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ON EXACT CONTROLLABILITY OF OPERATORS

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Dedicated to the memory of Constantin Apostol

1. Introduction. Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces, and let $A \in L(\mathcal{Y}), B \in L(\mathcal{X}, \mathcal{Y})$ (as usual, $L(\mathcal{X}, \mathcal{Y})$ stands for the Banach space of all linear bounded operators from $\mathcal{X}$ into $\mathcal{Y}$, and $L(\mathcal{Y})$ is the abbreviation of $L(\mathcal{Y}, \mathcal{Y})$). The pair (A, B) is called *exactly controllable* if

$$\mathcal{Y} = \bigcup_{n=1}^{\infty} \Big(\sum_{j=0}^{n-1} \operatorname{Im} A^{n-1-j} B \Big).$$

(Here and elsewhere we denote

$$\operatorname{Im} S = \{ Sx \mid x \in \mathcal{X} \}$$

for $S \in L(\mathcal{X}, \mathcal{Y})$.) This notion appears naturally in linear systems theory (an indication to that is given in the next section) and was studied by several authors [**1, 2, 3, 6, 8**]. In the finite dimensional case ($\dim \mathcal{Y} < \infty$) the notion of exact controllability is one of the most important in modern linear system theory and can be found in virtually every book on the subject (see, e.g., [**5, 10**]).

A crucial property of exactly controllable pairs in the finite dimensional case is the following fact (known as the pole, or spectrum, assignment theorem, see [**5, 9**]): A pair (A, B) is exactly controllable if and only if, for every m-tuple (here $m = \dim \mathcal{Y}$) of complex numbers $\lambda_1, \dots, \lambda_m$ there is $F \in L(\mathcal{Y}, \mathcal{X})$ such that $\sigma(A+BF) = \{\lambda_1, \dots, \lambda_m\}$. Recently, infinite dimensional versions of these results were proved in [**2, 8**].

In this paper we make more precise the spectrum assignment results for exactly controllable pairs proved in [**8**] by exhibiting the continuous dependence on the parameters involved.

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2. Exactly controllable pairs of operators. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let $A \in L(\mathcal{Y}), B \in L(\mathcal{X}, \mathcal{Y})$. Consider the linear system

$$x_n = Ax_{n-1} + Bu_{n-1}, \quad n = 1, 2, \ldots, \tag{2.1}$$

where $\{x_n\}_{n=0}^{\infty}$ is a sequence of vectors in $\mathcal{Y}$ and $\{u_n\}_{n=0}^{\infty}$ is a sequence of vectors in $\mathcal{X}$.

The equation (2.1) is often interpreted in terms of system theory. Thus, $\mathcal{Y}$ is assumed to represent the states of a system, while $\mathcal{X}$ represents controls (or inputs). The problem then becomes to choose the sequence of controls $\{u_n\}_{n=1}^{\infty}$ in a certain way to ensure desired properties of the system.

A system (2.1) (or, equivalently, the pair (A, B)) is called *exactly controllable* if any state x can be reached from any initial value x_0 in a finite number of steps. More precisely, this means the following: for any $x_0, x \in \mathcal{Y}$ there is an integer $m(\geq 0)$ and controls $u_0, \ldots, u_{m-1}$ such that if $x_1, \ldots, x_m$ are defined by (2.1) for $n = 1, \ldots, m$, then $x = x_m$. As the solution of (2.1) is given by

$$x_n = A^n x_0 + \sum_{j=0}^{n-1} A^{n-1-j} B u_j, \quad n = 1, 2, \ldots,$$

it follows that if (2.1) is exactly controllable then

$$\mathcal{Y} = \bigcup_{n=1}^{\infty} \Big(\sum_{j=0}^{n-1} \operatorname{Im} A^{n-1-j} B \Big). \tag{2.2}$$

Actually, Theorem 2.1 below implies that (2.2) is equivalent to exact controllability of (2.1).

THEOREM 2.1. (a) *A pair of Banach space operators $A \in L(\mathcal{Y})$ and $B \in L(\mathcal{X}, \mathcal{Y})$ is exactly controllable if and only if*

$$\mathcal{Y} = \Big(\sum_{j=0}^{m-1} \operatorname{Im} A^{m-1-j} B \Big)$$

for some integer m.

