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DERIVATIONS FROM SUBALGEBRAS OF OPERATOR ALGEBRAS: RESULTS AND PROBLEMS OLD AND NEW

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1. Introduction. Let $B \subseteq A$ be C^*-algebras. A linear map $\delta : B \to A$ is a derivation if $\delta(ab) = a\delta(b) + \delta(a)b$, for $a, b \in B$. If there is an element x of A for which $\delta(b) = (\operatorname{ad} x)(b) = xb - bx, b \in B$, we say that δ is *inner in* A, and *generated by* x. We will refer to any element x of A with this property as a *generator* of δ. If $B = A$, i.e., if δ is defined on all of A and maps A into itself, we will call δ a *derivation of* A. Because of its importance for what we will be discussing in the sequel, we recall that a *multiplier of* a C^*-algebra A is an element m of the enveloping von Neumann algebra A^{**} of A which multiplies A into itself ($mA \cup Am \subseteq A$). The set of all multipliers of A evidently forms a unital C^*-subalgebra of A^{**} which contains A as a closed, two-sided ideal and which is usually referred to as the *multiplier algebra of* A.

The purpose of this paper is to discuss some results and problems on derivations of operator algebras, with emphasis on those topics that have occupied the attention of the author for the past several years. Our dicussion centers around nine open problems that are posed at various places in the text. The main goal of the exposition is to discuss ideas and results that we hope motivate an interest in these problems, and which place them within the context of previous work on the subjects with which they deal. Because of this and the usual limitations of time and space, we have eschewed proofs, preferring instead to indicate precise references to places in the literature where proofs can be found by the interested reader.

2. Algebras with only inner derivations. The theory of derivations began to attract the attention of operator algebraists when I. Kaplansky proved in 1953 [**26**, Theorem 9] that all derivations of a type I von Neumann algebra are inner in the algebra. Thirteen years later, this was completed when S. Sakai [**43**], building on important preliminary work of R.V. Kadison [**22**], proved that a derivation of

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any von Neumann algebra is inner. This result has become one of the favorite chestnuts of the subject, and several elegant proofs are now available (Arveson [**4**, Theorem 4.1], Kadison [**23**], Pedersen [**38**, Corollary 8.6.6], Ringrose [**40**, Theorem 3.6]). Thus was initiated an intensive search for C^*-algebras with only inner derivations. It soon became clear that, in order to obtain a unified and smoothly-functioning treatment for the nonunital as well as unital case, the correct project was to find C^*-algebras A all of whose derivations are generated by elements of the multiplier algebra of A. Sakai ([**44**; II, Theorem 2, see also [**38**, Corollary 8.6.10], made the first advance along these lines by showing that derivations of a simple C^*-algebra are so generated, followed by the work of Akemann, Elliott, Pedersen, and Tomiyama ([**1**, Theorem 3.2], [**38**, Proposition 8.6.11]), in which all derivations of continuous-trace C^*-algebras with paracompact spectrum were shown to be generated by multipliers. Lance ([**28**, Theorem 2.1]) had proven this earlier for the C^*-tensor product of a commutative C^*-algebra and an elementary C^*-algebra, which shows that the paracompactness assumption is unnecessary when the fiber structure in the associated operator field is trivial. This gives rise to our first question:

PROBLEM 1. Are all derivations of continuous-trace C^*-algebras (with arbitrary spectrum) generated by multipliers?

After these results were obtained, attention focused on the separable case. From what we have just said, all separable C^*-algebras which are either simple or have continuous trace admit only derivations generated by multipliers, and it is a fairly simple matter to prove that the same holds true for restricted direct sums of these algebras. In 1977, G. Elliott [**14**] (with a small gap filled in shortly thereafter by Akemann and Pedersen [**3**, Theorem 2.4]) proved the following beautiful converse, which gives a complete solution for separable C^*-algebras:

THEOREM 2.1. ([**14**, Theorem 1], [**3**, Theorem 3.9]). *Let A be a separable C^*-algebra. Then every derivation of A is generated by a multiplier if and only if $A = A_1 \oplus A_2$, where A_1 has continuous trace and A_2 is the restricted direct sum of simple C^*-algebras.*

