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A NOTE ON THE CYCLIC COHOMOLOGY AND K-THEORY ASSOCIATED WITH DIFFERENCE OPERATORS

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ABSTRACT. The index map of K_0-theory associated with a difference operator is given. In the odd dimension case, a theorem on the cyclic cohomology is established.

1. This note is a continuation of the author's previous paper [**2**]. Let $\mathcal{A}$ and $\mathcal{A}_1$ be two algebras over $\mathbf{C}$ satisfying $\mathcal{A} \subset \mathcal{A}_1$. As it is introduced in [**2**], an operator δ from $\mathcal{A}$ into $\mathcal{A}_1$ is said to be a difference operator if δ is linear and satisfies

$$\delta(fg) = f\delta g + (\delta f)g - (\delta f)\delta g \tag{1}$$

for $f, g \in \mathcal{A}$.

In [**2**], the following theorem is proved.

THEOREM. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A}$ a subalgebra of $\mathcal{L}(\mathcal{H})$ and δ a difference operator from $\mathcal{A}$ into $\mathcal{L}(\mathcal{H})$ satisfying*

$$\delta f \in \mathcal{L}^p(\mathcal{H}), \quad f \in \mathcal{A},$$

where $p \geq 1$. Let n be an even number satisfying $n \geq p-1$ and

$$\psi_n(f_0, \dots, f_n) = \operatorname{tr}(\delta f_0 \cdots \delta f_n), \quad f_0, \dots, f_n \in \mathcal{A}.$$

Then ψ_n is a cyclic cocycle. If $n \geq p+1$, then ψ_n is in the cyclic cohomology class containing $bR_{n-2}\psi_{n-2}$, where R_k is the operation

$$(R_k\xi)(f_0, f_1, \dots, f_{k+1}) = \frac{2}{k+2}\sum_{j=0}^{k}(-1)^j(k-j+1)\xi(f_jf_{j+1}, f_{j+2}, \dots, f_{j+k+1})$$

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with $f_{j+k+2} = f_j, 0 \leq j \leq k-1$.

In §2 of this note, we will study the K_0-theory associated with the difference operator δ and the functional ψ_n.

In §3, we will study the cyclic cohomology in the case of odd n. The simplest case is the following. Suppose there is an operator $P \in \mathcal{L}(\mathcal{H})$ satisfying $P^2 = I$ and the anticommutator

$$\{P, \delta(f)\} = 0, \quad f \in \mathcal{A}.$$

Then define

$$\psi_n(f_0, f_1, \dots, f_n) = \operatorname{tr}(P\delta(f_0)\cdots\delta(f_n)).$$

But we will deal with a more general case.

Theorem 1 and Theorem 2 in this note may be considered as generalizations of the corresponding theorems in [**1**].

2. In order to study the K_0-theory associated with the difference operator δ from the algebra $\mathcal{A}$ to the algebra $\mathcal{A}_1$, we have to define an operator $\tilde{\delta}$ from $M_k(\mathcal{A})$ to $M_k(\mathcal{A}_1)$ by

$$\tilde{\delta}[a_{ij}] = [\delta a_{ij}] \quad \text{for} \quad [a_{ij}] \in M_k(\mathcal{A}).$$

It is obvious that

$$\begin{aligned}\tilde{\delta}([a_{ij}][b_{ij}]) &= \Big[\sum_\ell \delta(a_{i\ell}b_{\ell j})\Big] \\ &= \Big[\sum_\ell (\delta a_{i\ell})b_{\ell j}\Big] + \Big[\sum_\ell a_{i\ell}\delta b_{\ell j}\Big] - \Big[\sum_\ell (\delta a_{i\ell})\delta b_{\ell j}\Big] \\ &= (\tilde{\delta}[a_{ij}])[b_{ij}] + [a_{ij}]\tilde{\delta}[b_{ij}] - (\tilde{\delta}[a_{ij}])\tilde{\delta}[b_{ij}].\end{aligned}$$

Thus $\tilde{\delta}$ is a difference operator from $M_k(\mathcal{A})$ to $M_k(\mathcal{A}_1)$. For simplicity, the operator $\tilde{\delta}$ is still denoted by δ.

