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BIMODULES OVER CARTAN SUBALGEBRAS

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ABSTRACT. Given a Cartan subalgebra $\mathbf{A}$ of a von Neumann algebra $\mathbf{M}$, the techniques of Feldman and Moore are used to analyze the partial isometries v in $\mathbf{M}$ such that $v^*\mathbf{A}v$ is contained in $\mathbf{A}$. Orthonormal bases for $\mathbf{M}$ consisting of such partial isometries are discussed, and convergence of the resulting generalized fourier series is shown to take place in the Bures $\mathbf{A}$-topology. The Bures $\mathbf{A}$-topology is shown to be equivalent to the strong topology on the unit ball of $\mathbf{M}$. These ideas are applied to $\mathbf{A}$-bimodules in $\mathbf{M}$ to prove the existence of orthonormal bases for bimodules and to give a simplified and intuitive proof of the Spectral Theorem for Bimodules first proven by Muhly, Saito, and Solel.

1. Introduction. The notion of a *triangular subalgebra* of a von Neumann algebra $\mathbf{M}$ was introduced in a paper of Kadison and Singer [**7**]; it is defined to be an algebra $\mathbf{T}$ of operators in $\mathbf{M}$ such that $\mathbf{T}\cap\mathbf{T}^*$ is a maximal abelian subalgebra $\mathbf{A}$ of $\mathbf{M}$, called the *diagonal* of $\mathbf{T}$. In a recent paper of Muhly, Saito and Solel [**8**] (referred to here as MSS) triangular subalgebras whose diagonal is a Cartan subalgebra of $\mathbf{M}$ were considered and analyzed using a formalism developed by Feldman and Moore in [**4, 5**] which is summarized below. A subspace of $\mathbf{M}$ which is invariant under left and right multiplication by members of $\mathbf{A}$ is called an $\mathbf{A}$-*bimodule*; this class of subspaces includes subalgebras of $\mathbf{M}$ containing $\mathbf{A}$ and in particular triangular subalgebras with diagonal $\mathbf{A}$. In MSS a critical role was played by the Spectral Theorem for Bimodules in which σ-weakly closed $\mathbf{A}$-bimodules are characterized.

In the introduction of MSS a simple and elegant motivation of the Spectral Theorem for Bimodules is given in the finite-dimensional case, but the proof of this theorem in the context of a Cartan subalgebra doesn't follow the motivation given. In §5 of this paper it is shown that the motivation and proof of the finite dimensional case can in fact be carried through to the case of a Cartan subalgebra. Some preliminary

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results on partial isometries (§2), topologies (§3), and orthogonality over **A** (§4) will be needed. The heart of the proof of the Spectral Theorem for Bimodules given here is understanding the relationship between the Bures topology introduced in [**2**] and the standard topologies on a von Neumann algebra.

The Feldman-Moore Formalism.

Suppose **M** is a von Neumann algebra with Cartan subalgebra **A**. (A Cartan subalgebra **A** is a regular maximal abelian subalgebra for which a conditional expectation E: $\mathbf{M} \to \mathbf{A}$ exists. For **A** to be regular, the normalizer $N_{\mathbf{M}}(\mathbf{A})$ of **A**, defined to be the group of all unitaries $u \in \mathbf{M}$ such that $u^*\mathbf{A}u = \mathbf{A}$, must generate all of **M**.) Feldman and Moore [**4, 5**] provided a construction of the pair $(\mathbf{M}, \mathbf{A})$ in terms of a measurable equivalence relation. Let $(X, \mathcal{B}, \mu)$ be a standard Borel space with μ finite. R is a *countable standard relation* on X if R is a Borel subset of $X \times X, R$ is an equivalence relation on X, and all equivalence classes are countable. R is *ergodic* if all saturated Borel sets are either null or conull. If $(x, y) \in R$ we say that x is equivalent to y and write $x \sim y$. The equivalence class of x is denoted by $R(x)$. There are natural projection maps π_ℓ and π_r from R onto X with $\pi_\ell(x, y) = x$ and $\pi_r(x, y) = y$. A measure $\nu = \nu_r$ (*right counting measure*) may be defined on R by $\nu_r(B) = \int_X |\pi_r^{-1}(x) \cap B| d\mu(x)$. (A left counting measure ν_ℓ is defined in a similar way.) This leads to

$$\int_R f(x,y) d\nu(x,y) = \int_X \big(\sum_{y \sim x} f(y,x)\big) d\mu(x). \tag{1.1}$$

The operators of the algebra **M** are represented in terms of functions $T(x, y)$ on R. The representation may require the introduction of a complex-valued “2-cocycle” $s(x, y, z)$ defined on ordered triples for which $x \sim y \sim z$ and satisfying the properties (i) $|s(x, y, z)| = 1$; (ii) $s(t, z, x)s(z, y, x) = s(t, y, x)s(t, z, y)$; (iii) $s(x, y, z) = 1$ if any two of the arguments are equal (skew symmetry). In the simplest cases s is identically equal to 1.

A measurable function $T(x, y)$ is *left-finite* if $T(x, y)$ is bounded and there is an integer N such that, for each x and y in X, the sets $\{z : T(x, z) \neq 0\}$ and $\{z : T(z, y) \neq 0\}$ have cardinality $\leq N$. The function $T(x, y)$ then defines a bounded operator (also denoted T) on

$L^2(R, v)$ via the formula

$$T\xi(x,z) = \sum_{y\sim x} T(x,y)\xi(y,z)s(x,y,z). \tag{1.2}$$

The σ-weak closure of the collection of such operators is a von Neumann algebra on $L^2(R,\nu)$ denotes $M(R,s)$. The operators in $M(R,s)$ can be represented as functions in $L^2(R,\nu) \cap L^\infty(R,\nu)$ which continue to act via equation (1.2). The operations of multiplication and adjoint are implemented by the formulas

$$\begin{aligned} T_1T_2(x,z) &= \sum_{y\sim x} T_1(x,y)T_2(y,z)s(x,y,z) \text{ and} \\ T^*(x,y) &= \overline{T(y,x)}. \end{aligned} \tag{1.3}$$

