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# THE BOUNDEDNESS CONDITION OF DILATION THEORY CHARACTERIZES SUBNORMALS AND CONTRACTIONS

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TO THE MEMORY OF CONSTANTIN APOSTOL

**1. Introduction.** At the foundations of the general dilation theory on semigroups there are two conditions: positive definiteness PD and the boundedness condition BC /see definitions below/. In general, PD is considered to be more basic than BC, essentially because of the traditional and the most natural method of constructing the dilation Hilbert space by introducing the associated sesquilinear form, positivity of which is guaranteed by PD. The core of this method goes back to classical works of Kolmogoroff, Moore-Aronszajn, Krein, Koranyi-Sz.-Nagy, and others—see [**6**, KMKA Lemma] for references. An abstract version of this method can be found in [**11**], where it is also shown that, assuming PD, dilations can be constructed under conditions much weaker than BC, but these dilations are far from being bounded, even if semigroups in question have involutions. BC can be seen, in general, as the condition that guarantees boundedness of dilations. This general approach applies to a single operator theory in two important cases: unitary dilations of contractions and normal extensions of subnormal operators, which has been done by Sz.-Nagy [**9, 8**], following, for subnormals, Halmos's positivity condition [**2**].

In both cases BC is a consequence of PD. For a single contraction the associated PD function is defined on the group (of integers), which makes BC disappear. That, for a subnormal operator, the associated PD function satisfies BC, was proved by Bram [**1**] who used a deep result of Heinz [**3**]. Szafraniec [**7**] was able to show this without Heinz's result, but applying instead his remarkably simplified BC for $*$-semigroups, which is a consequence of a very careful and elaborate use of Schwarz's inequality. These problems for semigroups of contractions and subnormal semigroups are discussed in [**4**, **10**] and [**12**], respectively. Therefore it seems that BC is insignificant in these

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two cases. A close look, however, at the procedure how BC is proved in case of contraction semigroups [**10**], where one has to use contractivity once again, although it has already been used previously to prove PD, gives rise to quite a contrary suspicion. In this paper we shall show that, in both contraction and subnormal cases, BC is actually a condition equivalent to PD, hence it characterizes contractions and subnormals equally well as PD. To do this for subnormals (§3) we shall need some operator inequalities (§2) which seem to be interesting in themselves. The contraction case is treated in §4 for semigroups, following the approach of [**4, 10**].

$B(H)$ denotes the set of all linear bounded operators in a complex Hilbert space $H$. $I$ is the identity operator. Let $S$ be a set. $F(S,H)$ stands for the linear space of all functions from $S$ to $H$ vanishing off a finite subset of $S$. A function $A : S \times S \to B(H)$ is called *positive definite* PD if $\sum (A(s,t)f(s), f(t)) \geq 0$ for each $f \in F(S,H)$. $A$ will be called *symmetric* if $A(s,t)^* = A(t,s), s,t \in S$. PD implies symmetry (cf. [**5**, p. 18]). Let now $S$ be a semigroup (always with unit) and let $A : S \times S \to B(H)$ be a symmetric function. $A$ satisfies the *boundedness condition* BC if, for each $u \in S$, there is a non-negative real number $c(u)$ such that

$$\sum (A(us,ut)f(s), f(t)) \leq c(u) \sum (A(s,t)f(s), f(t)) \quad \text{for each } f \in F(S,H).$$

This inequality makes sense, namely, both its sides are real numbers, because $A$ is symmetric. Since the bounding constant $c(u)$ above is assumed to be non-negative, it is immediate that

(1.1) BC *is submultiplicative, i.e., if* BC *is satisfied for* $u, v \in S$ *with* $c(u), c(v)$*, respectively, then* BC *is satisfied for* $uv$ *with* $c(u)c(v)$.

Suppose now that $S$ is a $*$-semigroup, i.e., a mapping $* : S \to S$ is defined so that $(s^*)^* = s, (st)^* = t^*s^*, s,t \in S, 1^* = 1$. Let $\Phi : S \to B(H)$ be a function. Define $A_\Phi : S \times S \to B(H)$ by $A_\Phi(s,t) = \Phi(t^*s), s,t \in S$. Then $A_\Phi$ is symmetric if and only if $\Phi(s^*) = \Phi(s)^*, s \in S$.