If $e \in \operatorname{Proj} M_k(\mathcal{A})$ and $\delta e \in \mathcal{L}^p$, then δe is a compact operator. Define

$$P_e = \frac{1}{2\pi i}\int_\gamma (\lambda - e(\delta e)e)^{-1} d\lambda$$

and

$$Q_e = \frac{1}{2\pi i}\int_\gamma (\lambda + (1-e)(\delta e)(1-e))^{-1} d\lambda,$$

where γ is a counter-clockwise contour $|\lambda - 1| = \varepsilon > 0$, where ε is chosen such that

$$(\sigma(e(\delta e)e) \cup \sigma(-(1-e)(\delta e)(1-e))) \cap \{\lambda : 0 < |\lambda - 1| \leq \varepsilon\} = \phi.$$

It is obvious that P_e and Q_e are independent of ε.

Let δ_e be the operator from the range of P_e to the range of Q_e defined by

$$\delta_e = (1-e)(\delta e)e|_{\text{range of } P_e}.$$

In the proof of the following theorem, it will be shown that the range of $(1-e)(\delta e)eP_e$ is in the range of Q_e.

THEOREM 1. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A}$ a subalgebra of $\mathcal{L}(\mathcal{H})$ and δ be a difference operator from $\mathcal{A}$ into $\mathcal{L}(\mathcal{H})$ satisfying*

$$\delta f \in \mathcal{L}^p(\mathcal{H}) \quad \text{for } f \in \mathcal{A},$$

where $p \geq 1$. Then the index map $K_0(\mathcal{A}) \to \mathbf{Z}$ is given by

$$\text{Index}(\delta_e) = \text{tr}(\delta e)^q = \text{rank} P_e - \text{rank} Q_e \tag{2}$$

for every $e \in M_k(\mathcal{A})$ satisfying $e = e^2$, where $q \geq p$ is an odd number.

PROOF. Without loss of generality, we may assume that $k = 1$. Let $\mathcal{H}_1 = e\mathcal{H}$ and $\mathcal{H}_2 = (1-e)\mathcal{H}$. Denote

$$a_{11} = e(\delta e)e|_{\mathcal{H}_1}, \qquad a_{22} = -(1-e)(\delta e)(1-e)|_{\mathcal{H}_2},$$
$$a_{12} = e(\delta e)(1-e)|_{\mathcal{H}_2}, \qquad a_{21} = (1-e)(\delta e)e|_{\mathcal{H}_1}.$$

Then the operator δe may be written as a matrix

$$\delta e = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{22} \end{pmatrix}. \tag{3}$$

From (1) and $e = e^2$, it is easy to see that

$$(\delta e)^2 = e\delta e + (\delta e)e - \delta e. \tag{4}$$

From (3), (4) and

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{5}$$

we get

$$(\delta e)^2 = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}. \tag{6}$$

From (3) and (6), we obtain that

$$(\delta e)^{2m+1} = \begin{pmatrix} a_{11}^{m+1} & a_{11}a_{22}^m \\ a_{21}a_{11}^m & -a_{22}^{m+1} \end{pmatrix}, \tag{7}$$

and

$$a_{11} - a_{11}^2 = a_{12}a_{21}, \quad a_{22} - a_{22}^2 = a_{21}a_{12}, \tag{8}$$

$$a_{11}a_{12} = a_{12}a_{22}, \quad a_{21}a_{11} = a_{22}a_{21}. \tag{9}$$

Therefore $a_{jj}^{m+1} \in \mathcal{L}^p$, $j = 1, 2$, and

$$\operatorname{tr}((\delta e)^{2m+1}) = \operatorname{tr}(a_{11}^{m+1}) - \operatorname{tr}(a_{22}^{m+1}) \tag{10}$$

for $2m + 1 \geq p$.

If $\lambda \in \rho(a_{11}) \cap \rho(a_{22})$, then

$$(\lambda - a_{11})^{-1}a_{12} = a_{12}(\lambda - a_{22})^{-1}$$

and

$$a_{21}(\lambda - a_{11})^{-1} = (\lambda - a_{22})^{-1}a_{21}$$

by (9). Hence

$$P_e a_{12} = a_{12}Q_e, \quad a_{21}P_e = Q_e a_{21}. \tag{11}$$

The ranks of P_e and Q_e are finite, since a_{11} and a_{22} are compact. Thus, there is a natural number n such that

$$P_e(1-a_{11})^n = 0, \quad Q_e(1-a_{22})^n = 0.$$

By (8), it is easy to calculate that

$$\begin{aligned} P_e(1-a_{11}^{m+1}) &= P_e \sum_{j=0}^{m} a_{11}^j \sum_{k=1}^{n-1}((1-a_{11})^k - (1-a_{11})^{k+1}) \\ &= P_e \sum_{j=0}^{m} a_{11}^j \sum_{k=1}^{n-1}(1-a_{11})^{k-1} a_{12}a_{21}. \end{aligned}$$