(b) *The operator*

$$[B, AB, \ldots, A^{m-1}B] \in L(\mathcal{X}^m, \mathcal{Y})$$

is right invertible for some integer m *if and only if*

$$[\lambda I - A, B] \in L(\mathcal{Y} \oplus \mathcal{X}, \mathcal{Y})$$

is right invertible for every $\lambda \in \mathbf{C}$*.*

Part (a) is a consequence of the open mapping theorem (see [**6**]); the proof of part (b) is found in [**4**, Appendix] (see also [**7**]).

In case $\mathcal{X}$ is a Hilbert space, the conditions expressed in part (a) are obviously the same as in part (b).

3. Spectrum assignment theorems. An important problem in control is to bring about the desired behavior of the system (2.1) by using state feedback, that is, by putting $u_n = Fx_n, n = 1, 2, \ldots$, where $F \in L(\mathcal{Y}, \mathcal{X})$ is a suitable operator. The operator will be called the *feedback operator*. In particular, one is interested to find if there is a feedback operator F such that the system

$$x_n = Ax_{n-1} + Bu_{n-1}, \quad u_{n-1} = Fx_{n-1}, \quad n = 1, 2, \ldots, \tag{3.1}$$

is stable, i.e., the spectrum of $A + BF \in L(\mathcal{Y})$ lies in the open unit disc $\{\lambda \in \mathbf{C} \mid |\lambda| < 1\}$. More generally, it is of interest to find a feedback operator F such that $A + BF$ has its spectrum in a prescribed set in the complex plane. Also, it is important to have some control on the behavior of F; for instance, it is desirable to keep the norm of F moderate.

It turns out that if the system (2.1) is exactly controllable and the spaces $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, then, by using suitable F, one can make the spectrum of $A + BF$ to be any prescribed non-empty compact set in the complex plane. Moreover, F can be chosen to depend continuously on A, B and the prescribed compact set. We make this precise (and other related statements) in this section. *Everywhere in this section it will be assumed that* $\mathcal{X}$ *and* $\mathcal{Y}$ *are Hilbert spaces.*

Consider the set S_n of all n-tuples $\{\lambda_1, \ldots, \lambda_n\}$ of complex numbers with repetitions allowed. Two n-tuples obtained from each other by a permutation are considered the same element in S_n. One defines naturally a metric on S_n:

$$d_n(\{\lambda_1,\dots,\lambda_n\},\{\mu_1,\dots,\mu_n\}) = \inf \sup_{1\le i\le n} |\,\lambda_i - \mu_{\sigma(i)}|,$$

where the infimum is taken over all permutations σ of the set $\{1,\dots,n\}$. Consider also the set $C(\mathbf{C})$ of all non-empty compact subsets of the complex plane with the usual Hausdorff metric (here $M,\Lambda \in C(\mathbf{C})$):

$$d(M,\Lambda) = \max\{\max_{y\in\Lambda}\min_{x\in M} |\,x-y\,|\,, \max_{y\in M}\min_{x\in\Lambda} |\,x-y\,|\,\}$$

For $\lambda = \{\lambda_1,\dots,\lambda_n\} \in S_n$ denote by $||\lambda||$ the quantity $\max_{1\le j\le n}|\lambda_j|$, and for $\Lambda \in C(\mathbf{C})$, put

$$||\Lambda|| = \max_{z\in\Lambda} |z|.$$

The following is a more precise version of the main theorem in [**8**].

THEOREM 3.1. *Let $A \in L(\mathcal{Y}), B \in L(\mathcal{X},\mathcal{Y})$ be such that*

$$\sum_{k=0}^{n-1}(A^kB)(\mathcal{X}) = \mathcal{Y}. \tag{3.2}$$

(a) *Let $\lambda = \{\lambda_1,\dots,\lambda_n\} \in S_n$. Then there exist positive constants ε (depending on A,B only) and K (depending on A,B and $||\lambda||$) with the following property: For every pair $A' \in L(\mathcal{Y}), B' \in L(\mathcal{X},\mathcal{Y})$ such that*

$$||A-A'|| + ||B-B'|| < \varepsilon \tag{3.3}$$

and, for every $\mu = \{\mu_1,\dots,\mu_n\} \in S_n$, there is a feedback $F = F(A',B',\mu)$ such that

$$\prod_{j=1}^{n}(A'+B'F-\mu_jI) = 0$$

and the inequalities

$$||F(A,B,\lambda)|| \le K \tag{3.4}$$

and

$$(3.5)\quad ||F(A',B',\mu)-F(A,B,\lambda)|| \leq K(||A'-A||+||B'-B||+d_n(\lambda,\mu))$$

hold for any $\mu \in S_n$ *and any* A', B' *satisfying* (3.3).

