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ON SPECTRAL PROPERTIES OF ALMOST MATHIEU OPERATORS AND CONNECTIONS WITH IRRATIONAL ROTATION C^*-ALGEBRAS

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1. In [**2**] we introduced some concepts that led to a study of some almost periodic Schrödinger operators in the light of the C^*-algebras they are associated with in a natural way. More specifically, our topic is almost Mathieu operators $h(\alpha,\beta)$, in the setup of irrational rotation C^*-algebras. Let α be an irrational number and let u, v be two unitary operators such that $uv = e^{2\pi\alpha i}vu$. Then $h(\alpha,\beta)$ is the self-adjoint element $u + u^* + \beta(v + v^*)$ in the C^*-algebra $\mathcal{A}_\alpha$ generated by the operators u and v.

In what follows we shall discuss some problems related to this C^*-algebraic approach. First, we investigate (primarily from an algebraic point of view) the difference equation that characterizes all formal Fourier series in u and v which commute with $h(\alpha,\beta)$. Then we apply this to exhibit domains for β where $h(\alpha,\beta)$ fails to have point spectrum under certain representations of $\mathcal{A}_\alpha$ on the Hilbert space $\ell^2(\mathbf{Z})$. Finally, using rational interpolation, we give a characterization in terms of C^*-algebras of those operators $h(\alpha,\beta)$ which have a Cantor spectrum.

2. In the sequel we always assume that α is an irrational number and $\beta \notin \{-1,0,1\}$. A state ϕ on $\mathcal{A}_\alpha$ is called an eigenstate of $h(\alpha,\beta)$ for some $\chi \in \mathrm{Sp}(h(\alpha,\beta))$ (cf. [**2**]) if

$$\phi(ha) = \chi\phi(a) \quad \text{for all } a \in \mathcal{A}_\alpha.$$

The dimension of the linear subspace of the dual $\mathcal{A}_\alpha^*$ generated by the eigenstates for χ is called the multiplicity of $\chi \in \mathrm{Sp}(h(\alpha,\beta))$. The multiplicity of χ is always less than or equal to two. We consider the automorphisms σ of $\mathcal{A}_\alpha$ determined by $\sigma(u) = u^*, \sigma(v) = v^*$. Let $\lambda = e^{\pi\alpha i}$ and, for $p, q \in \mathbf{Z}$, let $S_{pq} = \lambda^{-pq}(u^p v^q + u^{-p}v^{-q}), T_{pq} = \lambda^{-pq}(u^p v^q - u^{-p}v^{-q})i$.

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If ϕ is an eigenstate for χ and we set

$$s_{pq} = \phi(S_{pq}), \quad t_{pq} = \phi(T_{pq}),$$

then $\{s_{pq}\}$ and $\{t_{pq}\}$ are double sequences of real numbers which solve the following system of linear equations:

(2.1)

(a) $\cos(\pi\alpha q)(x_{p-1,q} + x_{p+1,q}) + \beta\cos(\pi\alpha p)(x_{p,q-1} + x_{p,q+1}) = \chi x_{pq}$.

(b) $\sin(\pi\alpha q)(x_{p-1,q} - x_{p+1,q}) - \beta\sin(\pi\alpha p)(x_{p,q-1} - x_{p,q+1}) = 0$.

We consider the formal (Fourier) series

$$f = \sum_{p,q\in\mathbf{Z}} c_{pq}S_{pq} + \sum_{p,q\in\mathbf{Z}} d_{pq}T_{pq} \tag{2.2}$$

with complex coefficients c_{pq}, d_{pq}. We shall always assume that $c_{pq} = c_{-p,-q}$ and $d_{p,q} = -d_{-p,-q}$, thus rendering (2.2) into a unique representation. If g is a formal series with only finitely many non-vanishing coefficients (we shall refer to such a formal series as a polynomial), then we can extend the usual product of polynomials in the canonical way to define the products fg and gf. To any linear functional ψ on the $*$-algebra generated by u and v we associate the formal series

$$\sum_{p,q\in\mathbf{Z}} \psi(S_{pq})S_{pq} + \sum_{p,q\in\mathbf{Z}} \psi(T_{pq})T_{pq}.$$

If a is an element in $\mathcal{A}_\alpha$ then its Fourier series is

$$\frac{1}{4}\Big(\sum_{p,q\in\mathbf{Z}} \tau(aS_{pq})S_{pq} + \sum_{p,q\in\mathbf{Z}} \tau(aT_{pq})T_{pq}\Big),$$

where τ is the (unique) normalized trace on $\mathcal{A}_\alpha$. In this case the coefficients form two square summable double sequences. We record the following basic fact.