Similarly, we have

$$Q_e(1-a_{22}^{m+1}) = Q_e \sum_{j=0}^{m} a_{22}^j \sum_{k=1}^{n-1}(1-a_{22})^{k-1} a_{21}a_{12}.$$

By (9) and (11), it is easy to prove that

$$Q_e(1-a_{22}^{m+1}) = a_{21} P_e \sum_{j=0}^{m} a_{11}^j \sum_{k=1}^{n-1}(1-a_{11})^{k-1} a_{12}.$$

Therefore

$$\operatorname{tr}(P_e(1-a_{11}^{m+1}) - Q_e(1-a_{22}^{m+1})) = 0. \tag{12}$$

Denote $a_1 = (1-P_e)a_{11}$ and $a_2 = (1-Q_e)a_{22}$. Identities (9) and (11) imply that

$$a_1 a_{12} = a_{12} a_2 \quad \text{and} \quad a_{21} a_1 = a_2 a_{21}.$$

On the other hand, the operators

$$(1-a_1)|_{(1-P_e)\mathcal{H}_1} \quad \text{and} \quad (1-a_2)|_{(1-Q_e)\mathcal{H}_2} \tag{13}$$

are invertible in $(1-P_e)\mathcal{H}_1$ and $(1-Q_e)\mathcal{H}_2$ respectively. Denote the inverses of the operators in (13) by b_1 and b_2 respectively. Then

$$a_1 = b_1(1-P_e)a_{12}a_{21}, \quad a_2 = b_2(1-Q_e)a_{21}a_{12}$$

by (8).

By (9), it is easy to see that

$$b_2 a_{21} = a_{21} b_1.$$

Therefore

$$a_2 = a_{21} b_1 (1 - P_e) a_{12}.$$

Thus

$$a_1^{m+1} = Q a_{21}, \quad a_2^{m+1} = a_{21} Q,$$

where $Q = b_1(1 - P_e)(a_{12}a_{21}b_1(1 - P_e))^m a_{12}$. Hence

$$\operatorname{tr}((1 - P_e)a_{11}^{m+1} - (1 - Q_e)a_{22}^{m+1}) = 0. \tag{14}$$

From (12) and (14) we get

$$\begin{aligned}\operatorname{tr}(a_{11}^{m+1} - a_{22}^{m+1}) &= \operatorname{tr}(P_e - Q_e)\\ &= \operatorname{rank} P_e - \operatorname{rank} Q_e.\end{aligned}$$

On the other hand, from (11), it is easy to see that

$$\delta_e \mathcal{H}_1 = a_{21}(P_e \mathcal{H}_1) \subseteq Q_e \mathcal{H}_2.$$

Therefore

$$\operatorname{Index}(\delta_e) = \operatorname{rank} P_e - \operatorname{rank} Q_e,$$

which proves (2).

By the same method of proving the Proposition (14a), [**1**, Chapter II] and [**2**, Theorem 1], we may prove that $\operatorname{tr}(\delta e)^q$ depends only on the equivalence class of e. Theorem 1 is proved. □

3. For the case of odd n, we have to introduce two operators as follows. Let $\mathcal{A}$ and $\mathcal{A}_1$ be the two algebras, ρ be a linear operator from $\mathcal{A}$ to $\mathcal{A}_1$ and ε be a bilinear operator from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}_1$ satisfying the condition

$$\varepsilon(fg, h) - \varepsilon(f, gh) = \rho(f)\varepsilon(g, h) - \varepsilon(f, g)\rho(h). \tag{15}$$

For a given ρ, the simplest example of $\varepsilon(\cdot,\cdot)$ satisfying (15) is

$$\varepsilon(f,g) = (\rho(fg) - \rho(f)\rho(g))a, \tag{16}$$

where a is an element in $\mathcal{A}_1$ commuting $\{\rho(f) : f \in \mathcal{A}\}$.