**2. Some inequalities.** Here we discuss conditions that guarantee positivity of a self-adjoint operator.

(2.1) PROPOSITION. *Suppose* $T, M \in B(H)$, $M$ *is self-adjoint, and* $T$ *is a contraction such that* $T^n x \to 0$ *for each* $x \in H$. *If* $T^*MT \leq M$, *then* $M \geq 0$.

PROOF. The assumption $T^*MT \leq M$ implies

$$M \geq T^*MT \geq T^{*2}MT^2 \geq \cdots \geq T^{*n}MT^n \geq \ldots .$$

Let $x \in X$. Then $||T^{*n}MT^n x|| \leq ||M|| \, ||T^n x|| \to 0, \quad \text{as } n \to \infty$. By the sequence of inequalities above, $M \geq 0$. □

This proposition applies in particular if $||T|| < 1$.

The following consequence of a result of Heinz [**3**; Satz 3, p. 426] is the essential point in the elimination of the boundedness condition from Halmos's characterization of subnormality by Bram.

(2.2) (HEINZ). *If* $A, B \in B(H)$ *are positive operators and* $A^2 \leq B^2$, *then* $A \leq B$.

Is there a way to get a certain converse of this result so that it could be used to prove positivity? The most obvious converse would be:

(2.3) *If* $A, B \in B(H)$ *are self-adjoint,* $B^2 \leq A^2$, *and* $B \leq A$, *then* $A \geq 0$.

This statement fails if $A - B$ is not invertible even for $\dim H = 1$. It is true under additional assumptions:

(2.4) PROPOSITION. *If* $A, B \in B(H)$ *are self-adjoint,* $B^2 \leq A^2, B \leq A, AB = BA$, *and* $A - B$ *is invertible, then* $A \geq 0$.

PROOF. Since $A, B$ commute, the $C^*$-algebra with unit they generate is commutative. By the Gelfand-Naimark functional calculus, the continuous functions $f, g$ on the compact spectrum of this algebra, that correspond to $A, B$, respectively, are real, $g^2 \leq f^2, g \leq f$, and their values are not equal at any point of the spectrum, because $A - B$ is invertible. The conclusion is now immediate. □

**3. Subnormals.** In this section we shall show that the boundedness condition eliminated by Bram ((3.2)(ii) below) characterizes subnormality of a single operator. We shall use the dilation approach as in [**8, 12**], as well as the original Bram approach, both of which we now describe. Let $N$ be the additive semigroup of all non-negative integers. Let $S$ be the product semigroup $N \times N$ with involution defined by $(m, n)^* = (n, m)$, for all $(m, n) \in S$. Let $T \in B(H)$. Define $\Phi : S \to B(H)$ by $\Phi(m, n) = T^{*n}T^m, (m, n) \in S$. Let $A_\Phi$ be as defined after (1.1). Let $K$ be the direct sum of countably many copies of $H$, which may be seen as $l^2 \otimes H$. Let $M$ be the operator in $K$ given by the matrix whose $(m, n)$ entry is $\Phi(m, n)$. $M$ is symmetric on the dense subspace $F(N, H)$ of $K$. If $||T|| < 1$, then $M$ extends to a bounded operator on $K$, which will be denoted also by $M$ (cf. the first part of the proof of Theorem 1 in [**1**]). $T_0 = I \otimes T$ is the diagonal matrix operator in $K$ with all diagonal entries equal $T$. These notations are fixed throughout this section.

(3.1) PROPOSITION. *The following conditions are equivalent*:

(i) $A_\Phi$ *is* PD *on* $S$,

(ii) $\sum (T^m g(n), T^n g(m)) \geq 0$ *for each* $g \in F(N, H)$,

(iii) $(Mg, g) \geq 0$ *for each* $g \in F(N, H)$.

The proof can be found in [**1, 8** and **12**]. Each of these conditions is known to be equivalent to the subnormality of $T$.

Notice that $\Phi(n, m) = \Phi(m, n)^*, (m, n) \in S$. By the remarks at the end of §1, $A_\Phi$ is a symmetric function. Hence BC makes sense.