The functions of $L^\infty(X,\mu)$ may be interpreted as functions on R which are supported on the diagonal (which has positive ν-measure). These functions are left finite and hence in $M(R,s)$. Due to the skew symmetry of s, for a $\in L^\infty(X,\mu)$, formula (1.2) simplifies to

$$a\xi(x,z) = a(x)\xi(x,z). \tag{1.4}$$

The major result of [5] can be stated as follows:

If $\mathbf{M}$ *is a von Neumann algebra with Cartan subalgebra* $\mathbf{A}$ *then there exists a countable standard relation* R *and a cocycle* s *such that* $\mathbf{M} \simeq M(R,s)$ *and under this isomorphism* $\mathbf{A} \simeq L^\infty(X,\mu)$. $\mathbf{M}$ *is a factor if and only if* R *is ergodic.*

A Borel isomorphism $\varphi\colon X \to X$ with $\Gamma(\varphi) \subset R$ will be called an *R-automorphism* of X. If $\varphi\colon E \to F$ is a Borel isomorphism of two Borel subsets E and F of X with $\Gamma(\varphi) \subset R$, then φ will be called a *partial R-isomorphism.* By [**4**, p. 294–296] $M(R,s)$ is finite (respectively semifinite) if and only if a finite (respectively semifinite) measure μ on X can be chosen so that all partial R-isomorphisms are measure-preserving.

For further background material the reader is referred to the papers mentioned above, in particular §1 and §2 of MSS. Throughout this paper we deal only with algebras represented on separable Hilbert spaces.

2. Partial Isometries.

This section provides background material on partial isometries in the context of the Feldman-Moore formalism. It is well recognized that partial isometries in the generalized normalizer of $\mathbf{A} \cong L^\infty(X,\mu)$ are related to Borel automorphisms of X [**1, 5**]. Proposition 2.2 is implicit in the work of Feldman and Moore [**4, 5**] but it is such an important tool that an explicit proof is given here.

An operator $v \in \mathbf{M}$ is a partial isometry if $v^*v = p$ and $vv^* = q$ are projections. We then have $v = vp = qv$ and $v^* = pv^* = v^*q$. p is called the domain projection and q the range projection of v. We define $GN_{\mathbf{M}}(\mathbf{A})$ ("the generalized normalizer of $\mathbf{A}$ in $\mathbf{M}$") to be the set of all partial isometries v in $\mathbf{M}$ such that v^*v and vv^* belong to $\mathbf{A}$ and $v^*\mathbf{A}v = \mathbf{A}p$. If $v \in GN_{\mathbf{M}}(\mathbf{A})$ then also $v^* \in GN_{\mathbf{M}}(\mathbf{A})$, since $v\mathbf{A}v^* = v\mathbf{A}pv^* = v(v^*\mathbf{A}v)v^* = q\mathbf{A}q = \mathbf{A}q$. If $\mathbf{M}$ is finite then $v \in GN_{\mathbf{M}}(\mathbf{A})$ if and only if $v = up$ for some $u \in N_{\mathbf{M}}(\mathbf{A})$ and some projection $p \in \mathbf{A}$. [**9**]

Recall that E is the unique faithful normal conditional expectation from $\mathbf{M}$ to $\mathbf{A}$. The following lemma is an easy generalization of the fact that $u^*E(T)u = E(u^*Tu)$ for any unitary u in $N_{\mathbf{M}}(\mathbf{A})$.

Lemma 2.1. *$v^*E(T)v = E(v^*Tv)$ for all v in $GN_{\mathbf{M}}(\mathbf{A})$ and all T in* $\mathbf{M}$.

Proof. Let $v \in GN_{\mathbf{M}}(\mathbf{A})$, and let $p = v^*v$ and $q = vv^*$. Consider the map $\Phi : T \to vE(v^*Tv)v^*$ from $q\mathbf{M}q$ to $q\mathbf{A}q$. We claim that Φ is a faithful normal positive linear projection of norm one, and hence a conditional expectation [**14**]. Since the map $T \to qE(T)q$ is also a faithful normal conditional expectation from $q\mathbf{M}q$ to $q\mathbf{A}q$, by the uniqueness of Φ [**12**, 10.15(4)] we must have $qE(T)q = vE(v^*Tv)v^*$. Then $v^*E(T)v = v^*qE(T)qv = v^*vE(v^*Tv)v^*v = pE(v^*Tv)p = E(pv^*Tvp) = E(v^*Tv)$.

To verify the claim, first let $T \in q\mathbf{A}q$. Then $T \in \mathbf{A}$ so $vE(v^*Tv)v^* = vv^*Tvv^* = qTq = T$. Therefore Φ is a projection. If $T = 1$, then $\Phi(1) = vE(v^*v)v^* = vE(p)v^* = vpv^* = q$, and therefore Φ is of norm one. If $T \in q\mathbf{A}q$ with $T \geq 0$ and $vE(v^*Tv)v^* = 0$, then $0 = v^*vE(v^*Tv)v^*v = pE(v^*Tv)p = E(pv^*Tvp) = E(v^*Tv)$, hence $v^*Tv = 0$ since E is faithful. Therefore $0 = vv^*Tvv^* = qTq = T$, so Φ is faithful. The remainder of the claim is straightforward. □

PROPOSITION 2.2. *Let $v(x,y)$ be a complex-valued Borel function on R. Then v represents a partial isometry in $GN_{\mathbf{M}}(\mathbf{A})$ if and only if there is a partial R-isomorphism φ of X such that, for almost every (y,x) in R,*

(i) $v(y,x) = 0$ *unless* $(y,x) \in \Gamma(\varphi)$.

(ii) $|v(y,x)| = 1$ *when* $(y,x) \in \Gamma(\varphi)$.

In this case, $(v^*av)(x) = (a \circ \varphi^{-1})(x)v^*v(x)$.