Proposition 2.1. *A formal series commutes with $h(\alpha,\beta)$ if and only if its coefficients satisfy the equation* (2.1b) *for all $p, q \in \mathbf{Z}$.*

REMARK. The relative commutant of the C^*-algebra $C^*(h(\alpha,\beta))$ in $\mathcal{A}_\alpha$ is always abelian. To see this, note that it was shown in [**2**] that, for each $\chi \in \mathrm{Sp}(h(\alpha,\beta))$, there is exactly one σ-invariant eigenstate; consequently, $C^*(h(\alpha,\beta))$ is maximal abelian in the fixed-point algebra $\mathcal{A}_\alpha^\sigma$. Now let a and b be in the commutant. In order to show that a and b commute it suffices to consider the case that a and b are self-adjoint with $\sigma(a) = -a$, $\sigma(b) = -b$. We have

$$\begin{aligned}(ab-ba)(ba-ab) &= ab^2a - b(ab)a - a(ba)b + ba^2b \\ &= a^2b^2 - ba^2b - ab^2a + a^2b^2 = 0,\end{aligned}$$

whence $ab = ba$.

In the same way one can show that the relative commutant of $C^*(h(\alpha,\beta))$ in the von Neumann algebra associated with the trace τ is abelian.

Our ultimate interest is in those formal series commuting with $h(\alpha,\beta)$ whose coefficients are square summable. (We shall refer to formal series with square summable coefficients as square summable series or as being in ℓ^2.) A complete description can be given of all square summable series commuting with $h(\alpha,\beta)$ such that $d_{pq} = 0$ for all $p, q \in \mathbf{Z}$ in (2.2):

THEOREM 2.2. *The square summable series in S_{pq} commuting with $h(\alpha,\beta)$ are in one-one correspondence with the square integrable functions on the spectrum* $\mathrm{Sp}(h(\alpha,\beta))$ *of $h(\alpha,\beta)$ with respect to the integrated density of states.*

This theorem is a consequence of [**2**], Theorem 2.2(a). At the present stage we do not know whether there are any non-trivial square summable series in T_{pq} which commute with $h(\alpha,\beta)$. However, we can show the following.

PROPOSITION 2.3. *If $f = \sum_{p,q\in\mathbf{Z}} d_{pq}T_{pq}$ is in ℓ^2 and f commutes with $h(\alpha,\beta)$ then*

$$d_{pq} = \begin{cases} 0 & \text{for } |p| \geq |q| \text{ if } |\beta| > 1 \\ 0 & \text{for } |p| \leq |q| \text{ if } |\beta| < 1. \end{cases} \tag{2.3}$$

PROOF. We have a representation π_τ of $\mathcal{A}_\alpha$ on the Hilbert space $\ell^2(\mathbf{Z}^2)$ such that $\tau(a) = \langle \pi_\tau(a)\delta_{(0,0)}, \delta_{(0,0)}\rangle$ for all $a \in \mathcal{A}_\alpha$. To f there corresponds a normalized vector ξ in $\ell^2(\mathbf{Z}^2)$ such that

$$\langle S_{pq}\delta_{(0,0)}, \xi\rangle = 0, \quad \langle T_{pq}\delta_{(0,0)}, \xi\rangle = d_{pq}, \quad p, q \in \mathbf{Z}\,.$$

Since f commutes with $h(\alpha, \beta)$, the functional

$$\psi(a) = \langle a\delta_{(0,0)}, \xi\rangle, \quad a \in \mathcal{A}_\alpha,$$

is $h(\alpha,\beta)$-central (i.e., $\psi(ah(\alpha,\beta)) = \psi(h(\alpha,\beta)a)$). Thus ψ is in the weak closure of the linear span of all eigenstates of $h(\alpha,\beta)$. Since the condition (2.3) is satisfied for the eigenstates [**2**, §2], we conclude that (2.3) is true for f. □

Henceforth we assume that $|\beta| > 1$. The case $0 < |\beta| < 1$ is treated in a similar way. In order to analyze the solutions $\{x_{pq}\}$ of (2.1b) satisfying (2.3) we introduce an operator H_s on the space of all double sequences $x = \{x_{pq}\}_{p,q\in\mathbf{Z}}$ by

$$(H_s x)_{pq} = \cos(\pi\alpha q)(x_{p-1,q} + x_{p+1,q}) + \beta\cos(\pi\alpha p)(x_{p,q-1} + x_{p,q+1}) - s x_{pq}.$$

We shall use the following recursion formula, which produces the solutions of (2.1) satisfying (2.3) for $q > p \geq 0$.