Thus we need to next consider the nonseparable case. Here the results are much less definitive. D. Olesen [**35**] proved that every derivation of an AW^*-algebra is inner, which gives a very nice generalization of the von Neumann algebra result, and D. Olesen and G.K. Pedersen [**36**, Theorem 3.1] showed that every derivation of a countably generated, monotone sequentially closed C^*-algebra is inner. Both of these papers make crucial use of the theory of spectral subspaces for groups of $*$-automorphisms [**4**] and semicontinuity in the enveloping von Neumann algebra of a C^*-algebra. This circle of ideas can no doubt be exploited further in the hunt for generators, at least when the algebras involved possess closure properties with respect to a topology weaker than the norm. A completely different set of techniques, based on convergence properties of sequences of derivations and the Dixmier property in C^*-algebras, were used by C. Akemann and B.E. Johnson [**2**] to prove that a derivation of the C^*-tensor product of a commutative C^*-algebra and a von Neumann algebra is inner. It is thus natural to next consider the case of the (minimal) C^*-tensor product of two von Neumann algebras, and we hence pose

PROBLEM 2. Is every derivation of the minimal C^*-tensor product of two von Neumann algebras inner?

An affirmative answer to Problem 2 has been conjectured by Akemann and Johnson ([**2**], beginning of §4), and some partial results on it have been obtained by Tomiyama [**45**].

3. When are all derivations from subalgebras inner? In 1970, Kaplansky [**27**] presented a sequence of lectures on analytic and algebraic aspects of operator algebras which inaugurated the CBMS Regional Conference Series in Mathematics. Motivated by derivation results in the theory of central simple (algebraic) algebras and Sakai's theorem for simple C^*-algebras, he raised as one of his topics the problem of the structure of derivations from a C^*-subalgebra of a C^*-algebra A into A. In [**27**, p. 7], Kaplansky conjectured the following: If A is an AW^*-algebra and B is a C^*-subalgebra of A, every derivation from B into A is inner in A. When $A = B(H)$, the algebra of all bounded linear operators on a Hilbert space H, E. Christensen

[**8**] found some very interesting connections between this conjecture and the Kadison-Kastler theory of perturbations of operator algebras [**24**]. Using a remarkable inequality for bounded linear maps between operator algebras due to G. Pisier [**39**, Corollary 2.3], Christensen proved in [**7**, Corollary 5.4] that if A is a C^*-subalgebra of $B(H)$ with a cyclic vector, then every derivation of A into $B(H)$ is inner in $B(H)$. Because it is central to many of the interesting questions in this area, we thus have

PROBLEM 3. If A is a C^*-subalgebra of $B(H)$, is every derivation from A into $B(H)$ inner in $B(H)$?

Problem 3 has a noteworthy connection with Kadison's similarity problem for representations of C^*-algebras [**21**]. Let A be a C^*-algebra and suppose π is a bounded, Banach-algebra homomorphism of A into $B(H)$ for some Hilbert space H. The similarity problem asks, for an invertible $T \in B(H)$ such that the map $a \to T\pi(a)T^{-1}, a \in A$, is a $*$-representation of A on H. Now, suppose $A \subseteq B(H)$ and $\delta : A \to B(H)$ is a derivation. By a theorem of Ringrose [**41**, Theorem 2], δ is automatically bounded, and this together with the derivation identity shows that the map $\pi : A \to B(H \oplus H)$, defined by

$$\pi(a) = \begin{pmatrix} a & \delta(a) \\ 0 & a \end{pmatrix}, \quad a \in A,$$

is a bounded, Banach-algebra homomorphism. If this homomorphism is similar to a $*$-representation of A on $H \oplus H$, then δ is inner in $B(H)$ (for a quick proof of this implication, see the end of the introduction to [**16**]). Thus a positive solution to the similarity problem for representations will yield a positive solution to Problem 3. The similarity problem has been solved affirmatively for bounded homomorphisms with a cyclic vector by U. Haagerup [**16**], using an improvement of the same Pisier inequality for linear maps that Christensen used to solve Problem 3 for subalgebras of $B(H)$ with a cyclic vector.