PROOF. We first prove the "only if" part of the claim. Let $p = v^*v$ and $q = vv^*$. Suppose that $p = \chi(E)$ and $q = \chi(F)$ for Borel sets E and F in X. Let a $\in \mathbf{A}$ be represented by a measurable function $a(x)$ on X. Then, for $(x,z) \in R$, (1.3) and (1.4) give

$$(v^*av)(x,z) = \sum_{y\sim x} \overline{v(y,x)}\, a(y)\, v(y,z)\, s(x,y,z). \tag{2.1}$$

If $v^*av \in \mathbf{A}p$ for all $a \in \mathbf{A}$, then, for almost every $x \in E$, $(v^*av)(x,z) = 0$ whenever $z \neq x$. We may interpret this as saying that, for almost every $x \in E$, $a(y)$ and $\overline{v(y,x)}v(y,z)s(x,y,z)$ are orthogonal in $\ell^2(R(x))$ (where y ranges over $R(x)$). The resctictions of $a(y)$ to $R(x)$ for $a \in \mathbf{A}$ span $\ell^2(R(x))$. Therefore, since $s(x,y,z)$ is never 0, we conclude that:

$$\begin{aligned}&\text{For almost every}\, x \in E, \text{and for all}\, z \in R(x) \text{ with } z \neq x,\\ &\overline{v(y,x)}v(y,z) = 0 \text{ for all } y \in R(x). \text{ If } x \notin E, v(y,x) = 0.\end{aligned} \tag{2.2}$$

On the other hand $v^*v(x) = \chi(E)(x)$, and using $a = 1$ in (2.1) gives

$$\text{For almost every}\, x \in E, \sum_{y\sim x} |v(y,x)|^2 = 1. \tag{2.3}$$

Analogously, by considering also $v^* \in GN_{\mathbf{M}}(\mathbf{A})$, we can conclude that

$$\begin{aligned}&\text{For almost every} \quad x \in F \text{ and all} \quad z \in R(x) \text{ with } z \neq x,\\ &v(x,y)\overline{v(z,y)} = 0 \text{ for all} \quad y \in R(x). \text{ If}\, x \notin F, v(x,y) = 0.\end{aligned} \tag{2.4}$$

$$\text{For almost every} \quad x \in F, \sum_{y\sim x} |v(x,y)|^2 = 1. \tag{2.5}$$

Let $B = \{y \in X \mid v(y,x) \neq 0$ for more than one $x\}$. Consider the set $C = \{(y,x) \in R \mid v(y,x) \neq 0$ and $\exists\, z \in R(x), z \neq x$, with $v(y,z) \neq 0\}$. Then $\pi_\ell(C) = B$. Suppose that $\mu(\pi_r(C)) > 0$. Then there exists a nonnull subset D of X such that, for every $x \in D$, there are y and z in $R(x)$ with both $v(y,x)$ and $v(y,z)$ nonzero. But this contradicts (2.2), and hence $\mu(\pi_r(C)) = 0$. By [**4**, Proposition 2.1], we also have $\mu(\pi_\ell(C)) = \mu(B) = 0$. Combining this with (2.5) yields

$$\text{(2.6)} \qquad \begin{aligned} &\text{for almost every } x \in F, v(x,y) \neq 0 \text{ for exactly one} \\ &y \in R(X), \text{ and } |v(x,y)| = 1 \text{ for that } y. \end{aligned}$$

And, analogously, working from (2.4) and (2.3) (and interchanging roles of x and y),

$$\text{(2.7)} \qquad \begin{aligned} &\text{for almost every } y \in E, v(x,y) \neq 0 \text{ for exactly one} \\ &x \in R(y), \text{ and } |v(x,y)| = 1 \text{ for that } x. \end{aligned}$$

Because $v(x,y)$ can be modified on a null set without changing the operator v, we will assume v has been modified so that these results hold for all $x \in F$ and $y \in E$ respectively.

For $x \in E$, denote the unique y such that $v(y,x)$ is nonzero by $\psi(x)$. From (2.1) we then get

$$\text{(2.8)} \qquad (v^*av)(x) = \sum_{y \sim x} a(y)\,|v(y,x)|^2 = a(\psi(x))v^*v(x).$$

To show that ψ is a partial R-isomorphism, note that the map $a \mapsto v^*av$ is an algebraic isomorphism from $q\mathbf{A}$ to $p\mathbf{A}$, since, for $a, b \in q\mathbf{A}, v^*abv = v^*aqbv = v^*aqbv = (v^*av)(v^*bv)$. It then follows from [3, Appendix IV] that there is a Borel isomorphism η from F to E such that, for $x \in E, v^*av(x) = a(\eta(x))$. Therefore $\psi = \eta$, and by the definition of ψ we have $\Gamma(\psi) \subseteq R$, so ψ is a partial R-isomorphism. Now just take $\varphi = \psi^{-1}$.

The converse is straightforward and most of the details are omitted. If φ is a partial R-isomorphism, and if $v(x,y)$ is a measurable function on R satisfying conditions (i) and (ii) of this Proposition, then v is trivially left finite and so belongs to $\mathbf{M}$. Using the multiplication formulas one can show that v^*v and vv^* are respectively the projections in $\mathbf{A}$ corresponding to the domain and range of φ. Likewise $v^*av = (a \circ \varphi^{-1})v^*v$, so $v \in GN_{\mathbf{M}}(\mathbf{A})$. □

COROLLARY 2.3. *Let $u(x, y)$ be a complex-valued Borel function on R. Then u represents a unitary in $N_{\mathbf{M}}(\mathbf{A})$ if and only if there is an R-automorphism φ of X such that, for almost every (x, y) in R,*

(i) *$u(x, y) = 0$ unless $(x, y) \in \Gamma(\varphi)$.*

(ii) *$|u(x, y)| = 1$ for $(x, y) \in \Gamma(\varphi)$.*

The partial R-isomorphism associated with a partial isometry $v \in GN_{\mathbf{M}}(\mathbf{A})$ will be denoted φ_v. It follows directly from [**8**, Lemma 2.3(2)] that for $u, v \in GN_{\mathbf{M}}(\mathbf{A}), \varphi_{uv} = \varphi_v \circ \varphi_u$.

