and each $V$ in $\mathbf{F}, \int_V \beta(I)\xi(I)$ exists and is $\zeta(V)$ (see [**8**]). As the reader shall see, this fact and a very elementary "invariance of uniform absolute continuity" lemma, together with some simple well-known things about absolute continuity, are essentially all that are needed to obtain Theorem 5.1.

**2. Preliminary theorems and definitions.** We let $r(\mathbf{F})$ denote the set of all functions from $\mathbf{F}$ into $\exp(\mathbf{R})$. If $\gamma$ is a function from $\mathbf{F}$ into $\mathbf{R}$, then we regard $\gamma$ as equivalent to the element $\delta$ of $r(\mathbf{F})$ given, for each $I$ in $\mathbf{F}$, by $\delta(I) = \{\gamma(I)\}$. We let $b(\mathbf{F})$ denote the set of all elements of $r(\mathbf{F})$ with bounded range union.

We refer the reader to [**3**] and [**9**] for the notions of subdivision, refinement, integral, sum supremum functional, sum infimum functional and integral function that we shall use in this paper. The reader is also referred to [**3**] for a statement of Kolmogoroff's [**15**] differential equivalence theorem, as well as certain of its more immediate consequences, and is further referred to [**3**] for certain elementary integral existence assertions. When, in this paper, a given integral existence or equivalence assertion is a simple consequence of the material of this section, we shall feel free to simply write the assertion and leave the details to the reader.

We shall let "$E \ll D$" mean that $E$ is a refinement of $D$. If $V$ is in $\mathbf{F}, D \ll \{V\}$, and $\alpha$ is in $r(\mathbf{F})$, then the statement that $b$ is an $\alpha$-function on $D$ means that $b$ is a function with domain $D$ such that if $I$ is in $D$, then $b(I)$ is in $\alpha(I)$.

In the interests of consistency, we give the following minor modification of the notion of characteristic function of a set $V$ of $\mathbf{F}$:

Definition 2.1. If $V$ is in $\mathbf{F}$, then $X(V)$ is the function with domain $\mathbf{F}$ such that, for each $I$ in $\mathbf{F}, X(V)(I) = 1$ if $I \subseteq V$, and $X(V)(I) = 0$ otherwise.

We give some results about absolute continuity and quasi mutual singularity.

For each $\xi$ and $\mu$ in $ba(\mathbf{F})^+$, we let

$$\lambda_\mu(\xi) = \Big\{\Big(V, \sup\Big\{\int_V \min\{\xi(I), K\mu(I)\} : 0 \le K\Big\}\Big) : V \text{ in } \mathbf{F}\Big\}.$$

Theorem 2.A.1. [**1**]. *If each of $\xi$ and $\mu$ is in $ba(\mathbf{F})^+$, then $\lambda_\mu(\xi)$ is in $A_\mu \cap ba(F)^+, \xi - \lambda_\mu(\xi)$ is in $ba(\mathbf{F})^+$ and $\int_U \min\{\xi(I) - \lambda_\mu(\xi)(I), \mu(I)\} =$* 0.

Theorem 2.A.2. [**1**]. *If $\mu$ is in $ba(\mathbf{F})^+, \eta$ is in $A_\mu \cap ba(\mathbf{F})^+$ and $\int_U \min\{\eta(I), \mu(I)\} = 0$, then $\eta(U) = 0$.*

Corollary 2.A.1. [**1**]. *If each of $\xi$ and $\mu$ is in $ba(\mathbf{F})^+$, then $\xi$ is in $A_\mu$ if and only if $\xi = \lambda_\mu(\xi)$.*

The next two corollaries are very easy consequences of the immediately preceding two theorems, corollary and matters treated in the reference cited.