The problem we have been discussing can be formulated in general by considering the class of C^*-algebras A with the following property: for each C^*-subalgebra B of A, every derivation of B into A is generated by a multiplier of A. Such C^*-algebras were dubbed *hereditarily* *cohomologically trivial* (HCT for short) by A.J. Lazar, S.-K. Tsui, and the author in [**34**]. Using this terminology, Problem 3 can be succinctly rephrased as: is $B(H)$ HCT? The HCT algebras are evidently found among the C^*-algebras all of whose derivations are generated by multipliers, and thus by the results of §2, one should look for them first among the von Neumann algebras, the simple C^*-algebras, the C^*-algebras with continuous trace, or the separable C^*-algebras.

In [**30**] and [**34**], Lazar, Tsui, and the author determined the HCT C^*-algebras which are either separable or have continuous trace.

THEOREM 3.1. [**30**, Theorem 1.1]. *Let A be a separable C^*-algebra. Then A is* HCT *if and only if $A = A_1 \oplus A_2$, where A_1 is abelian and A_2 is the restricted direct sum of elementary C^*-algebras.*

THEOREM 3.2. [**34**, Theorem 1.1] *Let A be a C^*-algebra with continuous trace. Then A is* HCT *if and only if $A = A_1 \oplus A_2 \oplus A_3$, where*

(i) *A_1 is abelian*;

(ii) *A_2 is the restricted direct sum of a sequence $\{A_n\}$ of C^*-algebras such that, for each n, A_n is isomorphic to $C_0(X_n) \otimes M_{k_n}$, with X_n a locally compact, Hausdorff, extremely disconnected space, and M_{k_n} is the algebra of $k_n \times k_n$ matrices, $2 \leq k_n < \infty$;*

(iii) *A_3 is the restricted direct sum of a family of infinite-dimensional elementary C^*-algebras.*

We thus see in particular that a separable, simple C^*-algebra is HCT if and only if it is elementary, and that $C_0(X) \otimes M_n$ is HCT if and only if either $n = 1$ or $n \geq 2$ and X is locally compact, Hausdorff, and extremely disconnected. These simple consequences can be used to show that hereditary cohomological triviality does not pass to (a) tensor products, (b) inductive limits, (c) C^*-subalgebras, or (d) quotients, although it is inherited by (closed, two-sided) ideals and finite direct sums. An example which verifies (a) is given by $C([0,1]) \otimes M_2$, and any separable, infinite-dimensional UHF algebra shows that (b) holds. To get an example which verifies (c), let N denote the positive integers with the discrete topology, let X denote the Stone-Ĉech compactification of N, set $A = C(X) \otimes M_2$, and let B denote the C^*-subalgebra of A consisting of all elements b of A with $\{b(n) : n = 1, 2, 3, \dots\}$ converging in norm to a matrix of the form $\left(\begin{smallmatrix} * & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Since X is compact, Hausdorff, and extremely disconnected, A is HCT. The subalgebra B is separable, has continuous trace, and its spectrum is homeomorphic to the one-point compactification of N, with the single one-dimensional irreducible representation of B determining the point at infinity. Thus the set of one-dimensional irreducible representations of B is not open in the spectrum, and so B is not HCT. To see that (d) holds, let A be as in (c), and let I denote the ideal of all elements x of A for which $\{x(n) : n = 1, 2, 3, \dots\}$ converges in norm to 0. Then A/I is isomorphic to $C(X \backslash N) \otimes M_2$, and since $X \backslash N$ is not extremely disconnected [**15**, Problem 6R], A/I is not HCT.

One should next try to find some nonseparable, simple HCT C^*-algebras and/or some HCT von Neumann algebras. A folk theorem (proved explicitly by Christensen in [**6**, Theorem 5.1]) asserts that all *finite* von Neumann algebras are HCT. The only nonseparable, simple HCT algebras that the author therefore knows are the infinite-dimensional finite factors and the nonseparable elementary C^*-algebras. We thus pose

Problem 4. What are the nonseparable, simple C^*-algebras that are HCT?

Problem 5. Are there any properly infinite von Neumann algebras that are HCT?