(b) *Assume that* $\mathcal{Y}$ *is infinite dimensional, and let* $\Lambda \in C(\mathbf{C})$*. Then there exist positive constants* ε *(depending on* A, B *only) and* K *(depending on* A, B *and* $||\Lambda||$*) with the following property: For every pair* A', B' *of operators satisfying* (3.3) *and for every* $M \in C(\mathbf{C})$ *there is a feedback* $F = F(A', B', M)$ *such that*

$$\sigma(A' + B'F) = M$$

and

$$(3.6)\qquad ||F(A',B',\Lambda)|| \leq K,$$

$$(3.7)\qquad ||F(A',B',M)-F(A,B,\Lambda)|| \leq K(||A'-A||+||B'-B||+d(\Lambda,M)).$$

The following simple proposition will be used in the proof of Theorem 3.1.

PROPOSITION 3.2. (a) *Assume* $\dim \mathcal{Y} \geq n$*. Given* $\lambda \in S_n$*, for every* $\mu = \{\mu_1, \ldots, \mu_n\} \in S_n$ *there exists a normal operator* $N(\mu) \in L(\mathcal{Y})$ *such that* $\sigma(N(\mu)) = \mu$ *and*

$$||N(\mu) - N(\lambda)|| \leq d_n(\lambda,\mu).$$

(b) *Assume* $\dim \mathcal{Y} = \infty$*. Given* $\Lambda \in C(\mathbf{C})$*, for every* $M \in C(\mathbf{C})$ *there exists a normal operator* $N(M) \in L(\mathcal{Y})$ *such that* $\sigma(N(M)) = M$ *and*

$$||N(M) - N(\Lambda)|| \leq 2d(\Lambda, M)$$

for every $M \in C(\mathbf{C})$*.*

PROOF. We omit the proof of statement (a) (it can be done by taking $N(\mu)$ to be a diagonal matrix in a fixed orthonormal basis in $\mathcal{Y}$).

We prove part (b). Without loss of generality we can assume that $\mathcal{Y}$ is separable. Let $\{\lambda_j\}_{j=1}^{\infty}$ be a sequence of (not necessarily different) complex numbers from Λ such that the closure of the set $\{\lambda_1, \lambda_2, \dots\}$ is Λ itself.

We can assume that
$$\mathcal{Y} = \oplus_{i=0}^{\infty} \mathcal{Y}_i,$$
where $\mathcal{Y}_i$ is a Hilbert space with an orthonormal basis $\{e_j^{(i)}\}_{j=1}^{\infty}$. Define $N(\Lambda) \in L(\mathcal{Y})$ by $N(\Lambda)e_j^{(i)} = \lambda_j e_j^{(i)}$. Consider now $M \in C(\mathbf{C})$, which is different from Λ, so $d(M, \Lambda) > 0$. Let $y_1, y_2, \dots$ be a countable dense subset in M. For each λ_j let $x_j \in M$ be such that $|\lambda_j - x_j| = \min_{x \in M} |\lambda_j - x|$. For each y_i let λ_{k_i} be such that

$$|y_i - \lambda_{k_i}| \leq \min_{x \in \Lambda} (y_i - x) + d(\Lambda, M).$$

Form the operator $N(M)$ as follows:

$$\begin{aligned}
N(M)e_j^{(0)} &= x_j e_j^{(0)}, \qquad j = 1, 2, \dots, \\
N(M)e_{k_i}^{(i)} &= y_i e_{k_i}^{(i)}, \qquad i = 1, 2, \dots, \\
N(M)e_j^{(i)} &= x_j e_j^{(i)}, \quad \text{for } j \neq k_i \text{ and } i = 1, 2, \dots.
\end{aligned}$$