COROLLARY 2.A.2. [**1**]. *If each of $\zeta$ and $\mu$ is in $ba(\mathbf{F})^+$, $\eta$ is in $A_\mu \cap ba(\mathbf{F})^+$ and $\int_U \min\{\zeta(I), \mu(I)\} = 0$, then $\int_U \min\{\zeta(I), \eta(I)\} = 0$.*

COROLLARY 2.A.3. [**1**]. *If each of $\xi$ and $\mu$ is in $ba(\mathbf{F})^+$, then $\int_U \min\{\xi(I) - \lambda_\mu(\xi)(I), \lambda_\mu(\xi)(I)\} = 0$.*

THEOREM 2.A.3. [**8**]. *If each of $\xi, \zeta$ and $\xi - \zeta$ is in $ba(\mathbf{F})^+$, then $\int_U \min\{\xi(I) - \zeta(I), \zeta(I)\} = 0$ if and only if there is a function $\beta$ from $\mathbf{F}$ into $\{0,1\}$ such that if $V$ is in $\mathbf{F}$, then $\int_V \beta(I)\xi(I)$ exists and is $\zeta(V)$.*

We now state two theorems, the first of which is half of a continuity characterization theorem and the second of which is a combination of half of a previous absolute continuity characterization theorem and invariance of absolute continuity theorem.

THEOREM 2.A.4. [**7**]. *If $N$ is a positive integer, $\{[a_k, b_k]\}_{k=1}^N$ is a sequence of number intervals, $f$ is a real-valued continuous function with domain $[a_1, b_1] \times \cdots \times [a_N, b_N], \xi$ is in $ba(\mathbf{F})^+$ and $\{\alpha_k\}_{k=1}^N$ is a sequence of elements of $b(\mathbf{F})$ such that, for all $k = 1, \ldots, N$, the range union of $\alpha_k \subseteq [a_k, b_k]$ and $\int_U \alpha_k(I)\xi(I)$ exists, then $\int_U f(\alpha_1(I), \ldots, \alpha_N(I))\xi(I)$ exists.*

THEOREM 2.A.5. [**4, 5**]. *If $\alpha$ is in $b(\mathbf{F}), \mu$ is in $ba(\mathbf{F})^+, \xi$ is in $A_\mu \cap ba(\mathbf{F})^+$ and $\int_U \alpha(I)\mu(I)$ exists, then $\int_U \alpha(I)\xi(I)$ exists and $\int \alpha\xi$ is absolutely continuous with respect to $\int \alpha\mu$.*

We end this section by stating three theorems that we shall use in subsequent sections. The proof of the first is quite routine and we leave it to the reader. The proofs of the second and third are in [**9**].

THEOREM 2.1. *Suppose that $\mu$ is in $ba(\mathbf{F})^+, \xi$ is in $A_\mu \cap ba(\mathbf{F})^+, \beta$ is in $b(\mathbf{F}), \int_U \beta(I)\mu(I)$ exists, $\{\alpha_k\}_{k=1}^\infty$ is a sequence of elements of $b(\mathbf{F}), M > 0$, for each $n$, range union of $\alpha_n \subseteq [-M, M]$ and $\int_U \alpha_n(I)\mu(I)$ exists. Suppose that $\int_U |\beta(I) - \alpha_n(I)|\mu(I) \to 0$ as $n \to \infty$.*

*Then* $\int_U \beta(I)\xi(I)$ *exists and, for each* $n, \int_U \alpha_n(I)\xi(I)$ *exists* (*by Theorem* 2.A.1), *and* $\int_U |\beta(I) - \alpha_n(I)|\xi(I) \to 0$ *as* $n \to \infty$.

THEOREM 2.A.6. [**9**]. *If* $\mu$ *is in* $ba(\mathbf{F})$ *and* $M$ *is a closed and bounded subset of* $\mathbf{R}$, *then* $\{\int \alpha\mu : \alpha$ *a function from* $\mathbf{F}$ *into* $\exp(M), \int_U \alpha(I)\mu(I)$ *exists*$\}$ *is closed in* $ba(\mathbf{F})$ *with respect to variation norm.*

THEOREM 2.A.7. [**9**]. *Suppose that* $M$ *is a closed and bounded subset of* $\mathbf{R}, \alpha$ *is a function from* $\mathbf{F}$ *into* $\exp(M)$ *and* $\mu$ *is in* $ba(\mathbf{F})$. *Then there is a function* $\beta$ *from* $\mathbf{F}$ *into* $M$ *such that if* $V$ *is in* $F$ *then* $\int_V \beta(I)\mu(I)$ *exists and is* $\int_V L(\alpha\mu)(I), L$ *being the sum supremum functional.*

**3. A partitioning theorem**. In this section we prove Theorem 3.2, as stated in the introduction. We begin with a lemma, well-known and easily shown, and a theorem.