Another example is the following. Let δ be a difference operator from $\mathcal{A}$ to $\mathcal{A}_1, P \in \mathcal{A}_1$ satisfying $P^2 = 1$ and

$$P\delta f + (\delta f)P = 0 \quad \text{for } f \in \mathcal{A}.$$

Define $E = (1+P)/2, \rho(f) = EfE$ and

$$\varepsilon(f,g) = E\delta f \delta g.$$

Then ρ and ε satisfying (15), since $E^2 = E$ and $[E, \delta f \delta g] = 0$.

LEMMA 1. *Let ρ be a linear operator from algebra $\mathcal{A}$ to the algebra $\mathcal{A}_1, \varepsilon$ be a bilinear operator from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}_1$ satisfying* (15) *and $\hat{\varepsilon}(f,g) = \rho(fg) - \rho(f)\rho(g)$. Then*

$$\varepsilon(f,g)\hat{\varepsilon}(h,k) = \hat{\varepsilon}(f,g)\varepsilon(h,k), \quad f,g,h,k \in \mathcal{A}. \tag{17}$$

PROOF. From (15), we get

$$\rho(fg)\varepsilon(h,k) - \varepsilon(fg,h)\rho(k) = \varepsilon(fgh,k) - \varepsilon(fg,hk) \tag{18}$$

and

$$(\varepsilon(fg,h) - \varepsilon(f,gh))\rho(k) = \rho(f)\varepsilon(g,h)\rho(k) - \varepsilon(f,g)\rho(h)\rho(k). \tag{19}$$

From (18) and (19), it is easy to see that

$$\begin{aligned} &\rho(fg)\varepsilon(h,k) + \varepsilon(f,g)\rho(h)\rho(k) \\ &\quad = \varepsilon(fgh,k) + \rho(f)\varepsilon(g,h)\rho(k) + \varepsilon(f,gh)\rho(k) - \varepsilon(fg,hk). \end{aligned} \tag{20}$$

Similarly, we may prove that

$$\begin{aligned} &\varepsilon(f,g)\rho(hk) + \rho(f)\rho(g)\varepsilon(h,k) \\ &\quad = \varepsilon(f,ghk) + \rho(f)\varepsilon(g,h)\rho(k) + \rho(f)\varepsilon(gh,k) - \varepsilon(fg,hk). \end{aligned} \tag{21}$$

Subtracting (20) from (21), we get

$$\begin{aligned}&\varepsilon(f,g)\hat{\varepsilon}(h,k) - \hat{\varepsilon}(f,g)\varepsilon(h,k)\\&\qquad = \varepsilon(f,ghk) - \varepsilon(fgh,k) - (\varepsilon(f,gh)\rho(k) - \rho(f)\varepsilon(gh,k))\end{aligned}$$

which equals zero, by (15). This proves (17). □

If k is odd and ξ is a $k+1$-linear functional, then define the operation R_k

$$\begin{aligned}&(R_k\xi)(f_0, f_1, \ldots, f_{k+1})\\&\qquad = \frac{1}{2(k+2)}\sum_{j=0}^{k}(k-j+1)\xi(f_j f_{j+1}, f_{j+2}, \ldots, f_{j+k+1}),\end{aligned}$$

where $f_j = f_{j-k-2}$ for $j \geq k+2$.

THEOREM 2. *Let $\mathcal{A}_1$ be an algebra, $J \subset \mathcal{A}_1$ a two-side ideal, $p \in \mathbf{N}$ and τ a linear functional on J^p such that*

$$\tau(ab) = \tau(ba) \quad \text{for } a \in J^k, b \in J^q, k+q = p.$$

Let $\mathcal{A}$ be an algebra, $\rho : \mathcal{A} \to \mathcal{A}_1$ be a bilinear map and $\varepsilon(\cdot,\cdot)$ be a bilinear map from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}_1$ satisfying (15) and

$$\varepsilon(f,g) \in J, \quad \text{for } f, g \in \mathcal{A}.$$

Let ψ_n be the $n+1$ linear functional on $\mathcal{A}$ given by

$$\psi_n(f_0, f_1, \ldots, f_n) = \tau(\varepsilon_0\varepsilon_2\cdots\varepsilon_{n-1} - \varepsilon_1\varepsilon_3\cdots\varepsilon_n),$$

where $n = 2m-1, m \geq p$,

$$\varepsilon_j = \varepsilon(f_j, f_{j+1}), \quad j = 0, 1, 2, \ldots, 2p-1,$$

$$\varepsilon_j = \hat{\varepsilon}(f_j, f_{j+1}) = \rho(f_j f_{j+1}) - \rho(f_j)\rho(f_{j+1}), \quad j = 2p, \ldots, n,$$

and $f_{n+1} = f_0$. Then ψ_n is a cyclic n-cocycle.