(2.4)

$$\begin{pmatrix} x_{p,q+1} \\ x_{p-1,q} \end{pmatrix} = \frac{1}{\beta\sin\pi\alpha(q-p)} \begin{pmatrix} -\sin 2\pi\alpha q & -\beta\sin\pi\alpha(p+q) \\ \beta\sin\pi\alpha(p+q) & \beta^2\sin 2\pi\alpha p \end{pmatrix} \begin{pmatrix} x_{p+1,q} \\ x_{p,q-1} \end{pmatrix} + \frac{\chi x_{pq}}{\sin\pi\alpha(q-p)} \begin{pmatrix} \beta^{-1}\sin\pi\alpha q \\ -\sin\pi\alpha p \end{pmatrix}.$$

We apply this recursion simultaneously for all χ: Set

$$x_{p,q} = 0 \quad \text{for } p \geq q \geq 0, \quad x_{p,p+1} = (-\beta)^{-p} \quad \text{for } p \geq 0.$$

For each $k \geq 2$ we determine $x_{p,p+k}$ for $p \geq 0$ by means of (2.4), having determined $x_{p,p+k-1}$ and $x_{p,p+k-2}$ for $p \geq 0$ already. (The recursion formula (2.4) is redundant, reflecting the fact that the system (2.1) is overdetermined.) Finally, we set

$$x_{-p,q} = x_{p,q} \quad \text{for } p, q \geq 0,$$
$$x_{p,-q} = -x_{p,q} \quad \text{for } q \geq 0.$$

From (2.4) we read off the following facts.

Proposition 2.4. *There exists a* (*unique*) *double sequence of polynomials* $\{\omega_{pq}(\chi)\}$ *with the following properties*:

(a) $\omega_{01} = 1, \omega_{pq} = 0$ *for* $|p| \geq |q|$.

(b) $\omega_{-p,q} = \omega_{p,q}, \omega_{p,-q} = -\omega_{p,q}$ *for all* $p, q \in \mathbf{Z}$.

(c) $\omega_{pq}(\chi)$ *is a solution of* (2.1) *for all* χ.

Moreover, this double sequence has additional properties:

(d) ω_{pq} *has the degree* $|q| - |p| - 1$ *for* $|q| > |p|$.

(e) $\omega_{p,p+k}(\chi)$ *decays exponentially as* $k \to \infty$, *for all* $p \geq 0$. *More precisely, we have*

$$\overline{\lim_{k\to\infty}} \beta^k \omega_{p,p+k}(\chi) < \infty.$$

It is readily seen that if we prescribe arbitrary values for x_{0q}, then there is exactly one solution of (2.1b) satisfying (2.3) that takes those values in the corresponding positions. Conversely, any solution of (2.1b) with (2.3) can be obtained in this fashion. Thus 2.4(d) gives rise to the following corollary.

Corollary 2.5. *There is a one-one correspondence between the linear functionals on the vector space of all polynomials in one variable and the solutions of* (2.1b) *satisfying* (2.3).

We shall give a more specific description of the coefficients of the polynomials in 2.4 in terms of the operator H_s we have introduced earlier. Consider some fixed number s. Let $\omega^s_{pq}(\chi) = \omega_{pq}(\chi - s)$, and let $a^{(k)}_{pq}$ be the k-th coefficient of ω^s_{pq}. We have

$$(H^s\omega^s)(\chi) = (\chi - s)\omega^s(\chi).$$

It follows that $H^s a^{(k)} = a^{k-1}$ for $k \geq 2$ and $H^s a^{(1)} = 0$. We can summarize this as

THEOREM 2.5. *There is, up to a scaling factor, exactly one sequence* $a^{(0)}, a^{(1)}, \ldots$ *of double sequences such that*

(a) $a^{(k)}$ *solves* (2.1b),

(b) $a^{(k)}_{-p,q} = a^{(k)}_{pq}, \quad a^{(k)}_{p,-q} = -a^{(k)}_{-p,q}$,

(c) $a^{(k)}_{pq} = 0$ *for* $|q| < |p| + k$,

(d) $H^s a^{(k)} = a^{(k-1)}$, *where* $a^{(0)} = 0$.