LEMMA 3.1. *If each of* $\xi$ *and* $\mu$ *is in* $ba(\mathbf{F})^+$, *then* $\xi$ *is in* $A_\mu$ *if and only if, for each* $c > 0$ *and* $M > 0$, *there is a* $d > 0$ *such that if* $D \ll \{U\}, b$ *is a function from* $D$ *into* $[O, M]$ *and* $\sum_D b(I)\mu(I) < d$, *then* $\sum_D b(I)\xi(I) < c$.

THEOREM 3.1. *Suppose that each of* $\eta$ *and* $\mu$ *is in* $ba(\mathbf{F})^+, \eta$ *is in* $A_\mu$ *and* $0 < H$. *Then there is a function* $\beta$ *from* $\mathbf{F}$ *into* $\{0,1\}$ *such that* $\int_U \beta(I)\mu(I)$ *exists and such that if* $\gamma$ *is a function from* $\mathbf{F}$ *into* $\{0,1\}$ *such that* $\int_U \gamma(I)\mu(I)$ *exists, then if* $0 = \int_U \gamma(I)[1-\beta(I)]\mu(I)$ *then* $\int_U \gamma(I)\eta(I) \leq H\int_U \gamma(I)\mu(I)$, *and if* $0 = \int_U \gamma(I)\beta(I)\mu(I)$ *then* $\int_U \gamma(I)\eta(I) \geq H\int_U \gamma(I)\mu(I)$.

PROOF. There is a function $\phi$ from $\mathbf{F}$ into $\{0,1\}$ such that, for each $I$ in $\mathbf{F}, \phi(I) = 1$ if $\eta(I) \leq H\mu(I)$, and $\phi(I) = 0$ otherwise. By Theorem 2.A.7, there is a function $\beta$ from $\mathbf{F}$ into $\{0,1\}$ such that, for each $V$ in $\mathbf{F}, \int_V \beta(I)\mu(I)$ exists and is $\int_V L(\phi\mu)(I)$.

Now, suppose that $\gamma$ is a function from $\mathbf{F}$ into $\{0,1\}$ such that $\int_U \gamma(I)\mu(I)$ exists and $0 = \int_U \gamma(I)[1-\beta(I)]\mu(I)$. Suppose that $0 < c'$. Let $c = c'/(4(H+1))$. By Lemma 3.1, there is $d' > 0$ such that if

$D \ll \{U\}$, $b$ is a function from $D$ into $[0,1]$ and $\sum_D b(I)\mu(I) < d'$, then $\sum_D b(I)\eta(I) < c$. Let $d = \min\{d', c\}$. There is a $D \ll \{U\}$ such that if $E \ll D$, then $|\int_U \gamma(I)\mu(I) - \sum_E \gamma(J)\mu(J)| < c$, $|\int_U \gamma(I)\eta(I) - \sum_E \gamma(J)\eta(J)| < c$, $\sum_E |[L(\phi\mu)(J)/\mu(J)] - \beta(J)|\mu(J) < d/2$ and $\sum_E \gamma(J)(1-\beta(J))\mu(J) < d$, this last inequality implying that $\sum_E \gamma(J)$ $(1-\beta(J))\eta(J) < c$. For each $I$ in $D$, there is $E(I) \ll \{I\}$ such that $L(\phi\mu)(I) - \sum_{E(I)} \phi(J)\mu(J) < d/2N$, where $N$ is the number of elements of $D$, so that $\sum_D \sum_{E(I)} |[L(\phi\mu)(J)/\mu(J)] - \phi(J)|\mu(J) \leq \sum_D [L(\phi\mu)(I) - \sum_{E(I)} \phi(J)\mu(J)] < Nd/2N = d/2$, which implies that

$$\begin{aligned}
\sum_D \sum_{E(I)} \gamma(J)|\beta(J) - \phi(J)|\mu(J) \\
&\leq \sum_D \sum_{E(I)} |\beta(J) - \phi(J)|\mu(J) \\
&\leq \sum_D \sum E(I)|\beta(J) - [L(\phi\mu)(J)/\mu(J)]|\mu(J) \\
&+ \sum_D \sum E(I)|[L(\phi\mu)(J)/\mu(J)] - \phi(J)|\mu(J) \\
&< d/2 + d/2 = d,
\end{aligned}$$