COROLLARY 2.4. *The Weyl group $W(\mathbf{A})$* of $\mathbf{A}$ *(defined to be $N_{\mathbf{M}}(\mathbf{A})/U(\mathbf{A})$)) is isomorphic to the group of* R-automorphisms of X. (See also [**5**, Proposition 2.9(3)].)

3. Topologies. We define two new topologies on $\mathbf{M}$ based on the Cartan subalgebra $\mathbf{A}$.

DEFINITION 3.1. The *Bures* $\mathbf{A}$*-topology* on $\mathbf{M}$ is defined to be the locally convex topology on $\mathbf{M}$ determined by the seminorms $T \mapsto \omega \circ E(T^*T)^{1/2}$ where ω runs over the normal states of $\mathbf{A}$ (normal positive linear functionals with $\omega(1) = 1$) [**2**, p. 48]. When the Cartan subalgebra $\mathbf{A}$ is understood, we will simply refer to the "Bures topology."

DEFINITION 3.2. The *relative L^2 topology* on $\mathbf{M}$ is defined to be the topology induced on $\mathbf{M}$ by its natural embedding into $L^2(R, \nu)$ via the Feldman-Moore formalism. For brevity this will usually be called the "L^2 topology."

REMARKS. If $\mathbf{M}$ is finite, $L^2(\mathbf{M})$ is naturally isomorphic to $L^2(R, \nu)$ as a Banach space, so in this case the terminology is consistent. In this case at least the relative L^2 topology does not depend on $\mathbf{A}$. Nevertheless the general question of whether these topologies depend on $\mathbf{A}$ may be interesting.

The unit ball of $\mathbf{M}$ will be denoted by $\mathbf{M}_1$, and, similarly for any subspace of $\mathbf{M}$, a subscript of 1 will indicate the unit ball of that subspace.

PROPOSITION 3.3. *The Bures topology is stronger than the L^2 topology on $\mathbf{M}$ and is equal to the L^2 topology on $\mathbf{M}_1$.*

PROOF. Since both the Bures topology and the L^2 topology are metrizable on the unit ball, we need only deal with sequences. (For the Bures topology we may use the metric $\rho(T_1, T_2) = \sum_{n=1}^{\infty} 2^{-n}\omega_n \circ E((T_1 - T_2)^*(T_1 - T_2))$, where $\{\omega_n\}$ is a countable dense set in the state space of $\mathbf{A}$.) Let $\{T_n\}$ be a sequence in $\mathbf{M}$ converging to 0 in the Bures topology. Then, for all states ω on $\mathbf{A}$, $\omega \circ E(T_n^* T_n)$ converges to 0. Using $\mathbf{A} \cong L^\infty(X, \mu)$ and $\mathbf{A}_* \cong L^1(X, \mu)$ with $\mu(X) = 1$, we have $E(T_n^* T_n)(x) = \sum_{y\sim x} |T_n(y, x)|^2$ and $\omega \circ E(T_n^* T_n) = \int_X \omega(x)\bigl(\sum_{y\sim x} |T_n(y, x)|^2\bigr) d\mu(x)$. Since $\mu(X) = 1$ we may take $\omega(x)$ identically equal to 1. Then $\omega \circ E(T_n^* T_n) = \int_X \sum_{y\sim x} |T_n(y, x)|^2 d\mu(x) = \int_R |T_n(x, y)|^2 dv(x, y) = \|T_n\|_2^2$. Hence $T_n \to 0$ in the L^2 topology.

To prove the converse on $\mathbf{M}_1$, we need an elementary lemma. Let $w(x) \in L^1(X, \mu)^+$ and let $\varepsilon > 0$ be given. Then we may choose $\delta > 0$ so that, whenever $g(x) \in L^\infty(X, \mu)$ with $g \geq 0$, $\|g\|_\infty \leq 1$, and $\|g\|_1 < \delta, \int_X g(x)w(x)d\mu(x) < \varepsilon$. To see this, let $S = \{x | w(x) > M\}$ for $M > 0$, and choose M so that $\int_S w(x)d\mu(x) < \varepsilon/2$ [**10**; p. 85, Proposition 13]. Choose $\delta = \varepsilon/2M$, and let g(x) be as given. Then $\int_X g(x)w(x)d\mu(x) = \int_S g(x)w(x)d\mu(x) + \int_{X\setminus S} g(x)w(x)d\mu(x) \leq \int_S w(x)d\mu(x) + M\int_{X\setminus S} g(x)d\mu(x) < \varepsilon/2 + M\delta = \varepsilon$.

Now suppose that $\{T_n\}$ is a sequence in $\mathbf{M}_1$ which converges to 0 in the L^2 topology. Then $\|E(T_n^* T_n)\| \leq 1$ so $\sum_{y\sim x} |T_n(y, x)|^2 \leq 1$ for *a.e.* x. We know that $\|T_n\|_2^2 = \int_X \sum_{y\sim x} |T_n(y, x)|^2 d\mu(x) \to 0$. Let $g_n(x) = \sum_{y\sim x} |T_n(y, x)|^2$. If ω is a state on $\mathbf{A}$ and $\varepsilon > 0$ is given, choose δ as above and choose N so that $n \geq N$ implies $\int_X g_n(x)d\mu(x) < \delta$. Then $\omega \circ E(T_n^* T_n) = \int_X \omega(x)g_n(x)d\mu(x) < \varepsilon$. Therefore $T_n \to 0$ in the Bures topology. □

PROPOSITION 3.4. *Let* $\mathbf{M}$ *be a von Neumann algebra with Cartan subalgebra* $\mathbf{A}$. *Then the following topologies are equivalent on the unit ball* $\mathbf{M}_1$ of $\mathbf{M}$:

(i) *the strong topology,*

(ii) *the* σ*-strong topology,*

(iii) *the relative* L^2*-topology,*

(iv) *the Bures* $\mathbf{A}$*-topology.*

PROOF. The equivalence of (i) and (ii) is a standard result [**13**, Lemma II.2.5], and we have just shown the equivalence of (iii) and (iv). Since $\omega \circ E \in \mathbf{M}_*$ whenever $\omega \in A_*$ the set of seminorms defining the Bures topology is a subset of those defining the σ-strong topology, and hence the Bures topology is weaker than the σ-strong topology on $\mathbf{M}$.