Moreover, for any solution $\{x_{pq}\}$ *of* (2.1b) *satisfying* (2.3), *there is a sequence* $\{c_k\}$ *such that*

$$x_{pq} = \sum_{k=1}^{\infty} c_k a^{(k)}_{pq}.$$

We have

$$\varlimsup_{k\to\infty} \beta^k x_{p,p+k} < \infty.$$

3. In this section we use ideas from §2 to prove the absence of point spectrum for $h(\alpha,\beta)$, $0 < |\beta| < 1$, under certain representations of $\mathcal{A}_\alpha$. For each $z \in \mathbf{C}, |z| = 1$, let π_z be the representation of $\mathcal{A}_\alpha$ on $\ell^2(\mathbf{Z})$ determined by

$$(\pi_z(u)\xi)_n = \xi_{n+1}, \quad (\pi_z(v)\xi)_n = ze^{-2\pi\alpha i}\xi_n.$$

If we set $z = e^{-i\theta}$, then we have

$$(\pi_z(h(\alpha,\beta))\xi)_n = \xi_{n+1} + \xi_{n-1} + 2\beta\cos(2\pi\alpha n + \theta)\xi_n, \quad \xi \in \ell^2(\mathbf{Z}).$$

We shall establish

THEOREM 3.1. *If* $\theta \notin \alpha\mathbf{Z}$ *and* $0 < |\beta| < 1$, *then* $\pi_z(h(\alpha,\beta))$ *has no eigenvectors in* $\ell^2(\mathbf{Z})$.

PROOF. Assume that the contrary is true. So there is a $\theta \in \mathbf{R} \setminus \alpha\mathbf{Z}$ and there are $\chi \in \mathrm{Sp}(h(\alpha,\beta)), \xi \in \ell^2(\mathbf{Z})$ such that $||\xi|| = 1$ and

$$\pi_z(h(\alpha,\beta))\xi = \chi\xi.$$

If $\overline{\xi} = \{\overline{\xi}_n\}_{n\in\mathbf{N}}$ is the conjugate vector of ξ then $\overline{\xi}$ is also an eigenvector for χ. Thus we may assume that ξ is a real vector. In particular,

$$\langle (u - u^*)\xi, \xi\rangle \in \mathbf{R}.$$

Since $\theta \notin \alpha\mathbf{Z}$, it follows from [**2**, Theorem 3.2] that χ has multiplicity 2. In particular, the eigenstate

$$\phi_\xi(a) = \langle a, \xi, \xi\rangle, \quad a \in A,$$

is not σ-invariant. It follows that $\{\phi_\xi(T_{pq})\}$ is a non-trivial bounded solution of (2.1) satisfying (2.3). Thus,

$$\phi_\xi(T_{10}) = \phi_\xi(u - u^*) \neq 0.$$

Since ϕ_ξ is a self-adjoint functional and $u-u^*$ is skew-adjoint, $\phi_\xi(u-u^*)$ is a non-zero imaginary number, contradicting our earlier conclusion. □

REMARKS. (a) At the present stage we don't know whether point spectrum can occur for $\pi_z(h(\alpha,\beta)), 0 < |\beta| < 1$, if $\theta \in \alpha\mathbf{Z}$.

(b) It was shown in [**3**], by similar ideas, that $\pi_z(h(\alpha,\beta))$ never has eigenvectors for $|\beta| = 1$ and all θ.