If $m > p$ then ψ_n is in the cyclic cohomology class containing

$$bR_{n-2}\psi_{n-2}, \tag{22}$$

where b is the Hochschild coboundary operation.

PROOF. Similar to the proof of the proposition 4 of [**2**], let

$$\psi^+(f_0, \cdots, f_n) = \tau(\varepsilon_0\varepsilon_2 \ldots \varepsilon_{n-1})$$

and

$$\psi^-(f_0, \cdots, f_n) = \tau(\varepsilon_1\varepsilon_3 \ldots \varepsilon_n).$$

Then $\psi_n = \psi^+ - \psi^-$. It is easy to see that

$$\begin{aligned} b\psi^+&(f_0, f_1, \ldots, f_{n+1}) - \tau(\varepsilon(f_{n+1}f_0, f_1)\varepsilon_2 \cdots \varepsilon_{n-1}) \\ &= \sum_{j=0}^{p-1} \tau(\varepsilon_0 \cdots \varepsilon_{2j-2}(\varepsilon(f_{2j}f_{2j+1}, f_{2j+2}) \\ &\qquad - \varepsilon(f_{2j}, f_{2j+1}f_{2j+2}))\varepsilon_{2j+3} \cdots \varepsilon_n) \\ &\quad + \sum_{j=p}^{m-1} \tau(\varepsilon_0 \cdots \varepsilon_{2j-2}(\hat{\varepsilon}(f_{2j}f_{2j+1}, f_{2j+2}) \\ &\qquad\quad - \hat{\varepsilon}(f_{2j}, f_{2j+1}f_{2j+2}))\varepsilon_{2j+3} \cdots \varepsilon_n) \\ &= \sum_{j=0}^{m-1} \tau(\varepsilon_0 \cdots \varepsilon_{2j-2}(\rho(f_{2j})\varepsilon_{2j+1} - \varepsilon_{2j}\rho(f_{2j+2}))\varepsilon_{2j+3} \cdots \varepsilon_n) \\ &= \tau(\rho(f_0)\varepsilon_1 \cdots \varepsilon_n) - \tau(\varepsilon_0 \cdots \varepsilon_{n-1}\rho(f_{n+1})), \end{aligned} \tag{23}$$

where $\varepsilon_n = \hat{\varepsilon}(f_n, f_{n+1})$ for $m > p$ or $\varepsilon_n = \varepsilon(f_n, f_{n+1})$ for $m = p$. Similarly, if $m > p$, then

$$\begin{aligned} b\psi^-&(f_0, f_1, \ldots, f_{n+1}) \\ &\quad - \tau(\varepsilon_2 \cdots \varepsilon_{2p-2}\varepsilon(f_{2p}, f_{2p+1})\varepsilon_{2p+2} \cdots \hat{\varepsilon}(f_{n+1}, f_0f_1)) \\ &= \tau(\varepsilon_1 \cdots \hat{\varepsilon}(f_n, f_{n+1})\rho(f_0)) \\ &\quad - \tau(\rho(f_1)\varepsilon_2 \cdots \varepsilon(f_{2p}, f_{2p+1})\varepsilon_{2p+2} \cdots \hat{\varepsilon}(f_{n+1}, f_0)). \end{aligned} \tag{24}$$

If $m = p$, then

$$b\psi^-(f_0, \dots, f_{n+1}) - \tau(\varepsilon_2 \cdots \varepsilon_{2p-2}\varepsilon(f_{n+1}, f_0 f_1)) \\ = \tau(\varepsilon_1 \cdots \varepsilon(f_n, f_0)\rho(f_0)) - \tau(\rho(f_1)\varepsilon_2 \cdots \varepsilon(f_{2p}, f_0)).$$