We now show that the relative L^2-topology is stronger than the strong topology on $\mathbf{M}_1$, which will complete the proof. By [**5**, Proposition 2.3], the left-finite functions are dense in $L^2(R,\nu)$. Since any (topological) subspace of a separable metric space is separable, we may choose a countable set of left finite functions $\{\xi_k\}$ which is dense in $L^2(R,\nu)$. If $T \in \mathbf{M}$ is represented by the function $T(x,y)$, then $T(x,y) \in L^2(R,\nu)$. By [**5**, Proposition 2.1] $L_T(\xi_k)(x,z) = R_{\xi_k}(T)(x,z)$, so $\|T\xi_k\| \leq C_k\|T\|_2$, where C_k is a constant depending on ξ_k. Let $U = \{T \in \mathbf{M}_1 \mid \|(T-T_0)\eta_i\| < \varepsilon_i\}$ be a basic open set in the strong topology on $\mathbf{M}_1$, where $\eta_i \in L^2(R,\nu)$ for $i = 1,\dots,N$. For each i choose ξ_i from the set $\{\xi_k\}$ so that $\|\xi_i - \eta_i\| < \varepsilon_i/4$. It then follows that $V = \{T \in \mathbf{M}_1 \mid \|(T-T_0)\xi_i\| < \varepsilon_i/2\}$ is a subset of U. Let $C = \max_{1\leq i\leq N}(C_i)$ and $\varepsilon = \min_{1\leq i\leq N}(\varepsilon_i)$; then $W = \{T \in \mathbf{M}_1 \mid \|T-T_0\|_2 < \varepsilon/2C\}$ is a subset of V and an L^2 neighborhood of T_0. □

COROLLARY 3.5. *For a subspace* $\mathbf{S}$ *of* $\mathbf{M}$ *the following are equivalent*:

(i) $\mathbf{S}$ ($\mathbf{S}_1$) *is closed in the Bures* $\mathbf{A}$*-topology*;

(ii) $\mathbf{S}$ ($\mathbf{S}_1$) *is closed in the relative* L^2 *topology*;

(iii) $\mathbf{S}$ ($\mathbf{S}_1$) *is closed in the* σ*-weak topology.*

PROOF. For any locally convex vector topology on $\mathbf{M}$ in which $\mathbf{M}_1$ is closed, a subspace $\mathbf{S}$ is closed if and only if $\mathbf{S}_1$ is closed. Since $\mathbf{M}_1$ is closed in each of these topologies, it suffices to prove the equivalence for $\mathbf{S}_1$. But $\mathbf{S}_1$ is σ-weakly closed if and only if it is σ-strongly closed by [**13**, Theorem II.2.6(vi)]. The equivalence then follows from Proposition 3.4. □

COROLLARY 3.6. *A linear functional on* $\mathbf{M}$ *is Bures continuous (or* L^2*-continuous) if and only if it is* σ*-weakly continuous. (A linear functional is continuous if and only if its null space is closed.)*

4. Orthonormal bases. Two elements $x, y \in \mathbf{M}$ are *orthogonal over* $\mathbf{A}$ if $E(x^*y) = 0$. A family $\{v_k\}$ of partial isometries in $\mathbf{M}$ will be called an *orthonormal basis over* $\mathbf{A}$ [**2**, p. 49] if (i) $\{v_k\}$ are mutually orthogonal over $\mathbf{A}$; (ii) the finite sums $\sum a_k v_k$ with $a_k \in \mathbf{A}$ are dense in $\mathbf{M}$ in the Bures $\mathbf{A}$-topology. We wish to show that (if $\mathbf{A}$ is a Cartan subalgebra) $\mathbf{M}$ always has an orthonormal basis over $\mathbf{A}$ of partial isometries in $GN_{\mathbf{M}}(\mathbf{A})$.

LEMMA 4.1. u, $v \in GN_{\mathbf{M}}(\mathbf{A})$ *are orthogonal over* $\mathbf{A}$ *if and only if* $\nu(\Gamma(\varphi_u) \cap \Gamma(\varphi_v)) = 0$.

PROOF. By (1.3), $E(u^*v)(x) = \sum_{y \sim x} \overline{u(y,x)} v(y,x)$. This equals zero for $x \in X$ if and only if $\varphi_u^{-1}(x) \neq \varphi_v^{-1}(x)$ or at least one of them is undefined, since then $u(y,x)$ and $v(y,x)$ will be nonzero in different terms. $E(u^*v) = 0$ if and only if this holds for almost every x, which is equivalent to $\nu(\Gamma(\varphi_u) \cap \Gamma(\varphi_v)) = 0$. □

Thus in order for the partial isometries $\{v_k\} \subset GN_{\mathbf{M}}(\mathbf{A})$ to form an orthonormal basis over $\mathbf{A}$ for $\mathbf{M}$, it is necessary that $\{\Gamma(\varphi_{v_k})\}$ be disjoint. By [**8**, Lemma 2.1], one can always find partial R-automorphisms $\{\varphi_k\}$ whose graphs form a partition of R. We will show below that if $\{\Gamma(\varphi_{v_k})\}$ form a partition of R up to null sets, then $\{v_k\}$ are an orthonormal basis (Theorem 4.4). If we then construct partial isometries v_k such that $\varphi_{v_k} = \varphi_k$ via Proposition 2.2, we will have such an orthonormal basis.

In the following, we assume $\{v_k\} \subset GN_{\mathbf{M}}(\mathbf{A})$ to be partial isometries with $\{\Gamma(\varphi_{v_k})\}$ a partition of R. Let $T \in \mathbf{M}$ be represented by the function $T(x,y)$ on R, and, for integers k and N, define $a_k = E(Tv_k^*), T_k = a_k v_k$, and $T_N = \sum_{k=1}^N T_k$.