so that $\sum_D \sum_{E(I)} \gamma(J)|\beta(J) - \phi(J)|\eta(J) < c$. Now,

$$\begin{aligned}
\int_U \gamma(I)\eta(I) &< c + \sum_D \sum_{E(I)} \gamma(J)\eta(J) \\
&< c + c + \sum_D \sum_{E(I)} \gamma(J)q\beta(J)\eta(J) \\
&< 2c + c + \sum_D \sum_{E(I)} \gamma(J)\phi(J)\eta(J) \\
&\leq 3c + \sum_D \sum_{E(I)} \gamma(J)H\mu(J) \\
&\leq 3c + H[\int_U \gamma(I)\mu(I) + c] \\
&= (3+H)c + H\int_U \gamma(I)\mu(I) < c' + H\int_U \gamma(I)\mu(I).
\end{aligned}$$

It therefore follows that $\int_U \gamma(I)\eta(I) \leq H\int_U \gamma(I)\mu(I)$.

Now, suppose that $\gamma$ is a function from $\mathbf{F}$ into $\{0,1\}$ such that $\int_U \gamma(I)\mu(I)$ exists and $0 = \int_U \gamma(I)\beta(I)\mu(I)$. The initial part of this portion is nearly a repetition of the preceding paragraph, except that the inequality $\sum_E \gamma(J)(1-\beta(J))\mu(J) < d$ is replaced by $\sum_E \gamma(J)\beta(J)\mu(J) < d$ (which, by implication, replaces the inequality $\sum_E \gamma(J)(1-\beta(J))\eta(J) < c$ with $\sum_E \gamma(J)\beta(J)\eta(J) < c$). Accordingly, we have that

$$\begin{aligned}
&\int_U \gamma(I)\eta(I)\\
&\qquad > -c + \sum_D \sum_{E(I)} \gamma(J)\eta(J)\\
&\qquad \geq -c + \sum_D \sum_{E(I)} \gamma(J)(1-\phi(J))\eta(J)\\
&\qquad \geq -c + \sum_D \sum_{E(I)} \gamma(J)H(1-\phi(J))\mu(J)\\
&\qquad = -c + H\sum_D \sum_{E(I)} \gamma(J)\mu(J) - H\sum_D \sum_{E(I)} \gamma(J)\phi(J)\mu(J)\\
&\qquad \geq -c + H[\int_U \gamma(I)\mu(I) - c] - H[\sum_D \sum_{E(I)} \gamma(J)\beta(J)\mu(J) + c]\\
&\qquad \geq -c + H\int_U \gamma(I)\mu(I) - 3Hc\\
&\qquad > H\int_U \gamma(I)\mu(I) - c'.
\end{aligned}$$

Therefore $\int_U \gamma(I)\eta(I) \geq H\int_U \gamma(I)\mu(I)$. $\square$

Proof of Theorem 3.2. By Theorem 3.1, there is a sequence $\{\delta_n\}_{n=1}^{\infty}$ of functions from $\mathbf{F}$ into $\{0,1\}$ such that if $n$ is a positive integer then $\int_U \delta_n(I)\mu(I)$ exists, and if $\gamma$ is a function from $\mathbf{F}$ into $\{0,1\}$ such that $\int_U \gamma(I)\mu(I)$ exists, then if $0 = \int_U \gamma(I)[1-\delta_n(I)]\mu(I)$ then $\int_U \gamma(I)\xi(I) \leq n\int_U \gamma(I)\mu(I)$, and if $0 = \int_U \gamma(I)\delta_n(I)\mu(I)$ then $\int_U \gamma(I)\xi(I) \geq n\int_U \gamma(I)\mu(I)$.

We define the sequence $\{\beta_k\}_{k=1}^{\infty}$ as follows: $\beta_1 = \delta_1$, and if $n$ is a positive integer $> 1$, then $\beta_n = \left(\prod_{k=1}^{n-1}(1-\delta_k)\right)\delta_n$.