By Lemma 1, it is easy to see that if $m > p$ then

$$\tau(\varepsilon_2 \cdots \varepsilon_{2p-2}\varepsilon(f_{2p}, f_{2p+1})\varepsilon_{2p+2} \cdots \varepsilon_{n-1}\hat{\varepsilon}(f_{n+1}, f_0 f_1)) \\ = \tau(\varepsilon_2 \cdots \varepsilon_{n-1}\varepsilon(f_{n+1}, f_0 f_1)). \tag{25}$$

By (15), we have

$$\tau(\varepsilon(f_{n+1}f_0, f_1)\varepsilon_2 \cdots \varepsilon_{n-1}) - \tau(\varepsilon(f_{n+1}, f_0 f_1)\varepsilon_2 \cdots \varepsilon_{n-1}) \\ = \tau(\rho(f_{n+1})\varepsilon_0\varepsilon_2 \cdots \varepsilon_{n-1}) - \tau(\varepsilon(f_{n+1}, f_0)\rho(f_1)\varepsilon_2 \cdots \varepsilon_{n-1}). \tag{26}$$

Similar to (25), we may prove that if $m > p$ then

$$\tau(\varepsilon(f_{n+1}, f_0)\rho(f_1)\varepsilon_2 \cdots \varepsilon_{n-1}) \\ = \tau(\rho(f_1)\varepsilon_2 \cdots \varepsilon(f_{2p}, f_{2p+1}) \cdots \hat{\varepsilon}(f_{n+1}, f_0)). \tag{27}$$

From (23)–(27), it follows that

$$b\psi^+ - b\psi^- = 0$$

which proves that ψ is a cocycle if $m > p$. Similarly, we may prove that ψ is a cocycle if $m = p$.

Assume $n - 1 = 2k$ and $k > p$. Define

$$\phi(f_0, \dots, f_{n-1}) = \tau(\rho(f_0)\varepsilon_1 \cdots \varepsilon_{n-2} \\ + (\rho(f_{n-1})\varepsilon_0 - \varepsilon(f_{n-1}f_0, f_1))\varepsilon_2 \cdots \varepsilon_{n-3}). \tag{28}$$

First, we have to prove that

$$\phi(f_0, \dots, f_{n-1}) - \phi(f_1, \dots, f_{n-1}, f_0) = \psi_{n-2}(f_0 f_1, \dots, f_{n-1}). \tag{29}$$

By (28), it is obvious that

$$\phi(f_1, \dots, f_{n-1}, f_0) \\ = \tau(\rho(f_1)\varepsilon_2 \cdots \varepsilon_{2p-2}\varepsilon(f_{2p}, f_{2p+1})\varepsilon_{2p+2} \cdots \hat{\varepsilon}(f_{n-1}, f_0)) \\ + \tau((\rho(f_0)\varepsilon(f_1, f_2) - \varepsilon(f_0 f_1, f_2))\varepsilon_3 \cdots \varepsilon_{n-2}).$$

Therefore

$$\begin{aligned}\phi(f_0, \dots, f_{n-1}) - \phi(f_1, \dots, f_{n-1}, f_0) \\ = \tau(\varepsilon(f_0 f_1, f_2)\varepsilon_3 \cdots \varepsilon_{n-2} + (\rho(f_{n-1})\varepsilon(f_0, f_1) \\ - \varepsilon(f_{n-1}, f_0)\rho(f_1) - \varepsilon(f_{n-1} f_0, f_1))\varepsilon_2 \cdots \varepsilon_{n-3}),\end{aligned}$$

since

$$\varepsilon(f_{2p}, f_{2p+1})\varepsilon_{2p+2} \cdots \hat{\varepsilon}(f_{n-1}, f_0) = \varepsilon_{2p} \cdots \varepsilon_{n-3}\varepsilon(f_{n-1}, f_0)$$

by Lemma 1.

Hence

$$\begin{aligned}\phi(f_0, \dots, f_{n-1}) - \phi(f_1, \dots, f_{n-1}, f_0) \\ = \tau(\varepsilon(f_0 f_1, f_2)\varepsilon_3 \cdots \varepsilon_{n-2} - \varepsilon(f_{n-1}, f_0 f_1)\varepsilon_2 \cdots \varepsilon_{n-3}),\end{aligned}$$

which equals $\psi_{n-2}(f_0 f_1, \dots, f_{n-1})$ since

$$\varepsilon(f_{n-1}, f_0 f_1)\varepsilon_2 \cdots \varepsilon_{2p} = \hat{\varepsilon}(f_{n-1}, f_0 f_1)\varepsilon_2 \cdots \varepsilon(f_{2p}, f_{2p+1}),$$

by Lemma 1 again.