Lemma 4.2. *$E(T_k^* T_m) = 0$ for $m \neq k$, and $T_k^* T_k \in \mathbf{A}$ is represented by the function $|a_k \circ \varphi_k^{-1}(x)|^2$.*

Proof. $E(T_k^* T_m) = E(v_k^* a_k^* a_m v_m) = (v_k^* a_k^* a_m v_k) E(v_k^* v_m) = 0$ if $m \neq k$. Since $T_k^* T_k = v_k^* a_k^* a_k v_k$, it is in $\mathbf{A}$ and (2.1) gives $T_k^* T_k(x) = \sum_{y \sim x} |a_k(y)|^2 |v_k(y,x)|^2 = |a_k \circ \varphi_k^{-1}(x)|^2$. □

Lemma 4.3.

(i) $E(T_N^* T_N) = E(T^* T_N) = E(T_N^* T)$, *and each is represented by the function* $\sum_{k=1}^N |a_k \circ \varphi_k^{-1}(x)|^2$.

(ii) $E(T^*T)$ *is represented by the function* $\sum_{k=1}^\infty |a_k \circ \varphi_k^{-1}(x)|^2$.

(iii) $E(T_N^* T_N)$ *converges* σ*-weakly to* $E(T^*T)$ in $\mathbf{A}$.

Proof.

(i)

$$E(T_N^* T_N) = E\big((\sum_{k=1}^N T_k)^* (\sum_{m=1}^N T_m)\big) = \sum_{k=1}^N \sum_{m=1}^N E(T_k^* T_m) = \sum_{k=1}^N T_k^* T_k,$$

$$\begin{aligned} E(T_N^* T) &= \sum_{k=1}^N E(v_k^* a_k^* T) = \sum_{k=1}^N v_k^* E(a_k^* T v_k^*) v_k \\ &= \sum_{k=1}^N v_k^* a_k^* E(T v_k^*) v_k = \sum_{k=1}^N T_k^* T_k, \\ E(T^* T_N) &= \sum_{k=1}^N E(T^* a_k v_k) = \sum_{k=1}^N v_k^* E(v_k T^*) a_k v_k \\ &= \sum_{k=1}^N v_k^* E(T v_k^*)^* a_k v_k = \sum_{k=1}^N T_k^* T_k. \end{aligned}$$

(ii) Since $a_k(x) = \sum_{y\sim x} T(x,y)\overline{v_k(x,y)}$,

$$\begin{aligned} |a_k(x)|^2 &= |\sum_{y\sim x} T(x,y)\overline{v_k(x,y)}|^2 \\ &= \sum_{y\sim x} |T(x,y)|^2 |v_k(x,y)|^2 \quad (v_k(x,y)\overline{v_k(x,z)} = 0 \text{ unless } z = y) \\ &= |T(x,\varphi_k(x))|^2. \end{aligned}$$

Therefore $|a_k \circ \varphi_k^{-1}(x)|^2 = |T(\varphi_k^{-1}(x),x)|^2$, and hence $\sum_{k=1}^{\infty} |a_k \circ \varphi_k^{-1}(x)|^2 = \sum_{k=1}^{\infty} |T(\varphi_k^{-1}(x),x)|^2 = \sum_{y\sim x} |T(y,x)|^2 = E(T^*T)(x)$.

(iii) From (i) and (ii), $E(T_N^* T_N)$ increases monotonically to $E(T^*T)$ as $N \to \infty$. If $\omega \in \mathbf{A}_*^+$ is represented by a function in $L^1(X,\mu)^+$, it is then a consequence of the Monotone Convergence Theorem on (X,μ) that $\omega \circ E(T_N^* T_N)$ converges to $\omega \circ E(T^*T)$. Since the positive functionals span $\mathbf{A}_*$, this holds for any $\omega \in \mathbf{A}_*$. □

THEOREM 4.4. *Let $\{v_k\}$ be a collection of partial isometries in $GN_{\mathbf{M}}(\mathbf{A})$ whose graphs form a partition of R, and let $T \in \mathbf{M}$. Then $\sum_{k=1}^{N} E(Tv_k^*)v_k$ converges to T in the Bures $\mathbf{A}$-topology.*

PROOF. Let $\omega \in \mathbf{A}_*$. Then, by Lemma 4.3(i), $\omega \circ E((T-T_N)^*(T-T_N)) = \omega \circ E(T^*T) - \omega \circ E(T_N^* T_N)$. By Lemma 4.3(iii), this converges to zero as $N \to \infty$. □

COROLLARY 4.5. *A collection of partial isometries in $GN_{\mathbf{M}}(\mathbf{A})$ is an orthonormal basis over $\mathbf{A}$ for $\mathbf{M}$ if and only if their graphs form a partition of R up to null sets.*

COROLLARY 4.6. *One can always choose an orthonormal basis over $\mathbf{A}$ for $\mathbf{M}$ from the members of $GN_{\mathbf{M}}(\mathbf{A})$, and any set in $GN_{\mathbf{M}}(\mathbf{A})$ which is orthogonal over $\mathbf{A}$ can be extended to an orthonormal basis over $\mathbf{A}$.* (This follows from Corollary 4.5 and the discussion following Lemma 4.1.)

QUESTIONS. (i) Popa [**9**, Proposition 2.2] has also proven this last result without the Feldman-Moore formalism. An even more interesting result of his [**9**, Corollary 2.6] is that if $\mathbf{M}$ is a Π_1 factor, there is an orthonormal basis over $\mathbf{A}$ consisting of unitaries in $N_{\mathbf{M}}(\mathbf{A})$. In the language of ergodic equivalence relations this says that if X has a finite R-invariant measure, then R has a partition into sets each of which is the graph of an R-automorphism of X. Is there a direct proof of this fact which is simpler than a line-for-line translation of Popa's proof?