4. From now on we assume that $\alpha_1, \alpha_2, \ldots$ is a sequence of rational numbers converging to the irrational number $\alpha = \alpha_\infty$. We set

$$(u\xi)_n = \xi_{n+1}, \quad (v_{\alpha_k}\xi)_n = e^{-2\pi\alpha_k n i}\xi_n, \quad \xi \in \ell^2(\mathbf{Z})$$

and let $\mathcal{A}_\alpha$ be the C^*-algebra generated by u and v_α. Then $\{\mathcal{A}_{\alpha_n}\}_{n\in\mathbf{N}\cup\{\infty\}}$ is a continuous field of C^*-algebras on the one point compactification of $\mathbf{N}$, where the (non-commutative) polynomials in the generators u and v_α form a generating family of continuous operator fields [**1**]. Let $\mathcal{B}$ denote the C^*-algebra of continuous operator fields. We choose a sequence $\beta_1, \beta_2, \ldots$ in $\mathbf{R}$, such that $\beta = \beta_\infty = \lim_{n\to\infty} \beta_n$

and (after a possible adjustment of the sequence $\{\alpha_n\}$) the gaps in the spectrum of the operator

$$h(\alpha_n, \beta_n) = u + u^* + \beta_n(v_{\alpha_n} + v_{\alpha_n}^*)$$

are all open (this is equivalent to the requirement that the multiplicity for any $\chi \in \mathrm{Sp}(h(\alpha_n, \beta_n))$, as defined in §2, be less than or equal to two). We denote by $\mathcal{C}$ the C^*-subalgebra of $\mathcal{B}$ generated by the operator field $n \mapsto h(\alpha_n, \beta_n)$ and the continuous complex functions on $\mathbf{N} \cup \{\infty\}$. We shall need the following.

LEMMA 4.1. *Let $\mathcal{I}_1, \mathcal{I}_2, \dots$ be a sequence of closed intervals such that $\mathcal{I}_n$ is a connected component of $\mathrm{Sp}(h(\alpha_n, \beta_n))$ for each $n \in \mathbf{N}$. Then $\lim_{n\to\infty}(\mathrm{length}\,\mathcal{I}_n) = 0$.*

PROOF. For each $n \in \mathbf{N}$ the C^*-algebra $\mathcal{A}_{\alpha_n}$ is isomorphic to the C^*-algebra of continuous matrix-valued functions on the circle $\mathbf{T}$ generated by

$$z \mapsto z \underbrace{\begin{pmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & & \cdots & 0 \end{pmatrix}}_{q_n}$$

and

$$z \mapsto \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & e^{-2\pi\alpha_n i} & & \\ \vdots & & & \\ 0 & \cdots & & e^{-2\pi(q_n-1)\alpha_n i} \end{pmatrix}$$

where $\alpha_n = p_n/q_n$, with p_n and q_n relatively prime. Let e_n be the spectral projection of $h(\alpha_n, \beta_n)$ corresponding to the interval $\mathcal{I}_n$. Since all gaps are open by assumption, it follows that $e_n(z)$ is a one dimensional projection in the matrix algebra $M_{q_n}(\mathbf{C})$ for all $z \in \mathbf{T}^-$. In particular the C^*-algebra $e_n \mathcal{A}_{\alpha_n} e_n$ is abelian.

Now assume that the assertion is not true. It follows that there is a non-degenerate closed interval $\mathcal{I}$ and a sequence $n_1, n_2, \ldots$ in $\mathbf{N}$ such that

$$\mathcal{I} \subset \bigcap_{k \in \mathbf{N}} \mathcal{I}_{n_k}.$$

We may assume that $n_k = k$. Take a continuous function f which is non-zero on $\mathcal{I}$ but vanishes outside $\mathcal{I}$. We have

$$\overline{f(h(\alpha_n, \beta_n))\mathcal{A}_{\alpha_n} f(h(\alpha_n, \beta_n))} \subseteq e_n \mathcal{A}_{\alpha_n} e_n.$$

In particular the C^*-algebra on the left-hand side is abelian. It follows that $\overline{f(h(\alpha, \beta))\mathcal{A}_\alpha f(h(\alpha, \beta))}$ is also abelian (and non-zero), which is impossible since $\mathcal{A}_\alpha$ is a simple C^*-algebra having no abelian hereditary subalgebras.

We shall prove the following.