(3.2) PROPOSITION. *The following conditions are equivalent*:

(i) $A_\Phi$ *satisfies* BC,

(ii) *There is* $c \geq 0$ *such that*

$$\sum (T^{m+1}g(n), T^{n+1}g(m)) \leq c \sum (T^m g(n), T^n g(m)), \quad g \in F(N, H),$$

(iii) *There is* $c \geq 0$ *such that* $(T_0^* M T_0 g, g) \leq c(Mg, g), g \in F(N, H)$.

Proof. The equivalence of (ii) and (iii) is obvious (the constant $c$ is the same in both conditions).

Let $s = (m, m'), t = (n, n'), u = (p, p') \in S$. Then

$$t^* u^* us = (n' + p' + p + m, n + p + p' + m'),$$

and

$$A_\Phi(us, ut) = \Phi(t^* u^* us) = T^{*n+p+p'+m'} T^{n'+p'+p+m}.$$

Let now $f \in F(S, H)$. Define $g \in F(N, H)$ by $g(n) = \sum_m T^m f(m, n)$. Then

$$\begin{aligned} &\sum (A_\Phi(us, ut) f(s), f(t)) \\ &= \sum (T^{n'+p+p'+m} f(m, m'), T^{n+p+p'+m'} f(n, n') \\ &= \sum (T^{n'} T^{p+p'} g(m'), T^{m'} T^{p+p'} g(n')). \end{aligned} \tag{3.3}$$

If $A_\Phi$ satisfies BC, then, by (3.3), the inequality in (ii) holds with $c = c(u)$, where $u = (1, 0)$. Conversely, if (ii) holds, then BC is satisfied for $u = (1, 0)$ with the constant $c$. By (1.1), i.e., the submultiplicativity of BC, and (3.3), $A_\Phi$ satisfies BC. □

(3.4) Theorem. *The following conditions are equivalent*:

(i) $\sum (T^m g(n), T^n g(m)) \geq 0$, *for each* $g \in F(N, H)$,

(ii) *For each* $c > ||T||^2$

$$\sum (T^{m+1}g(n), T^{n+1}g(m)) \leq c \sum (T^m g(n), T^n g(m)), \ g \in F(N, H),$$

(iii) *There is a* $c > ||T||^2$ *such that*

$$\sum (T^{m+1}g(n), T^{n+1}g(m)) \leq c \sum (T^m g(n), T^n g(m)), \ g \in F(N, H).$$

PROOF. (i) $\Rightarrow$ (ii) has been proved by Bram [**1**, Theorem 1]. For the sake of completeness we give here a shorter proof, essentially following Bram's arguments. The equality (3.5) is a new ingredient which makes the proof more transparent. As shown at the end of the proof of Theorem 1 in [**1**], we may assume $||T|| < 1$. Then $M$ is bounded. Let $U_+$ denote the unilateral shift of multiplicity one in $l^2$. Then $U_+ \otimes I$ is the unilateral shift of multiplicity $\dim H$ in $l_2 \otimes H = K$. The interesting point is that the special form of the matrix $M$ implies:

$$MT_0 = (U_+ \otimes I)^* M. \tag{3.5}$$

Now it is immediate that $(T_0^* M T_0)^2 \le M^2$. For if $x \in K$, then, by (3.5),

$$||T_0^* M T_0 x|| \le ||T_0^* (U_+ \otimes I)^* M x|| \le ||Mx||.$$

By (3.1) ((ii) $\Rightarrow$ (iii)), $M$ is positive. To finish the proof apply (2.2) and (3.2) ((iii) $\Rightarrow$ (ii)).

Since (ii) $\Rightarrow$ (iii) is obvious, it remains to prove (iii) $\Rightarrow$ (i). As above, we assume $||T|| < 1$. By (3.2)((ii) $\Rightarrow$ (iii)), $T_0^* M T_0 \le M$, because $M$ is bounded. Since $||T_0|| = ||T|| < 1$, we now apply the crucial Proposition (2.1) to conclude that $M$ is positive. (i) follows now from (3.1) ((iii) $\Rightarrow$ (ii)). □

(3.6) COROLLARY. *The following conditions are equivalent*:

(i) $T$ *is subnormal*,

(ii) $A_\Phi$ *is* PD *on the* $*$-*semigroup* $S$,

(iii) $A_\Phi$ *satisfies* BC.