Clearly, if $n$ is a positive integer, then $\int_U \beta_n(I)\mu(I)$ exists.

Suppose that $m$ and $n$ are positive integers, $m < n$ and $I$ is in $\mathbf{F}$. Since $\delta_m(I)$ is a factor of $\beta_m(I)$ and $1-\delta_m(I)$ is a factor of $\beta_n(I)$ and $\delta_m(I)(1-\delta_m(I)) = 0$, it follows that $\beta_m(I)\beta_n(I) = 0$.

Now, suppose that $\gamma$ is a function from $\mathbf{F}$ into $\{0,1\}$ such that $\int_U \gamma(I)\mu(I)$ exists and $n$ is a positive integer such that $\int_U \gamma((I))(1-\beta_n(I))$ $\mu(I) = 0$. If $n = 1$, then $\beta_n = \beta_1 = \delta_1$, so that $(1-1)\int_U \gamma(I)\mu(I) = 0 \leq \int_U \gamma(I)\xi(I) \leq 1\int_U \gamma(I)\mu(I)$. Suppose that $1 < n$. Since $\beta_n = \big(\prod_{k=1}^{n-1}(1-\delta_k)\big)\delta_n$, it follows that

$$\begin{aligned}
0 &\leq \max\Big\{\int_U \gamma(I)(1-\delta(I))\mu(I), \int_U \gamma(I)\delta_{n-1}(I)\mu(I)\Big\}\\
&= \max\Big\{\int_U \gamma(I)(1-\delta_n(I))\mu(I), \int_U \gamma(I)(1-(1-\delta_{n-1}(I)))\mu(I)\Big\}\\
&\leq \int_U \gamma(I)(1-\Big(\prod_{k=1}^{n-1}(1-\delta_k(I))\Big)\delta_n(I))\mu(I) = 0,
\end{aligned}$$

so that $(n-1)\int_U \gamma(I)\mu(I) \leq \int_U \gamma(I)\xi(I) \leq n\int_U \gamma(I)\mu(I)$.

Finally, we show that $\sum_{k=1}^{\infty}\int_U \beta_k(I)\mu(I) = \mu(U)$. First, $\delta_1 + (1-\delta_1) = 1$. If $m$ is a positive integer and $\big(\sum_{k=1}^{m}\beta_k\big) + \big(\prod_{k=1}^{m}(1-\delta_k)\big) = 1$, then

$$\begin{aligned}
&\Big(\sum_{k=1}^{m+1}\beta_k\Big) + \Big(\prod_{k=1}^{m+1}(1-\delta_k)\Big)\\
&\qquad = \Big(\sum_{k=1}^{m}\beta_k\Big) + \Big(\prod_{k=1}^{m}(1-\delta_k)\Big)\delta_{m+1} + \Big(\prod_{k=1}^{m}(1-\delta_k)\Big)(1-\delta_{m+1})\\
&\qquad = \Big(\sum_{k=1}^{m}\beta_k\Big) + \prod_{k=1}^{m}(1-\delta_k) = 1.
\end{aligned}$$

Thus, by induction, for all positive integers $n$,

$$\Big(\sum_{k=1}^{n}\beta_k\Big)+\prod_{k=1}^{n}(1-\ \delta_k)=1,$$

so that

$$\begin{aligned}
&\sum_{k=1}^{n+1}\int_U \beta_k(I)\mu(I)\\
&\qquad=\int_U \Big[\Big(\sum_{k=1}^{n}\beta_k(I)\Big)+\Big(\prod_{k=1}^{n}(1-\delta_k(I))\Big)\delta_{n+1}(I)\Big]\mu(I)\\
&\qquad=\int_U \Big[\Big(\sum_{k=1}^{n}\beta_k(I)\Big)+\Big(\prod_{k=1}^{n}(1-\delta_k(I))\Big)\\
&\qquad\qquad-\Big(\pi_{k=1}^{n}(1-\delta_k(I))\Big)(1-\delta_{n+1}(I))\Big]\mu(I)\\
&\qquad=\int_U \Big[1-\prod_{k=1}^{n+1}(1-\delta_k(I))\Big]\mu(I)\\
&\qquad=\mu(U)-\int_U \Big(\prod_{k=1}^{n+1}(1-\delta_k(I))\Big)\mu(I).
\end{aligned}$$