Define

$$v(f_0, f_1, \dots, f_{n-1}) = \frac{1}{n}\sum_{j=0}^{n-1} \phi(f_j, \dots, f_{j+n-1}),$$

where $f_j = f_{j-n}$ for $j \geq n$. From (29), it is easy to see that

$$v = \phi - 2R_{n-2}\psi_{n-2}. \tag{30}$$

Now, we have to calculate $b\phi$. Define

$$\hat{\phi}(f_0, \dots, f_{n-1}) = \tau(\rho(f_0)\varepsilon_1\varepsilon_3 \cdots \varepsilon_{n-2})$$

and

$$\tilde{\phi}(f_0, \dots, f_{n-1}) = \tau((\rho(f_{n-1})\varepsilon(f_0, f_1) - \varepsilon(f_{n-1} f_0, f_1))\varepsilon_2 \cdots \varepsilon_{n-3});$$

then $\phi = \hat{\phi} + \tilde{\phi}$. By (15) and Lemma 1, it is easy to calculate that

$$\begin{aligned}
b\hat{\phi}&(f_0, \dots, f_n) \\
&= \tau\Big(\rho(f_0 f_1)\varepsilon_2 \cdots \varepsilon(f_{2p}, f_{2p+1}) \cdots \varepsilon_{n-1} \\
&\qquad - \sum_{j=1}^{p-1} \rho(f_0)\varepsilon_1 \cdots (\varepsilon(f_{2j-1}f_{2j}, f_{2j+1}) - \varepsilon(f_{2j-1}, f_{2j}f_{2j+1}))\varepsilon_{2j+2} \\
&\qquad\qquad \cdots \varepsilon(f_{2p}, f_{2p+1}) \cdots \varepsilon_{n-1} - \rho(f_0)\varepsilon_1 \\
&\qquad\qquad \cdots (\varepsilon(f_{2p-1}f_{2p}, f_{2p+1}) - \varepsilon(f_{2p-1}, f_{2p}f_{2p+1}))\varepsilon_{2p+2} \cdots \varepsilon_{n-1} \\
&\qquad - \sum_{j=p+1}^{k} \rho(f_0)\varepsilon_1 \cdots (\hat{\varepsilon}(f_{2j-1}f_{2j}, f_{2j+1}) - \hat{\varepsilon}(f_{2j-1}, f_{2j}f_{2j+1}))\varepsilon_{2j+2} \\
&\qquad\qquad \cdots \varepsilon_{n-1} - \rho(f_n f_0)\varepsilon_1 \cdots \varepsilon_{n-2}\Big) \\
&= \tau\Big(\rho(f_0 f_1)\varepsilon_2 \cdots \varepsilon(f_{2p}, f_{2p+1}) \cdots \varepsilon_{n-1} \\
&\qquad - \sum_{j=1}^{p-1} \rho(f_0)\varepsilon_1 \cdots \varepsilon_{2j-3}(\rho(f_{2j-1})\varepsilon(f_{2j}, f_{2j+1}) \\
&\qquad\qquad\qquad - \varepsilon(f_{2j-1}, f_{2j})\rho(f_{2j+1}))\varepsilon_{2j+2} \\
&\qquad\qquad \cdots \varepsilon_{n-1} - \rho(f_0)\varepsilon_1 \cdots (\rho(f_{2p-1})\varepsilon(f_{2p}, f_{2p+1}) \\
&\qquad\qquad\qquad - \varepsilon(f_{2p-1}, f_{2p})\rho(f_{2p+1}))\varepsilon_{2p+2} \cdots \varepsilon_{n-1} \\
&\qquad - \sum_{j=p+1}^{k} \rho(f_0)\varepsilon_1 \cdots \varepsilon_{2j-3}(\rho(f_{2j-1})\hat{\varepsilon}(f_{2j}, f_{2j+1}) \\
&\qquad\qquad\qquad - \hat{\varepsilon}(f_{2j-1}, f_{2j})\rho(f_{2j+1}))\varepsilon_{2j+2} \\
&\qquad\qquad \cdots \varepsilon_{n-1} - \rho(f_n f_0)\varepsilon_1 \cdots \varepsilon_{n-2}\Big) \\