**4. Contraction semigroups.** Firstly let us recall the basic construction from [**10**]. Let $G$ be a commutative group ordered by a subsemigroup $G_+$. This means that $G_+ \cap (-G_+) = \{0\}, G_+ \cup (-G_+) = G$. The order in $G$ is defined by: $m \le n$ if $n - m \in G_+$ (for $n, m \in G$). In $G_+ \times G_+$ we define an involution by $(m,n)^* = (n,m)$, for $m, n \in G_+$, and an algebraic operation $\# :$ for $(j,k), (m,n) \in G_+ \times G_+$,

$$(j,k)\#(m,n) = \begin{cases} (m+j-n, k) & \text{if } j \ge n, \\ (m, k+n-j) & \text{if } j < n. \end{cases}$$

It is proved in [**10**, Proposition 1] that $(G_+ \times G_+, \#, *)$ is a $*$-semigroup with unit $(0, 0)$, where 0 is the unit of $G$. This semigroup is denoted by $G^\#$. Proposition 2 of [**10**] states that coisometric semigroup homomorphisms of $G_+$ into $L(H)$ (each value is a coisometry) are in a bijective correspondence with $*$-semigroup homomorphisms of $G^\#$.

Let $\pi : G_+ \to L(H)$ be a semigroup homomorphism. Define $\Phi : G^\# \to L(H)$ by $\Phi(m, n) = \pi(n)^*\pi(m), (m, n) \in G^\#$. Let $A_\Phi$ be as defined after (1.1) with $S = G^\#$. These notations will be preserved throughout this section.

(4.1) THEOREM. *The following conditions are equivalent*:

(i) $\pi$ *is contractive, i.e.,* $||\pi(n)|| \leq 1$ *for each* $n \in G_+$,

(ii) $A_\Phi$ *is* PD *on* $G^\#$,

(iii) $A_\Phi$ *satisfies* BC.

The conditions (i) and (ii) have been proved to be equivalent in [**10**]. It is also shown there that each of them is equivalent to the existence of the $*$-dilation of the function $A_\Phi$ which, in turn, is equivalent to the existence of the coisometric extension of $\pi$. The main point here is to prove that BC is equivalent to each (i), (ii) above. On the way to prove this equivalence we shall be able to give also a completely straightforward proof of (i) $\Leftrightarrow$ (ii), which, unlike the one given in [**10**], is entirely self-contained (and we get it here "for free"). The beginning of this proof is influenced by Sz.-Nagy's proof of his PD condition that characterizes a single contraction in [**8**, p. 28].

PROOF. Firstly notice that $A_\Phi$ is a symmetric function. For if $(m, n) \in G^\#$: then $\Phi((m, n)^*) = \Phi(n, m) = \pi(m)^*\pi(n) = (\pi(n)^*\pi(m))^* = \Phi(m, n)^*$. Hence if $f \in F(G^\#, H)$, then

$$\Omega_f = \sum_{\alpha,\beta}(A_\Phi(\beta, \alpha)f(\beta), f(\alpha)) = \sum_{\alpha,\beta}(\Phi(\alpha^*\#\beta)f(\beta), f(\alpha))$$

is a real number.

Let us take $f \in F(G^\#, H)$ and write $\Omega_f$ in full. Let $\alpha = (k, j)$,

$\beta = (m, n)$. Then

$$A_\Phi(\beta, \alpha) = \Phi(\alpha^* \# \beta) = \begin{cases} \pi(k)^*\pi(m)\pi(j-n) & \text{if } j \geq n, \\ \pi(n-j)^*\pi(k)^*\pi(m) & \text{if } j < n. \end{cases} \tag{1}$$

Hence

$$\begin{aligned} \Omega_f &= \sum_{j \geq n} (\pi(m)\pi(j-n)f(m,n), \pi(k)f(k,j)) \\ &\qquad + \sum_{j<n} (\pi(m)f(m,n), \pi(k)\pi(n-j)f(k,j)) \\ &= \sum_{j \geq n} (\pi(j-n)h(n), h(j)) + \sum_{j<n} (h(n), \pi(n-j)h(j)), \end{aligned}$$

where $h : G_+ \to H$ is defined by

$$h(n) = \sum_p \pi(p) f(p, n), \quad n \in G_+. \tag{2}$$

This computation of $\Omega_f$ up to this point has been done in [**10**].