Now, $\int_U \big(\Pi_{k=1}^{n+1}(1-\delta_k(I))\big)\mu(I)\le\int_U(1-\delta_{n+1}(I))\mu(I)$. Since $\delta_{n+1}=\delta_{n+1}\delta_{n+1}$, it follows that $\int_U(1-\delta_{n+1}(I))\delta_{n+1}(I)\mu(I)=\int_U(\delta_{n+1}(I)-\delta_{n+1}(I)\delta_{n+1}(I))\mu(I)=0$, so that $(n+1)\int_U(1-\delta_{n+1}(I))\mu(I)\le\int_U(1-\delta_{n+1}(I)\xi)(I)\le\xi(U)$, so that $\int_U(1-\delta_{n+1}(I))\mu(I)\le(1/(n+1))\xi(U)$. Clearly, then, $\int_U \big(\Pi_{k=1}^{n+1}(1-\delta_k(I))\big)\mu(I)\to 0$ as $n\to\infty$, so that $\sum_{k=1}^{\infty}\int_U\beta_k(I)\mu(I)=\mu(U)$. □

## 4. A uniform absolute continuity characterization theorem.

In this section we prove Theorem 4.1, as stated in the introduction.

We begin by stating a lemma which can be easily established by a routine induction argument.

LEMMA 4.1. *Suppose that $n$ is a positive integer and $\{\eta_k\}_{k=1}^{n+1}$ is a sequence of elements of $ba(\mathbf{F})^+$ such that if $r$ and $s$ are distinct* *positive integers* $\leq n+1$, *then* $\int_U \min\{\eta_r(I), \eta_s(I)\} = 0$. *Then, if* $V$ *is in* $\mathbf{F}$, $0 = \int_V \min\{\eta_1(I), \ldots, \eta_{n+1}(I)\}$ and $\sum_{k=1}^{n+1} \eta_k(V) = \int_V \max\{\eta_1(I), \ldots, \eta_{n+1}(I)\}$.

Proof of Theorem 4.1. Suppose that (1) is true and the hypothesis of (2) is satisfied. From Lemma 4.1 it follows that, for each positive integer $n$, $\sum_{k=1}^{n+1} \int_U \beta_k(I)\mu(I) = \int_U \max\{\beta_1(I), \ldots, \beta_{n+1}(I)\}\mu(I) \leq \mu(U)$, which implies that $\int_U \beta_n(I)\mu(I) \to 0$ as $n \to \infty$.

Now suppose that $0 < c$. There is a $d > 0$ such that if $V$ is in $\mathbf{F}$, $\mu(V) < d$ and $\xi$ is in $G$, then $\xi(V) < c/2$. There is a positive integer $N$ such that if $m$ is a positive integer $\geq N$, then $\int_U \beta_m(I)\mu(I) < d$. Suppose that $m$ is a positive integer $\geq N$ and $\xi$ is in $G$. There is $D \ll \{U\}$ such that if $E \ll D$, then $\int_U \beta_m(I)\xi(I) < c/2 + \sum_E \beta_m(W)\xi(W)$ and $\sum_E \beta_m(W)\mu(W) < d$, so that if $E^* = \{W : W \text{ in } E, \beta_m(W) = 1\}$, then $\sum_{E^*} \mu(W) = \sum_E \beta_m(W)\mu(W) < d$, so that $c/2 > \sum_{E^*} \xi(W) = \sum_E \beta_m(W)\xi(W)$, so that $\int_U \beta_m(I)\xi(I) < 2c/2 = c$.

Therefore (1) implies (2).

Now suppose that (2) is true, but that $G$ is not uniformly absolutely continuous with respect to $\mu$. Then there is $c > 0$ such that if $d > 0$ then there is $V$ in $\mathbf{F}$ and $\xi$ in $G$ such that $\mu(V) < d$, but $\xi(V) \geq c$. It therefore follows that if $\eta$ is in $G$ and $0 < h$, then there is $d > 0$ such that if $V$ is in $\mathbf{F}$ and $\mu(V) < d$ then $\eta(V) < h$ and there is $W$ in $\mathbf{F}$ and $\zeta$ in $G$ such that $\mu(W) < d$, but $\zeta(W) \geq c$, and, since $\mu(W) < d, \eta(W) < h$. It therefore follows by induction that there is a sequence $\{W_k\}_{k=1}^{\infty}$ of sets of $\mathbf{F}$ and a sequence $\{\xi_k\}_{k=1}^{\infty}$ of elements of $G$ such that, for each nonnegative integer $m$ and positive integer $k$, $\xi_k(W_{k+m}) \geq c$ if $m = 0$, and $\xi_k(W_{k+m}) < c/2^{m+1}$ if $m > 0$.