While Theorems 3.1 and 3.2 are definitive, they are somewhat disappointing in that the class of HCT examples which results is quite small. Larry Brown proposed in conversation with the author that a perhaps better definition of hereditary cohomological triviality would relax the insistence on generators from the multiplier algebra. Following a suggestion of Professor Brown, the author considered in [**46**], for a C^*-subalgebra B of a C^*-algebra A, the elements x of the enveloping von Neumann algebra of A which multiply B into A, i.e., for which $xB \cup Bx \subseteq A$. The set of such elements is denoted by $M(B, A)$, and we say that A is *weakly* HCT if, for each C^*-subalgebra B of A, every derivation of B into A is generated by an element of $M(B, A)$. Evidently an HCT algebra is weakly HCT, and both definitions coincide when A is unital. In [**46**], the separable C^*-algebras that are weakly HCT were determined as follows:

THEOREM 3.3. [**46**, Theorem 2.6]. *Let A be a separable C^*-algebra. Then A is weakly* HCT *if and only if $A = A_1 \oplus A_2$, where*

(i) *A_1 has continuous trace and the set of all irreducible representations of A_1 which act on a Hilbert space of dimension at least 2 is discrete in the spectrum of A_1;*

(ii) *A_2 is the restricted direct sum of a sequence of elementary C^*-algebras.*

It follows that, at least in the separable case, an HCT algebra differs from a weakly HCT algebra only at its continuous-trace summand. The simplest example of a weakly HCT algebra that is not HCT is thus the algebra B constructed in the paragraph which follows Theorem 3.2. It can be shown [**46**, Proposition 2.7] that an n-homogeneous C^*-algebra that is weakly HCT is in fact HCT. Thus it appears on the basis of the evidence available so far that although the two classes are different, the HCT C^*-algebras and the weakly HCT C^*-algebras are very closely related.

4. Objects of cohomological dimension 0 in the C*- and W*-categories. In the previous section, everything arose from asking when all derivations from subalgebras of a C^*-algebra are inner. In this section, we reverse quantifiers in this question and ask for what C^*-algebras A is it true that, when A is a subalgebra of a C^*-algebra B, each derivation of A into B is inner in B? In order to discuss this precisely, the notation and basic terminology of the Hochschild cohomology of algebras [**19**] will be useful.

Let $\mathcal{A}$ be a linear, associative algebra over the complex numbers $\mathbf{C}$, M a two-sided, linear $\mathcal{A}$-module. A linear map $\delta : \mathcal{A} \to M$ is a derivation if $\delta(ab) = a\delta(b) + \delta(a)b$, for $a, b \in A$, and a derivation δ is inner if, for some $m \in M, \delta(a) = ma - am, a \in \mathcal{A}$. The set of all derivations of $\mathcal{A}$ into M forms an abelian group $Z^1(\mathcal{A}, M)$ under vector space addition, and the subset $B^1(\mathcal{A}, M)$ of all inner derivations is a subgroup. We set $H^1(\mathcal{A}, M)$ equal to the quotient group $Z^1(\mathcal{A}, M)/B^1(\mathcal{A}, M)$. Thus every derivation of $\mathcal{A}$ into M is inner in M if and only if $H^1(\mathcal{A}, M) = (0)$. $\mathcal{A}$ is said to have *cohomological dimension* 0 if $H^1(\mathcal{A}, M) = (0)$ for all linear, two-sided $\mathcal{A}$-modules M. It is a simple matter [**11**, Lemma 1] to show that if $\mathcal{A}$ is a finite direct sum of full matrix algebras over **C**, then $\mathcal{A}$ has cohomological dimension 0, and classical results of Hochschild [**18**, Theorem 2.3] and Rosenberg and Zelinsky [**42**, Theorem 1] combine to demonstrate the converse.