Clearly, $N(M)$ is normal, the spectrum of $N(M)$ is M and

$$\begin{aligned}
||N(M) - N(\Lambda)|| = \sup\{&|\lambda_j - x_j|, \quad j = 1, 2, \dots; \\
&|y_i - \lambda_{k_i}|, \; i = 1, 2, \dots\} \\
\leq\ & 2d(M, \Lambda). \quad \square
\end{aligned}$$

Proof of Theorem 3.1. We follow the approach developed in the proof of the main theorem in [**8**]. Apply induction on n. If $n = 1$, then (3.2) becomes $B\mathcal{X} = \mathcal{Y}$, i.e., B is right invertible. Choose ε so small that B' is right invertible for every $B' \in L(\mathcal{X}, \mathcal{Y})$ satisfying $||B' - B|| < 2\varepsilon$, and put
$$F(A', B', \mu) = B'^{-1}(N(\mu) - A'), \quad \mu \in S_1,$$
where $N(\mu)$ is taken from Proposition 3.2 and B'^{-1} is some right inverse of B' chosen so that $||B'^{-1} - B^{-1}|| \leq K_0 ||B' - B||$, where the constant K_0 depends on B only. The estimates (3.4) and (3.5) are easily verified.

Analogously one proves part (b) in case $n = 1$.

Assume now Theorem 3.1 is proved with n replaced by $n - 1$. Let A, B be operators for which (3.2) holds. As in [**2**] or [**8**], we show that there is a (closed) subspace $\mathcal{M}_1 \subset \mathcal{Y}$ such that

$$\sum_{k=0}^{n-1} A^k(\mathcal{M}_1) = \mathcal{Y} \tag{3.8}$$

$$BC = I_{\mathcal{M}_1} \tag{3.9}$$

for some operator $C \in L(\mathcal{M}_1, \mathcal{X})$.

Indeed (see [**8**, p. 539]), one can take

$$\mathcal{M}_1 = B_\varepsilon(\mathcal{X}),$$

where

$$B_\varepsilon = B(I - E(\varepsilon)),$$

$E(t), 0 \leq t \leq \infty$, is the spectral resolution for the positive semidefinite operator B^*B, and $\varepsilon > 0$ is chosen sufficiently close to zero.

Without loss of generality, we can assume $\mathcal{M}_1 \neq \mathcal{Y}$ (otherwise B is right invertible and we can repeat the proof given above for the case $n = 1$).

Choose $\varepsilon > 0$ so small that, for every A', B' satisfying (3.3), we have

$$\sum_{k=0}^{n-1} A'^k(\mathcal{M}_1) = \mathcal{Y}$$

and

$$B'C' = I_{\mathcal{M}_1} \tag{3.10}$$

for some operator $C' \in L(\mathcal{M}_1, \mathcal{X})$; moreover, the estimate

$$||C - C'|| \leq K_0||B' - B||$$

holds where the positive constant K_0 depends on B and $\mathcal{M}_1$ only.

With respect to the orthogonal decomposition $\mathcal{Y} = \mathcal{M}_1 \oplus \mathcal{M}_1^{\perp}$ write

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Then (3.8) implies

$$\sum_{k=0}^{n-2} A_{22}^k A_{21}(\mathcal{M}_1) = \mathcal{M}_1^{\perp}.$$

So, applying the induction hypothesis, we find $K_1, \varepsilon_1 > 0$ such that, for any operators A'_{22}, A'_{21} with

$$||A'_{22} - A_{22}|| + ||A'_{21} - A_{21}|| < \varepsilon_1$$

and any $\tilde{\mu} = \{\mu_1, \ldots, \mu_{n-1}\} \in S_{n-1}$, there is an operator $D = D(A'_{22}, A'_{21}, \tilde{\mu}) \in L(\mathcal{M}_1^{\perp}, \mathcal{M}_1)$ such that

$$\prod_{j=1}^{n-1} (A'_{22} + A'_{21}D - \mu_j I) = 0 \tag{3.11}$$

and

$$||D(A_{22}, A_{21}, \tilde{\lambda})|| \leq K_1, \tag{3.12}$$

$$\begin{aligned} &||D(A'_{22}, A'_{21}, \tilde{\mu}) - D(A_{22}, A_{21}, \tilde{\lambda})|| \\ &\qquad \leq K_1(||A'_{22} - A_{22}|| + ||A'_{21} - A_{21}|| + d_{n-1}(\tilde{\mu}, \tilde{\lambda})). \end{aligned} \tag{3.13}$$

Here $\tilde{\lambda} = \{\lambda_1, \ldots, \lambda_{n-1}\}$ and the entries $\lambda_1, \ldots, \lambda_n$ are enumerated so that $||\tilde{\lambda}|| = ||\lambda||$.