(ii) In the case of the classical crossed product construction of a countable discrete group acting on an abelian von Neumann algebra $\mathbf{A}$, the unitaries corresponding to the group elements form an orthonormal basis for the crossed product algebra over $\mathbf{A}$ [**7**, Proposition 3]. Thus in many nonfinite cases it is still true that one can find an orthonormal basis over $\mathbf{A}$ consisting of unitaries in $N_{\mathbf{M}}(\mathbf{A})$. Is there an example where it is known that an orthonormal basis of unitaries cannot exist?

5. Bimodules. If $\mathbf{A}$ is a ring, then an $\mathbf{A}$-*bimodule* is a vector space on which left and right multiplication by members of $\mathbf{A}$ is defined, with the usual module laws holding. We restrict attention to the case where $\mathbf{A}$ is a Cartan subalgebra of a von Neumann algebra $\mathbf{M}$. In MSS bimodules are restricted to be subspaces of $\mathbf{M}$, but here we will consider also subspaces of $L^2(R,v)$. The Spectral Theorem for Bimodules [**8**, Theorem 2.5] states that any σ-weakly closed $\mathbf{A}$-bimodule in $\mathbf{M}$ is of the form $\mathcal{S}(B) = \{T \in \mathbf{M} \,|\, T(x,y) = 0 \text{ for } (x,y) \notin B\}$ for some Borel set $B \subset R$.

We are now in a position to give a proof of this Theorem that follows the motivation in §1 of MSS. As a matter of fact we only need the topological results of §3. Our strategy is:

(i) Let $\mathcal{S}$ be an $\mathbf{A}$-bimodule which is a closed subspace of $L^2(R,v)$. Then there is a Borel set $B \subset R$ such that $\mathcal{S} = \{f \in L^2(R,v) | f(x,y) = 0$ for $(x,y) \notin B\}$. This is pointed out in MSS [**8**, p. 15–16].

(ii) Any σ-weakly closed $\mathbf{A}$-bimodule $\mathcal{S}$ in $\mathbf{M}$ is in fact the intersection of $\mathbf{M}$ with an $\mathbf{A}$-bimodule which is a closed subspace of $L^2(R,v)$.

(iii) For an $\mathbf{A}$-bimodule $\mathcal{S}$ to be the intersection of $\mathbf{M}$ with a closed subspace of $L^2(R,v)$ is equivalent to $\mathcal{S} = \mathcal{S}^c \cap \mathbf{M}$, where $\mathcal{S}^c$ is the closure of $\mathcal{S}$ in $L^2(R,v)$. Note that if $\mathcal{S}$ is an $\mathbf{A}$-bimodule then so is $\mathcal{S}^c$, since multiplication by a fixed element of $\mathbf{A}$ is continuous in the topology of $L^2(R,v)$.

THEOREM 5.1. (SPECTRAL THEOREM FOR BIMODULES) *Let* $\mathcal{S}$ *be a* σ-weakly closed $\mathbf{A}$-*bimodule in* $\mathbf{M}$. *Then there is a Borel set* $B \subseteq R$ *such that* $\mathcal{S} = \mathcal{S}(B) = \{T \in \mathbf{M} \mid T(x,y) = 0$ *for* $(x,y) \notin B\}$.

PROOF. Since $\mathcal{S}$ is σ-weakly closed, it is also closed in the relative L^2 topology by Corollary 3.5. Therefore, by (iii) above, $\mathcal{S} = \mathcal{S}^c \cap M$ (thereby verifying (ii)). By (i) above, we have $\mathcal{S} = \{T \in L^2(R,v) \mid T(x,y) = 0$ for $(x,y) \notin B\}$ for some Borel set B, and we are done. □

The remainder of this section is devoted to showing that a σ-weakly closed $\mathbf{A}$-bimodule has an orthonormal basis of partial isometries in $GN_{\mathbf{M}}(\mathbf{A})$. This generalizes results of Popa [**9**, Proposition 2.2] (given as Corollary 4.6 in this paper) and Muhly-Saito-Solel [**8**, Corollary 2.7]. The crucial first step is the following Lemma, which is proven in MSS [**8**, Lemma 2.6].

LEMMA 5.2. *Let* $\mathcal{S}$ *be a* σ-*weakly closed* $\mathbf{A}$-*bimodule in* $\mathbf{M}$, *and let* $v \in GN_{\mathbf{M}}(\mathbf{A})$. *If* $T \in \mathcal{S}$ *then also* $E(Tv^*)v \in \mathcal{S}$.

A simple calculation shows that $E(\mathcal{S}v^*) = \{E(Tv^*) \mid T \in \mathcal{S}\}$ is a two sided ideal in $\mathbf{A}$. If $\mathcal{S}$ is σ-weakly closed, then so is $E(\mathcal{S}v^*)$. In particular, let $\{f_n\}$ be a sequence in $E(\mathcal{S}v^*)$ converging σ-weakly to f. Then $f_n v \in \mathcal{S}$ for all n and $f_n v$ converges σ-weakly to fv, so also $fv \in \mathcal{S}$. Let $q = vv^*$; since $f_n = E(T_n v^*)$ for some $T_n \in \mathcal{S}, f_n q = E(T_n v^* q) = E(T_n v^*) = f_n$. By continuity we also have $fq = f$. Then $E((fv)v^*) = E(fq) = E(f) = f$, and hence $f \in E(\mathcal{S}v^*)$. Since $E(\mathcal{S}v^*)$ is σ-weakly closed, $E(\mathcal{S}v^*) = \mathbf{A}r$ for some projection $r \in \mathbf{A}$ [**11**,1.10.5].

LEMMA 5.3 . *Let* $\mathcal{S}$ *be a nonzero* σ-*weakly closed* $\mathbf{A}$-*bimodule in* $\mathbf{M}$. *Then* $\mathcal{S}$ *contains a nonzero partial isometry in* $GN_{\mathbf{M}}(\mathbf{A})$.