Using Hochschild cohomology as his model, a cohomology theory for Banach algebras was defined and studied by B.E. Johnson in [**20**]. The target modules in this theory are taken to be two-sided, dual Banach $\mathcal{A}$-modules over a Banach algebra $\mathcal{A}$, and thus the objects of cohomological dimension 0 here are the Banach algebras $\mathcal{A}$ for which $H^1(\mathcal{A}, M) = (0)$ for all such modules M. Johnson showed [**20**, Theorem 2.5] that the L_1-group algebra of a locally compact group has cohomological dimension 0 if and only if the group is amenable, and he hence called a Banach algebra *amenable* if it has cohomological dimension 0 in this sense. Johnson also proved in Section 7 of [**20**] that an inductive limit of type I C^*-algebras is amenable. In [**9**], A. Connes verified a suspicion commonly held at the time that all amenable C^*-algebras are nuclear, and in 1983, Haagerup [**17**] proved that all nuclear C^*-algebras are amenable (for an elegant proof of this latter implication, see Effros [**10**]).

In view of the foregoing results as well as the considerations that were made in the previous section, a very natural question to ask is the one that was posed at the beginning of this section, i.e., in the notation just introduced, for what C^*-algebras A do we have

$$(*) \qquad H^1(A, B) = (0), \quad \text{for all } C^*\text{-algebras } B \text{ which contain } A?$$

More precisely, $(*)$ means that if A and B are C^*-algebras and $\pi : A \to B$ is a $*$-isomorphism, then every derivation of $\pi(A)$ into B is inner in B. From the point of view of operator algebras, this is the question that should be asked first, and indeed it was. In 1970, G. Elliott, in a study of derivations of matroid C^*-algebras, posed it as Problem 17.1 of [**11**]. Since a finite-dimensional C^*-algebra is a finite direct sum of matrix algebras, all finite-dimensional C^*-algebras satisfy $(*)$, and Elliott asked if the converse holds. In [**31**], A.J. Lazar, S.-K. Tsui, and the author answered this in the affirmative.

THEOREM 4.1. [**31**, Theorem 2.1], *A C^*-algebra satisfies $(*)$ if and only if it is finite-dimensional.*

We now wish to formulate the W^*-analog of Theorem 4.1. This requires finding the W^*-algebras M such that, whenever N is a W^*-algebra and $\pi : M \to N$ is a σ-continuous $*$-isomorphism, every derivation of $\pi(M)$ into N is inner in N, i.e., $H^1(M, N) = (0)$ for all W^*-algebras N containing M.

To orient ourselves to this, we return to the paper [**9**] of Connes. The main result there is a characterization of the injective W^*-algebras as the W^*-amenable ones. This means that a W^*-algebra M is injective if and only if, for each dual, normal, two-sided Banach M-module X (see [**25**, §2] for a definition of this), $H^1(M, X) = (0)$. Connes proved that C^*-algebraic amenability implies nuclearity by noticing that the enveloping von Neumann algebra A^{**} of an amenable C^*-algebra A is W^*-amenable in this sense, then using the above characterization to conclude that A^{**} is injective, whence A is nuclear by the results [**5**] of Choi and Effros.

To formulate a W^*-version of Theorem 4.1, we now notice that any W^*-algebra N which contains M is a dual, normal, Banach M-module with respect to the usual linear and algebraic operations in N. If M is injective, it hence follows that $H^1(M, N) = (0)$. An affirmative answer to the following question would thus give the W^*-algebra analog of Theorem 4.1 and improve upon one direction in Connes' amenability characterization of injectivity:

PROBLEM 6. If M is a W^*-algebra and $H^1(M, N) = (0)$ for all W^*-algebras N containing M, is M injective?

5. Extending derivations from subalgebras of separable C^*-algebras. Innerness of a derivation from a subalgebra B to an algebra A can be interpreted as an extension theorem, one which says that the derivation can be extended to a derivation of the entire algebra A, and in the best possible way. The extension problem for derivations is thus of interest in the search for generators.

We will concentrate our attention on separable C^*-algebras. If $B \subseteq A$ are separable C^*-algebras, when can a derivation of B into A be extended to a derivation of A? Progress has been made when B is a unital hereditary C^*-subalgebra of A, in which case there is a projection e of A with $B = eAe$, and A is approximately finite-dimensional (AF). We then have the following result:

Theorem 5.1. [**32**, Theorem 2.2]. *If A is a separable, AF, C^*-algebra and e is a projection in A, then every derivation of eAe into A can be extended to a derivation of A.*

Elliott had proven earlier [**12**, Theorem 4.5] that each derivation of eAe can be so extended, and his result is a crucial part of the proof of Theorem 5.1. It can be shown [**32**, pp. 111–114] that Theorem 5.1 fails if the hereditary subalgebra is not unital, even when A is UHF. Hence:

Problem 7. If A is a separable C^*-algebra and e is a projection in A, can every derivation of eAe into A (or into eAe) be extended to a derivation of A?