&= \tau(\hat{\varepsilon}(f_0, f_1)\varepsilon_2 \cdots \varepsilon_{2p-2}\varepsilon(f_{2p}, f_{2p+1})\varepsilon_{2p+2} \cdots \varepsilon_{n-1} - \varepsilon_1\varepsilon_3 \cdots \varepsilon_n) \\
&= \tau(\varepsilon_0\varepsilon_2 \cdots \varepsilon_{n-1} - \varepsilon_1 \cdots \varepsilon_n) = \psi_n(f_0, f_1, \dots, f_n).
\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
&b\tilde{\phi}(f_0,\dots,f_n)\\
&\quad= \tau\Big((\rho(f_n)\varepsilon(f_0f_1,f_2)-\varepsilon(f_0,f_1f_2))-\varepsilon(f_nf_0f_1,f_2)+\varepsilon(f_nf_0,f_1f_2))\varepsilon_3\\
&\qquad\quad \cdots\varepsilon_{n-2}+\sum_{j=1}^{p-1}(\rho(f_n)\varepsilon_0-\varepsilon(f_nf_0,f_1))\varepsilon_2\\
&\qquad\quad \cdots(\varepsilon(f_{2j}f_{2j+1},f_{2j+2})-\varepsilon(f_{2j},f_{2j+1}f_{2j+2}))\cdots\varepsilon_{n-2}\\
&\qquad\quad +\sum_{j=p}^{k-1}(\rho(f_n)\varepsilon_0-\varepsilon(f_nf_0,f_1))\varepsilon_2\\
&\qquad\qquad\quad \cdots(\hat{\varepsilon}(f_{2j}f_{2j+1},f_{2j+2})-\hat{\varepsilon}(f_{2j},f_{2j+1}f_{2j+2}))\cdots\varepsilon_{n-2}\\
&\qquad\quad +(\rho(f_{n-1}f_n)\varepsilon_0-\rho(f_{n-1})\varepsilon(f_nf_0,f_1)\Big)\varepsilon_2\cdots\varepsilon_{n-3}\\
&\quad= \tau\Big((\rho(f_n)(\rho(f_0)\varepsilon_1-\varepsilon_0\rho(f_2))-\rho(f_nf_0)\varepsilon_1+\varepsilon(f_nf_0,f_1)\rho(f_2))\varepsilon_3\cdots\varepsilon_{n-2}\\
&\qquad\quad +\sum_{j=1}^{p-1}(\rho(f_n)\varepsilon_0-\varepsilon(f_nf_0,f_1))\varepsilon_2\\
&\qquad\qquad\quad \cdots(\rho(f_{2j})\varepsilon(f_{2j},f_{2j+1})-\varepsilon(f_{2j},f_{2j+1})\rho(f_{2j+2}))\cdots\varepsilon_{n-2}\\
&\qquad\quad +\sum_{j=p}^{k-1}(\rho(f_n)\varepsilon_0-\varepsilon(f_nf_0,f_1))\varepsilon_2\\
&\qquad\qquad\quad \cdots(\rho(f_{2j})\hat{\varepsilon}(f_{2j+1},f_{2j+2}))-\hat{\varepsilon}(f_{2j},f_{2j+1})\rho(f_{2j+2})\cdots\varepsilon_{n-2}\\
&\qquad\quad +(\rho(f_{n-1}f_n)\varepsilon_0-\rho(f_{n-1})\varepsilon(f_nf_0,f_1))\varepsilon_2\cdots\varepsilon_{n-3})\Big)\\
&\quad= \tau(-\hat{\varepsilon}(f_n,f_0)\varepsilon_1\cdots\varepsilon_{n-2}+\hat{\varepsilon}(f_{n-1},f_n)\varepsilon_0\cdots\varepsilon_{n-3})\\
&\quad= \psi_n(f_0,f_1,\dots,f_n).
\end{aligned}
$$

Therefore

$$b\phi = 2\psi_n. \tag{31}$$

From (30) and (31), it follows that

$$\psi_n = bR_{n-2}\psi_{n-2} + b\Big(\frac{1}{2}v\Big)$$

which proves theorem 2. □

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