PROOF. Let $\{v_k\}$ be an orthonormal basis for $\mathbf{M}$ of partial isometries in $GN_{\mathbf{M}}(\mathbf{A})$. Then $\mathcal{J}_k = E(\mathcal{S}v_k^*) = \mathbf{A}r_k$ for projections r_k in $\mathbf{A}$. In particular $r_k \in \mathcal{J}_k$, and therefore, by Lemma 5.2, $r_k v_k \in \mathcal{S}$. For some k, $r_k v_k$ must be nonzero, as otherwise, for each T in $\mathcal{S}$, $T =$

$\sum_{k=1}^{\infty} E(Tv_k^*)v_k = \sum_{k=1}^{\infty} E(Tv_k^*)r_k v_k = 0$. But each nonzero $r_k v_k$ is easily checked to be in $GN_{\mathbf{M}}(\mathbf{A})$. $\square$

PROPOSITION 5.4. *Let $\mathcal{S}$ be a nonzero σ-weakly closed $\mathbf{A}$-bimodule in $\mathbf{M}$. Then $\mathcal{S}$ has an orthonormal basis consisting of partial isometries in $GN_{\mathbf{M}}(\mathbf{A})$. Furthermore, any orthonormal set in $\mathcal{S}$ chosen from $GN_{\mathbf{M}}(\mathbf{A})$ can be extended to an orthonormal basis.*

PROOF. Let $\{v_i\}$ be a maximal family of partial isometries in $GN_{\mathbf{M}}(\mathbf{A})$ chosen from $\mathcal{S}$ which are mutually orthogonal. Then the subspaces $\mathbf{A}v_i$ are mutually orthogonal. Suppose that $\mathcal{S} \not\subset \oplus \mathbf{A}v_i$, and let $\mathcal{R} = \{T \in \mathcal{S} \,|\, T \perp \mathbf{A}v_i \text{ for each } i\}$. We claim that $\mathcal{R}$ is again a nonzero σ-weakly closed $\mathbf{A}$-bimodule, and hence contains a member of $GN_{\mathbf{M}}(\mathbf{A})$, leading to a contradiction. To complete the proof we prove this claim.

If $\{T_\alpha\}$ is a net in $\mathcal{R}$ converging σ-weakly to T, then T is in $\mathcal{S}$ since $\mathcal{S}$ is σ-weakly closed. Since $E(T_\alpha^* a v_i) = 0$ for every i, α, and every $a \in \mathbf{A}$, and since E is σ-weakly continuous, we may take the limit in α to conclude that $E(T^* a v_i) = 0$ and hence $T \in \mathcal{R}$. For any $a \in \mathbf{A}$, aT and Ta are in $\mathcal{S}$, and we have $E((aT)^* \mathbf{A}v_i) = E(T^* a^* \mathbf{A}v_i) = E(T^* \mathbf{A}v_i) = 0$ and $E((Ta)^* \mathbf{A}v_i) = a^* E(T^* \mathbf{A}v_i) = 0$, so aT and Ta are also in $\mathcal{R}$. To see that $\mathcal{R}$ is nonzero, let $T \in \mathcal{S}$ with $T \notin \oplus \mathbf{A}v_i$, and let $\{w_i\}$ be an orthogonal set in $GN_{\mathbf{M}}(\mathbf{A})$ which extends $\{v_i\}$ to form an orthonormal basis over $\mathbf{A}$. Then, for at least one of the w_i, we must have $E(Tw_i^*) \neq 0$. But then, by Lemma 5.2, $E(Tw_i^*)w_i \in \mathcal{S}$, and hence $E(Tw_i^*)w_i \in \mathcal{R}$. $\square$

COROLLARY 5.5. *Let $\mathcal{S}$ be a nonzero σ-weakly closed $\mathbf{A}$-bimodule in $\mathbf{M}$. Let $\mathcal{S}_F$ be the subspace of left-finite elements of $\mathcal{S}$. Then $\mathcal{S}_F$ is σ-weakly dense in $\mathcal{S}$.*

PROOF. By Proposition 5.4 we know that $\mathcal{S}_F$ is Bures dense in $\mathcal{S}$, since finite linear combinations of elements of $GN_M(\mathbf{A})$ are left finite. By Corollary 3.5, the σ-weak closure of $\mathcal{S}_F$ is equal to the Bures closure of $\mathcal{S}_F$, which is $\mathcal{S}$. $\square$

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REFERENCES

1. W. Arveson, *Operator algebras and measure preserving automorphisms,* Acta Math. **118** (1967), 95–109.

2. D. Bures, *Abelian subalgebras of von Neumann algebras,* Mem. Amer. Math. Soc., **110** (1971).

3. J. Dixmier, *Von Neumann algebras,* North-Holland, Amsterdam, 1981.

4. J. Feldman and C.C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras* I, Trans. Amer. Math. Soc. **234** (1977), 289–324.

5. ——— and ———, *Ergodic Equivalence Relations, Cohomology, and von Neumann Algebras* II, Trans. Amer. Math. Soc. **234** (1977) 325–359.

6. R. Kadison, I.M. Singer, *Triangular operator algebras,* Amer. J. Math. **82** (1960), 227–259.

7. R. Mercer, *Convergence of Fourier series in discrete crossed products of von Neumann algebras,* Proc. Amer. Math. Soc. **94** (1985), 254–257.

8. P. Muhly, K. Saito, B. Solel, *Coordinates for triangular operator algebras,* Ann. of Math. (2) **127** (1988), 245–278.

9. S. Popa, *Notes on Cartan subalgebras in type* II_1 *factors,* Math. Scand. **57** (1985), 171–188.

10. H. Royden, *Real analysis,* 2nd Ed., MacMillan, New York, 1968.

11. S. Sakai, *C^*-Algebras and W^*-algebras*, Springer, New York, 1971.

12. S. Stratila, *Modular theory in operator algebras,* Abacus Press, Tunbridge Wells, England, 1981.

13. M. Takesaki, *Theory of operator algebras* I, Springer, New York, 1979.

14. J. Tomiyama, *On the projections of norm one in W^*-algebras,* Proc. Jap. Acad. **33** (1957), 608–612.

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