By Lemma 2.1 of [**32**], if A is any C^*-algebra, if e is a projection in A, and if $H^1(eAe, eAe) = (0)$, then $H^1(eAe, A) = (0)$. Thus if A is simple or has continuous trace, Problem 7 has an affirmative answer.

What if the subalgebra from which we wish to extend is an ideal? Simple examples show that a derivation of an ideal may not itself have an extension, so in order to get some positive results, we need to weaken our notion of extendability. In [**12**], Elliott considered the problem of extension modulo derivations generated by a multiplier. If $B \subseteq A$ are C^*-algebras, this means that if δ is a derivation of B, we seek a multiplier m of B such that the derivation $\operatorname{ad} m|_B + \delta$ of B extends to a derivation of A. G.K. Pedersen has pointed out in [**37**] that extensions of this type from ideals of A would be of great use in the problem of uniformly approximating derivations of A by derivations that restrict to an inner derivation on some nonzero ideal. Elliott proved in [**12**, Theorem 3] that, whenever$A$ is separable and AF, each derivation of an ideal of A extends modulo a multiplier derivation to a derivation of A, but the general (separable) case here remains open.

PROBLEM 8. If A is a separable C^*-algebra, if I is a closed, two-sided ideal of A, and if δ is a derivation of I, is there a multiplier m of I such that $\operatorname{ad} m|_I + \delta$ extends to a derivation of A?

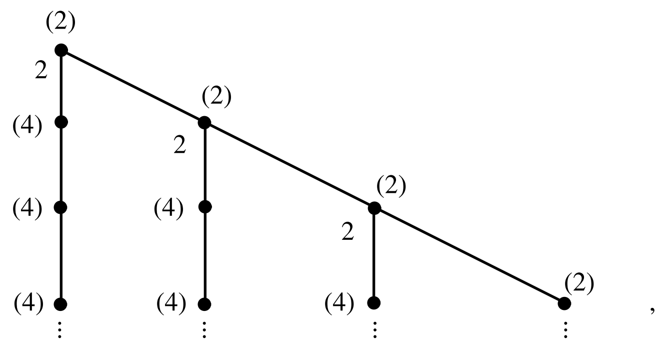
In [**12**, Problem 3.4], Elliott asked if his affirmative answer to Problem 8 when A is AF was still valid if the ideal was replaced by a hereditary C^*-subalgebra. Lazar, Tsui, and the author gave two examples in [**32**] which solves this problem in the negative. These examples can be described most easily in terms of Bratteli diagrams: for this we will follow the notation and terminology of [**29**, §2].

Let A be an AF algebra and let B be a hereditary C^*-subalgebra of A. By Theorem 3.1 and Remark 3.2 of [**13**], B is AF, and there is a generating sequence $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \ldots$ of finite-dimensional C^*-subalgebras of A such that B is generated by $\{B_n = B \cap A_n\}$. We call $\{B_n\}$ the *generating sequence of* B *associated with* $\{A_n\}$. For $n = 1, 2, 3, \ldots$, let

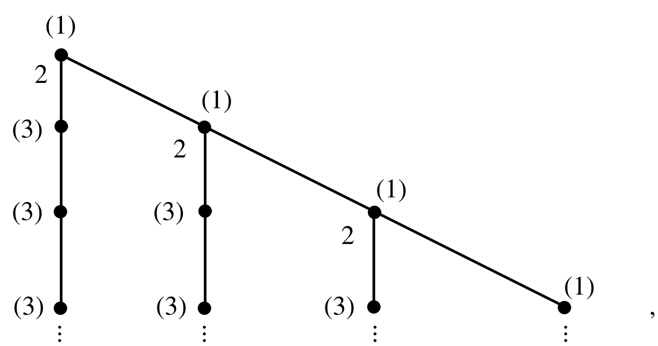
$$A_n = \oplus_{i=1}^{k_n} A_{in}, \quad B_n = \oplus_{i=1}^{k_n} B_{in}$$