Now given A', B' satisfying (3.3), write

$$A' = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix}$$

with respect to the orthogonal decomposition $\mathcal{Y} = \mathcal{M}_1 \oplus \mathcal{M}_1^{\perp}$, and, for $\mu = \{\mu_1, \ldots, \mu_n\} \in S_n$, put

$$\begin{aligned} F(A', B', \mu) =& [C'(DA'_{21} + \mu_{\sigma(n)}I - A'_{11}), C'(DA'_{22} - \mu_{\sigma(n)}D - A'_{12})] \\ &: \mathcal{M}_1 \oplus \mathcal{M}_1^{\perp} \to X, \end{aligned}$$

where C' is taken from (3.10), $D = D(A'_{22}, A'_{21}, \tilde{\mu}), \tilde{\mu} = \{\mu_{\sigma(1)} \dots, \mu_{\sigma(n-1)}\}$, and the permutation σ of $\{1, \dots, n\}$ is chosen so that

$$d_n(\lambda, \mu) = \sup_{1 \le i \le n} |\lambda_i - \mu_{\sigma(i)}|.$$

This choice of σ ensures that

$$d_{n-1}(\tilde{\lambda}, \tilde{\mu}) \le d_n(\lambda, \mu) \quad \text{and} \quad d_1(\lambda_n, \mu_{\sigma(n)}) \le d_n(\lambda, \mu).$$

It can be checked (as in [**8**, pp. 540–541]) that

$$\prod_{j=1}^{n} (A' + B'F(A', B', \mu) - \mu_{\sigma(j)} I] = 0. \tag{3.14}$$

Finally, using (3.12) and (3.13), one verifies the estimates (3.4) and (3.5).

For the part (b) assume that $\mathcal{Y}$ is infinite dimensional. Then, in view of (3.8), $\mathcal{M}_1$ is infinite dimensional as well. By Proposition 3.2(b), for any $M \in C(\mathbf{C})$, choose a normal operator $N(M)$ such that $\sigma(N(M)) = M$ and

$$||N(M) - N(\Lambda)|| \le 2d(\Lambda, M).$$

Put

$$\begin{aligned} F(A', B', M) = [&C'(DA'_{21} + N(M) - A'_{11}), C'(DA'_{22} - N(M)D - A'_{12}] \\ &\in L(\mathcal{M}_1 \oplus \mathcal{M}_1^{\perp}, \mathcal{Y}), \end{aligned}$$

where $D = D(A'_{22}, A'_{21}, M)$ is an operator which exists by the induction hypothesis. Then $A' + B'F(A', B', M)$ is similar to

$$T = \begin{bmatrix} N(M) & 0 \\ A'_{21} & A'_{22} + A'_{21}D \end{bmatrix}$$

(cf. the proof of the theorem in [**8**]). Since $\sigma(N(M)) = M, \sigma(A'_{22} + A'_{21}D) = M$ and $N(M)$ is normal, we have that $\sigma(T) = M$. The estimates (3.4) and (3.5) are proved in the same way as in the part (a). $\square$

We remark that the converse of Theorem 3.1 is true as well: spectrum assignability by feedback implies exact controllability. Actually, rather weak assumptions in terms of spectrum assignability allow one to deduce that the system is exactly controllable (see [**8**] for some results of this kind).