THEOREM 4.2. *The spectrum $h(\alpha, \beta)$ is totally disconnected if and only if the Gelfand spectrum of the abelian C^*-algebra $\mathcal{C}$ is homeomorphic to a subset of $\mathbf{R}$.*

PROOF. We note that the Gelfand spectrum Ω of $\mathcal{C}$ is homeomorphic to the following compact subset of $\mathbf{R}^2$:

$$\bigcup_{n=1}^{\infty} \left\{ \left(s, \frac{1}{n}\right) \middle| \, s \in \mathrm{Sp}(h(\alpha_n, \beta_n)) \right\} \cup \{(s, 0) | s \in \mathrm{Sp}(h(\alpha, \beta))\}.$$

It follows from 4.1 that the lengths of the connected components of $\mathrm{Sp}(h(\alpha_n, \beta_n))$ converge to zero uniformly in n. First assume that $\mathrm{Sp}(h(\alpha, \beta))$ is totally disconnected. Then

$$\mathbf{R} \setminus \mathrm{Sp}(h(\alpha, \beta)) = \bigcup_{k=1}^{\infty} F_k \cup F_{-\infty} \cup F_{+\infty},$$

where $F_1, F_2, \ldots$ are finite open intervals and $F_{-\infty}, F_{+\infty}$ are infinite open intervals, $F_{-\infty}$ is bounded from above, F_∞ is bounded from below. Let

$$a_k := \inf F_k, \quad k \in \mathbf{N} \cup \{\infty\}, \quad b_k := \sup F_k, \quad k \in \mathbf{N} \cup \{-\infty\}.$$

For each $k \in \mathbf{N} \bigcup\{\infty\}$, $n \in \mathbf{N}$, let $\mathcal{I}_n^{(k)}$ be a connected component of $\mathrm{Sp}(h(\alpha_n,\beta_n)) \times \{1/n\}$ which has minimum distance from $(a_k,0)$. Similarly, for each $k \in \mathbf{N} \cup \{-\infty\}$, $n \in \mathbf{N}$, let $\mathcal{J}_n^{(k)}$ be a connected component of $\mathrm{Sp}(h(\alpha_n,\beta_n)) \times \{1/n\}$ which has minimum distance from $(b_k,0)$. Then $\{\mathcal{I}_n^{(k)}\}_{n\in\mathbf{N}}$ converges to $(a_k,0)$ and $\{\mathcal{J}_n^{(k)}\}_{n\in\mathbf{N}}$ converges to $(b_k,0)$. By passing to subsequences and filling in some left-over intervals, if necessary, we may assume that any two sequences of intervals with different limits are mutually disjoint and that the union of all these intervals is equal to Ω. For each $k \in \mathbf{N} \cup \{\infty\}$ we can find a sequence $\{\hat{\mathcal{I}}_n^{(k)}\}$ of mutually disjoint compact subintervals of F_k converging to a_k and, for each $k \in \mathbf{N} \cup \{-\infty\}$, we can find a sequence $\{\hat{\mathcal{J}}_n^{(k)}\}$ of mutually disjoint compact subintervals of F_k converging to b_k. For $k \in \mathbf{N}$ we assume that $\{\hat{\mathcal{I}}_n^{(k)}\}$ and $\{\hat{\mathcal{J}}_n^{(k)}\}$ are mutually disjoint. Any mapping from $\cup_{n=1}^{\infty}\mathrm{Sp}(h(\alpha_n,\beta_n)) \times \{1/n\}$ into $M = \cup_{\substack{k\in\mathbf{N}\cup\{\infty\}\\ n\in\mathbf{N}}} \mathcal{I}_n^{(k)} \cup \cup_{\substack{k\in\mathbf{N}\cup\{-\infty\}\\ n\in\mathbf{N}}} \mathcal{J}_n^{(k)}$ that maps $\mathcal{I}_n^{(k)}$ into $\hat{\mathcal{J}}_n{}^{(k)}$ and $\mathcal{J}_n^{(k)}$ onto $\hat{\mathcal{J}}_n^{(k)}$, homeomorphically, extends (uniquely) to a homeomorphism from Ω onto $M \cup \mathrm{Sp}(h(\alpha,\beta))$.

Now let us assume that $\mathrm{Sp}(h(\alpha,\beta))$ is not totally disconnected. Then there is an open interval $\mathcal{I} \subset \mathrm{Sp}(H(\alpha,\beta))$. Pick a point $a \in \mathcal{I}$ and a sequence $\{a_n\}$ converging to a with $a_n \in \mathrm{Sp}(h(\alpha_n,\beta_n))$. The compact subset $\{(a_n,1/n) \,|\, n \in \mathbf{N}\} \cup (\mathcal{J} \times \{0\})$ of Ω cannot be embedded topologically into $\mathbf{R}$. Therefore Ω is not homeomorphic to a subset of $\mathbf{R}$. □

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