denote the Wedderburn decompositions of A_n and B_n (we suppose here that $B_{in} \subseteq A_{in}, i = 1, 2, \ldots, k_n$, and hence some direct summands of B_n could be $\{0\}$). Let f_{in} and e_{in} denote the units of A_{in} and B_{in}, respectively, so that $B_{in} = e_{in} A_n e_{in}, i = 1, 2, \ldots, k_n, e_n = \oplus_i e_{in}$ is the unit of B_n, and $f_n = \oplus_i f_{in}$ is the unit of $A_n, n = 1, 2, 3, \ldots$.

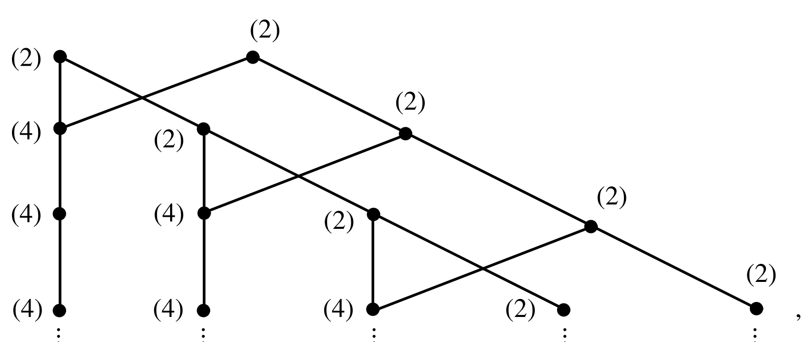
EXAMPLE 1. [**32**, Example 1]. Let A_1 be the AF algebra with Bratteli diagram



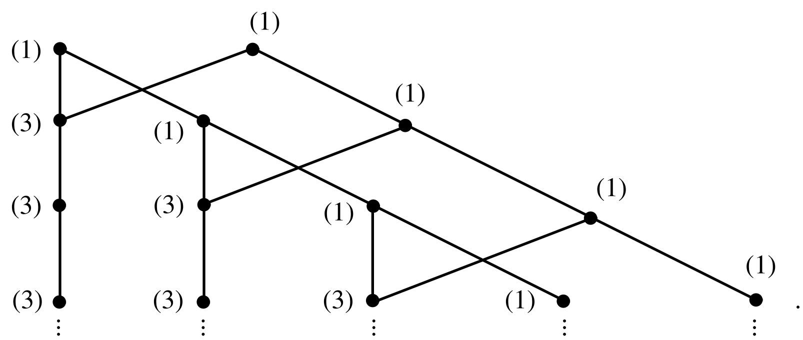
and let B_1 denote the hereditary C^*-subalgebra whose associated generating sequence has Bratteli diagram



EXAMPLE 2. [**32**, Example 2]. Let A_2 denote the AF algebra with Bratteli diagram



and let B_2 denote the hereditary C^*-subalgebra whose associated generating sequence has Bratteli diagram



In Theorems 3.2 and 3.5 of [**32**], a derivation δ_i of B_i was constructed such that, for each multiplier m of $B_i, \operatorname{ad} m|_{B_i} + \delta_i$ does not extend to a derivation of $A_i, i = 1, 2$. To see what the obstruction to the extension is in each example, consider the Wedderburn decompositions

$$A_n^{(1)} = \oplus_{i=1}^n A_{in}^{(1)},\ B_n^{(1)} = \oplus_{i=1}^n B_{in}^{(1)},\ A_n^{(2)} = \oplus_{i=1}^{n+2} A_{in}^{(2)},\ B_n^{(2)} = \oplus_{i=1}^{n+2} B_{in}^{(2)}$$