If each of $m$ and $n$ is a positive integer, then $\xi_n(\cup_{k=1}^m W_{n+k}) \leq \sum_{k=1}^m \xi_n(W_{n+k}) \leq \sum_{k=1}^m (c/2^{k+1}) < c/2$, so that $W_n$ is not a subset of $\cup_{k=1}^m W_{n+k}$; furthermore, letting $Y_{n,m} = W_n - [W_n \cap (\cup_{k=1}^m W_{n+k})] = W_n - \cup_{k=1}^m [W_n \cap W_{n+k}]$, we see that $\xi_n(Y_{n,m}) = \xi_n(W_n) - \xi(\cup_{k=1}^m [W_n \cap W_{n+k}]) > c - c/2 = c/2$.

Clearly, if each of $s, t$ and $n$ is a positive integer and $s \leq t$, then $Y_{n,t} \subseteq Y_{n,s} \subseteq W_n$.

Now suppose that each of $p, q, s$ and $t$ is a positive integer and $0 < q - p \leq t$. Then $Y_{p,q-p} = W_p - \cup_{k=1}^{q-p}[W_p \cap W_{p+k}]$, so that if

$x$ is in $Y_{p,q-p}$, then $x$ is not in $W_{p+q-p} = W_q$ implying that, if $x$ is in $Y_{p,t}$, then $x$ is in $Y_{p,q-p}$. Consequently, $x$ is not in $W_q$ and so is not in $Y_{q,s}$. Therefore, $0 = \int_U \min\{X(Y_{p,t})(I), X(Y_{q,s})(I)\}\mu(I)$.

Suppose that $n$ is a positive integer. From the remark preceding the above paragraph, we see that if each of $s$ and $t$ is a positive integer and $s \leq t$, then $\int X(Y_{n,s})\mu - \int X(Y_{n,t})\mu$ is in $ba(\mathbf{F})^+$, so that, by routine considerations, there is an element $\eta_n$ of $ba(\mathbf{F})^+$ such that $\int_U |\eta_n(I) - \int_I X(Y_{n,v})(J)\mu(J)| \to 0$ as $v \to \infty$. From Theorem 2.A.6 it follows that there is a function $\beta_n$ from $\mathbf{F}$ into $\{0,1\}$ such that if $V$ is in $\mathbf{F}$, then $\int_V \beta_n(I)\mu(I)$ exists and is $\eta_n(V)$, so that $\int_U |\beta_n(I) - X(Y_{n,v})(I)|\mu(I) \to 0$ as $v \to \infty$. This last remark implies two things. First, if $p, q, s$ and $t$ are positive integers and $0 < q-p \leq t$, then $\int_U \min\{\beta_p(I), \beta_q(I)\}\mu(I) = \int_U \min\{\beta_p(I), \beta_q(I)\}\mu(I) - \int_U \min\{X(Y_{p,t})(I), X(Y_{q,s})(I)\}\mu(I) \leq \int_U |\beta_p(I) - X(Y_{p,t})(I)|\mu(I) + \int_U |\beta_q(I) - X(Y_{q,s})(I)|\mu(I) \to 0$ as $\min\{t,s\} \to \infty$, which implies that $\int_U \min\{\beta_p(I), \beta_q(I)\}\mu(I) = 0$, so that, by hypothesis, $\int_U \beta_w(I)\xi(I) \to 0$ as $w \to \infty$ for $\xi$ in $G$, uniformly on $G$. Second, by Theorem 2.1, for all positive integers $n$ and $v$, $\delta_{n,v} = \int_U |\beta_n(I) - X(Y_{n,v})(I)|\xi_n(I) \to 0$ as $v \to \infty$, so that $\int_U \beta_n(I)\xi_n(I) \geq \int_U X(Y_{n,v})(I)\xi_n(I) - \int_U |X(Y_{n,v})(I) - \beta_n(I)|\xi_n(I) = \xi_n(Y_{n,v}) - \delta_{n,v} \geq c/2 - \delta_{n,v}$, which implies that $\int_U \beta_n(I)\xi_n(I) \geq c/2$, a contradiction to the statement that $\int_U \beta_n(I)\xi(I) \to 0$ as $n \to \infty$ for $\xi$ in $G$, uniformly on $G$.