It is clear that $h$ vanishes off a finite subset $\{j_0, \ldots, j_\mu\}$ of $G_+$. Assume that $j_0 < j_1 < \cdots < j_\mu$. Define $g : G_+ \to H$ by

$$g(j) = \sum_{n<j} \pi(j-n)h(n) \quad \text{if } j > j_0, \quad \text{and } g(j) = 0 \quad \text{if } j \leq j_0. \tag{3}$$

This function has the following properties:

(4) If $p = 0, \ldots, \mu-1$ and $j_p < j \leq j_{p+1}$, or if $p = \mu$ and $j > \mu$, then

$$g(j) = \sum_{0 \leq i \leq p} \pi(j - j_i), \text{ and}$$

$$g(j) = \pi(j - j_p)[g(j_p) + h(j_p)];$$

$$g(j) = \pi(j - j_0)h(j_0) \quad \text{if } j_0 < j \leq j_1. \tag{5}$$

The property (5) follows from (4) and from the definition of $g$. The first part of (4) is clear. Here is the proof of the second part.

$$\begin{aligned} g(j) &= \sum_{0\le i\le p} \pi(j-j_i)h(j_i) = \pi(j-j_p)\sum_{0\le i\le p}\pi(j_p-j_i)h(j_i) \\ &= \pi(j-j_p)\left[\sum_{0\le i\le p-1}\pi(j_p-j_i)h(j_i)+h(j_p)\right] \\ &= \pi(j-j_p)[g(j_p)+h(j_p)]. \end{aligned}$$

Now we shall continue computing $\Omega_f$:

$$\begin{aligned} \Omega_f &= \sum_n ||h(n)||^2 + \sum_{j>n}(\pi(j-n)h(n),h(j)) + \sum_{j<n}(h(n),\pi(n-j)h(j)) \\ &= \sum_n ||h(n)||^2 + \sum_{j>j_0}\left[\sum_{n<j} 2\mathrm{Re}\,(\pi(j-n)h(n),h(j))\right] \\ &= \sum_n ||h(n)||^2 + \sum_{j>j_0} 2\mathrm{Re}\,(g(j),h(j)) \\ &= ||h(j_0)||^2 + \sum_{1\le i\le\mu}(||g(j_i)+h(j_i)||^2 - ||g(j_i)||^2) \\ &= ||h(j_0)||^2 - ||g(j_i)||^2 + \sum_{1\le i\le \mu-1}||g(j_i)+h(j_i)||^2 \\ &\quad - \sum_{2\le i\le\mu}||g(j_i)||^2 + ||g(j_\mu)+h(j_\mu)||^2 \\ &= ||h(j_0)||^2 - ||g(j_0)||^2 + \sum_{1\le i\le\mu-1}(||g(j_i)+h(j_i)||^2 - ||g(j_{i+1})||^2) \\ &\quad + ||g(j_\mu)+h(j_\mu)||^2. \end{aligned}$$

Using (4) and (5) we get

(6)
$$\begin{aligned} \Omega_f =& [||h(j_0)||^2 - ||\pi(j_1-j_0)h(j_0)||^2] \\ &+ \sum_{1\le i\le\mu-1}[||g(j_i)+h(j_i)||^2 - ||\pi(j_{i+1}-j_i)(g(j_i)+h(j_i))||^2] \\ &+ ||g(j_\mu)+h(j_\mu)||^2. \end{aligned}$$

At this point we actually have proved that (i) $\Leftrightarrow$ (ii). Indeed, if $\pi(n)$ is a contraction for each $n$, then $\Omega_f \ge 0$. Conversely, let us fix arbitrarily $x \in H, n \in G_+$, and let us choose $f$ so that $h(0) = x, h(n) = -\pi(n)h(0), h(j) = 0$ elsewhere. Take $\mu = 1, j_0 = 0, j_1 = n$. Then in the sum in (6) only the first two terms are non-zero, by (5) and the choice of $h(n)$. Thus

$$0 \le \Omega_f = ||x||^2 - ||\pi(n)x||^2.$$

Now let us fix arbitrarily $\delta = (c, d) \in G^{\#}$. The product $\delta^*\#\delta = (c, c)$ does not depend upon the second element of the pair $(c, d)$. Let $\alpha = (k, j), \beta = (m, n) \in G^{\#}$. Since # is associative [**10**, Proposition 1],

$$\begin{aligned}(\alpha^*\#\delta^*)\#(\delta\#\beta) &= \alpha^*\#(\delta^*\#\delta)\#\beta = (j,k)\#(c,c)\#(m,n)\\ &= \begin{cases} (j,k)\#(m,n) & \text{if } j \ge c \text{ or } n \ge c \\ (m+c-n, k+c-j) & \text{if } j < c \text{ and } n < c. \end{cases}\end{aligned}$$