of the generating sequences $\{A_n^{(i)}\}$ and $\{B_n^{(i)}\}$ of A_i and B_i respectively, $i = 1, 2$. In Example 1, the obstruction to extending δ_1 in the desired way comes from the fact that the portion of $B_n^{(1)}$ that is orthogonal to $B_{n-1}^{(1)}$ enters $B_n^{(1)}$ *via a partial embedding of multiplicity greater than* 1, that is, $e_n^{(1)} - e_{n-1}^{(1)} \neq 0, e_{n-1,n-1}^{(1)} \neq 0, e_n^{(1)} - e_{n-1}^{(1)} \leq f_{n-1,n-1}^{(1)}$, and $A_{n-1,n-1}^{(1)}$ is partially embedded in $A_{n-1,n}^{(1)}$ with multiplicity $2 > 1, n = 2, 3, \ldots$. In Example 2, all partial embeddings are of multiplicity 1, and so an obstruction to extensions like the one in the first example does not occur, but what prevents extension of δ_2 is the fact that the portion of $B_n^{(2)}$ that is orthogonal to $B_{n-1}^{(2)}$ enters $B_n^{(2)}$ *from two distinct direct summands from the previous level*, that is $e_n^{(2)} - e_{n-1}^{(2)} \neq 0, e_{n,n-1}^{(2)} \neq 0 \neq e_{n+1,n-1}^{(2)}, e_n^{(2)} - e_{n-1}^{(2)} \leq f_{n,n-1}^{(2)} \oplus f_{n+1,n-1}^{(2)}$, and $e_n^{(2)} - e_{n-1}^{(2)}$ is majorized by a sum of no fewer units from $A_{n-1}^{(2)}, n = 2, 3, \ldots$.

Examples 1 and 2 can now be used as a guide to obtaining a condition on hereditary subalgebras which allow derivation extensions of the desired type by looking for situations in which the characteristics of these examples do not occur. Specifically, consider an AF C^*-algebra A and a hereditary C^*-subalgebra B with associated generating sequences and Wedderburn decompositions expressed in the notation specified in the paragraph which precedes Example 1. We say that B satisfied *Condition E* if the following holds: for $n = 1, 2, 3, \ldots, m = 1, 2, \ldots, n-1, i = 1, 2, \ldots, k_n$, each nonzero projection $e_{in} - e_{in}e_m e_{in}$ is the sum of mutually orthogonal, minimal projections p of A_{in} each of which satisfies at least one of the following conditions:

(E1) p is orthogonal to every summand A_{jm} with $B_{jm} \neq \{0\}$;

(E2) there is a $j \in \{1, 2, \ldots, k_n\}$ with $B_{jm} \neq \{0\}$ such that $p \leq f_{jm}$ and A_{jm} is partially embedded in A_{in} with multiplicity 1.

Lazar, Tsui, and the author proved in [**33**] that derivation extensions modulo multiplier derivations are always available in the presence of Condition E:

THEOREM 5.2. [**33**, Theorem 2.1]. *Let A be a separable, AF C^*-algebra, and suppose B is a hereditary C^*-subalgebra of A which satisfies Condition E. Then every derivation of B extends modulo a multiplier derivation to a derivation of A.*

Notice that, in Example 1, (E1) and (E2) both fail at level n for $i = m = n - 1$, and they both fail in Example 2 at level n for $i = n, m = n - 1$. Indeed, Theorem 5.2 can be interpreted as saying that, generally speaking, a derivation of a hereditary subalgebra can be extended in the indicated way if the subalgebra does not possess obstructions similar to those of Examples 1 and 2. Note also that every closed, two-sided ideal of A satisfies Condition E ((E1) always holds), and so Theorem 5.2 is a nontrivial generalization of Elliott's extension theorem. However, Theorem 5.2 is by no means the whole story. If A is simple, then every hereditary C^*-subalgebra B of A is also simple, and so by Sakai's theorem every derivation of B is generated by a multiplier of B. Thus every derivation of B, modulo a multiplier derivation, is in fact *identically zero*, and B can be very far indeed from satisfying condition E. Our final question stems from these remarks:

PROBLEM 9. What are conditions on a hereditary C^*-subalgebra B of a separable, AF C^*-algebra which are both necessary and sufficient for derivation extensions from B modulo multiplier derivations?

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