Therefore (2) implies (1).

Therefore (1) and (2) are equivalent. $\square$

**5. A uniform absolute continuity theorem**. In this section we prove Theorem 5.1, as stated in the introduction.

We assume the hypothesis of the theorem and prove a lemma.

LEMMA 5.1. *If $\gamma$ is a function from $\mathbf{F}$ into $\{0,1\}$ such that $\int_U \gamma(I)\eta(I)$ exists, then* (*clearly by Theorem* 2.A.5) *$\int_U \gamma(I)\xi(I)$ exists for each $\xi$ in $G$, and $\int \gamma G$, i.e., $\{\int \gamma\xi : \xi \text{ in } G\}$, is uniformly absolutely continuous with respect to $\int \gamma\eta$.*

PROOF. Suppose that $0 < c$. There is $d > 0$ such that if $W$ is in $\mathbf{F}$ and $\eta(W) < d$ and $\xi$ is in $G$, then $\xi(W) < c/2$. Now, suppose that $V$ is in $\mathbf{F}$, $\int_V \gamma(J)\eta(J) < d$ and $\xi$ is in $G$. There is $D \ll \{V\}$ such that if $E \ll D$, then $d > \sum_E \gamma(I)\eta(I) = \sum_{E'} \eta(I)$, where $E' = \{I : I \text{ in } E, \gamma(I) = 1\}$, so that $c/2 > \sum_{E'} \xi(I) = \sum_E \gamma(I)\xi(I)$, which implies that $\int_V \gamma(J)\xi(J) \leq c/2 < c$.

Therefore $\int \gamma G$ is uniformly absolutely continuous with respect to $\int \gamma\eta$. □

We now prove Theorem 5.1.

PROOF OF THEOREM 5.1. Since, by Corollary 2.A.3, $\int_U \min\{\eta(I) - \lambda_\mu(\eta)(I), \lambda_\mu(\eta)(I)\} = 0$, it follows from Theorem 2.A.3 that there is a function $\beta$ from $\mathbf{F}$ into $\{0, 1\}$ such that if $V$ is in $\mathbf{F}$, then $\int_V \beta(I)\eta(I)$ exists and is $\lambda_\mu(\eta)(V)$.

From Lemma 5.1 it follows that each of $\int \beta G$ and $\int (1 - \beta)G$ is uniformly absolutely continuous with respect to $\int \beta\eta$ and $\int (1 - \beta)\eta$, respectively.

Now, suppose that $\xi$ is in $G$. We shall show that $\int \beta\xi = \xi$. By Theorem 2.A.1, $0 = \int_U \min\{\eta(I) - \lambda_\mu(\eta)(I), \mu(I)\} = \int_U \min\{\int_I (1 - \beta(I))\eta(I), \mu(I)\}$, so that, since $\int (1 - \beta)\xi$ is absolutely continuous with respect to $\int (1 - \beta)\eta$, it follows from Corollary 2.A.2 that $0 = \int_U \min\{\int_I (1 - \beta(I))\xi(I), \mu(I)\}$. From the hypothesis it clearly follows that $\int (1 - \beta)\xi$ is in $A_\mu$. Therefore, by Theorem 2.A.2, $0 = \int_U (1 - \beta(I))\xi(I)$, so that $\int \beta\xi = \xi$.

It immediately follows, since $\int \beta G$ is uniformly absolutely continuous with respect to $\int \beta\eta$, that $G$ is uniformly absolutely continuous with respect to $\int \beta\eta$ and so, trivially, with respect to $\mu$. □

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Mathematics Department, North Texas State University, Denton, TX 76203