Next, consider behavior of the feedback in the presence of analytic dependence of the operators A and B on a complex parameter. Here also the feedback can be chosen to be well-behaved, at least locally:

THEOREM 3.3. *Let $\Omega \subset \mathbf{C}$ be an open set, and let $A : \Omega \to L(\mathcal{Y})$, $B : \Omega \to L(\mathcal{X}, \mathcal{Y})$ be operator valued functions which are analytic in Ω. Assume that $z_0 \in \Omega$ is such that*

$$\sum_{k=0}^{n-1} (A(z_0))^k B(z_0)(\mathcal{X}) = \mathcal{Y}$$

for some n. Then, for every n-tuple of scalar analytic functions $\lambda_1(z), \dots, \lambda_n(z)$ defined on an open neighborhood U of z_0, there is an analytic operator function $F : V \to L(\mathcal{Y}, \mathcal{X})$, where $V \subset U$ is an open neighborhood of z_0 such that

$$\prod_{j-1}^{n} (A(z) + B(z)F(z) - \lambda_j(z)I) = 0$$

for all $z \in V$.

PROOF. We mimic the ideas used in the proof of Theorem 3.1. Proceed by induction on n. We omit the consideration of the case $n = 1$, which is easy.

Assume Theorem 3.3 is proved with n replaced by $n - 1$. As in the proof of Theorem 3.1, find a closed subspace $M_1 \subset Y$ such that

$$\sum_{k=0}^{n-1} (A(z_0))^k (\mathcal{M}_1) = \mathcal{Y} \tag{3.15}$$

and

$$B(z_0)C = I_{\mathcal{M}_1} \tag{3.16}$$

for some $C \in L(\mathcal{M}_1, \mathcal{X})$. Let us examine the equality (3.16). Since C is left invertible, $\operatorname{Im} C$ is a (closed) subspace. Write $B(z)$ in the 2×2 block operator matrix form

$$B(z) = \begin{bmatrix} B_{11}(z) & B_{12}(z) \\ B_{21}(z) & B_{22}(z) \end{bmatrix} : (\operatorname{Im} C) \oplus (\operatorname{Im} C)^{\perp} \to \mathcal{M}_1 \oplus \mathcal{M}_1^{\perp}$$

(so, for instance, $B_{11}(z) \in L(\operatorname{Im} C, \mathcal{M}_1)$). The equality (3.16) implies that $B_{21}(z_0) = 0$ and that $B_{11}(z_0)$ is invertible (its inverse is C considered as an operator from $\mathcal{X}$ onto $\operatorname{Im} C$). So $B_{11}(z)$ is invertible for all $z \in V_1$, where V_1 is a sufficiently small neighborhood of z_0. Let

$$S(z) = \begin{bmatrix} I_{\mathcal{M}_1} & 0 \\ -B_{21}(z)B_{11}(z)^{-1} & I_{\mathcal{M}_1^{\perp}} \end{bmatrix} : \mathcal{M}_1 \oplus \mathcal{M}_1^{\perp} \to \mathcal{M}_1 \oplus \mathcal{M}_1^{\perp}, \ z \in V_1.$$

Obviously, $S(z)$ is invertible, and $S(z)B(z) = \left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right]$. Replacing $A(z)$ by$S(z)A(z)S(z)^{-1}$ and $B(z)$ by $S(z)B(z)$ we can assume that $B_{21}(z)$ is zero for $z \in V_1$. This means

$$B(z)C = I_{\mathcal{M}_1}, \quad z \in V_1$$

Taking V_1 smaller if necessary we can ensure that also

$$\sum_{k=0}^{n-1} (A(z))^k (\mathcal{M}_1) = \mathcal{Y}, \quad z \in V_1.$$

Now we repeat the construction given in the proof of Theorem 3.1. $\square$

The results of this and preceding sections admit dual statements, which can be obtained by passing to the adjoint operators. For example, the following is the dual statement to the part (b) of Theorem 2.1 (see [**4**, Appendix]).

THEOREM 3.4. *Let $A \in L(\mathcal{Y})$ and $C \in L(\mathcal{Y}, \mathcal{X})$ be Banach space operators. Then the operator*

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m-1} \end{bmatrix} \in L(\mathcal{Y}, \mathcal{X}^m)$$

is left invertible for some m if and only if the operator

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} \in L(\mathcal{Y}, \mathcal{Y} \oplus \mathcal{X})$$

is left invertible for all $\lambda \in \mathbf{C}$.

It is an open question whether Theorem 3.1 (under the assumption that $[B, AB, \ldots, A^{n-1}B]$ is right invertible) is valid for Banach space operators.

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