Therefore
(7)

$$\begin{aligned}A_\Phi(\delta\#\beta, \delta\#\alpha) &= \Phi(\alpha^*\#\delta^*\#\delta\#\beta)\\ &= \begin{cases} A_\Phi(\beta,\alpha) & \text{if } j \ge c \text{ or } n \ge c \\ \pi(k+c-j)^*\pi(m+c-n) & \text{if } j < c, n < c. \end{cases}\end{aligned}$$

For $f \in F(G^{\#}, H)$ define

$$\Omega_f(\delta) = \sum_{\alpha,\beta} (A_\Phi(\delta\#\beta, \delta\#\alpha)f(\beta), f(\alpha)).$$

From (7) we see that

$$\begin{aligned}\Omega_f(\delta) = &\sum_{\alpha,\beta: j<c, n<c} (A_\Phi(\delta\#\beta, \delta\#\alpha)f(\beta), f(\alpha))\\ &+ \sum_{\alpha,\beta: j\ge c \text{ or } n\ge c} (A_\Phi(\beta,\alpha)f(\beta), f(\alpha)).\end{aligned}$$

which shows that the summands in $\Omega_f$ and $\Omega_f(\delta)$ corresponding to $j \ge c$ or $n \ge c$ are equal. Hence we may disregard them, because for our purpose of examining BC we shall be interested only in estimating the quantity $\Omega_f - \Omega_f(\delta)$. Thus with no loss of generality we may assume that, given $\delta = (c,d)$, we consider only functions $f \in F(G^{\#}, H)$ such that the corresponding functions $h$ defined by (2) vanish off a subset $\{j_0, \dots, j_\mu\}$ satisfying $j_0 < \dots < j_\mu < c$. Let us fix such $f$. Then

$$\begin{aligned}\Omega_f(\delta) &= \sum_{j<c, n<c} (A_\Phi(\delta\#\beta, \delta\#\alpha)f(\beta), f(\alpha)) \\ &= \sum_{j<c, n<c} (\pi(c-n)h(n), \pi(c-j)h(j)) \\ &= ||\sum_{j<c} \pi(c-j)h(j)||^2 = ||g(c)||^2 = ||\pi(c-j_\mu)(g(j_\mu)+h(j_\mu))||^2.\end{aligned}$$

The last two equalities follow from (3) and (4), respectively. Finally, by (6), we get

$$\begin{aligned}\Omega_f - \Omega_f(\delta) =& [||h(j_0)||^2 - ||\pi(j_1 - j_0)h(j_0)||^2] \\ &+ \sum_{1\le i \le \mu-1} [||g(j_i) + h(j_i)||^2 \\ &\qquad - ||\pi(j_{i+1} - j_i)(g(j_i) + h(j_i))||^2] \\ &+ ||g(j_\mu) + h(j_\mu)||^2 - ||\pi(c - j_\mu)(g(j_\mu) + h(j_\mu))||^2.\end{aligned}$$

Now, if each $\pi(n)$ is a contraction, then $\Omega_f - \Omega_f(\delta) \ge 0$. Hence $A_\Phi$ satisfies BC. Conversely, suppose that BC holds. Let us fix $x \in H$ and $n \in G_+$. Let $\mu = 0, j_0 = 0, c = n, h(j_0) = x, h(j) = 0$ elsewhere. Then the above sum reduces to its last two terms. Since $g(j_0) = 0$,

$$0 \le \Omega_f - \Omega_f(\delta) = \|x||^2 - ||\pi(n)x||^2.$$

Hence each $\pi(n)$ is a contraction. □

The last part of this proof shows that BC seems to "fit" contractivity even better than PD.

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