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ALGEBRA IS EVERYWHERE

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ABSTRACT. To various degrees, the invertibility or singularity of an operator between two different spaces can be reduced to that of a normed algebra element.

If an n-tuple $a \in A^n$ of normed algebra elements can be represented as a bounded linear operator $\mathrm{row}(L_a) : A^n \to A$ between normed spaces, and also as a bounded linear operator $\mathrm{col}(L_a) : A \to A^n$, then it is only fair that we should try to represent a bounded linear operator $T : X \to Y$ between different normed spaces by a system of normed algebra elements. In this note we see how various degrees of "invertibility" and "non-singularity" for $T \in \mathrm{BL}(X, Y)$ can be expressed in terms of the same thing for a related single element of the normed algebra $\mathrm{BL}(X \times Y, X \times Y)$ of operators on the cartesian product space, which we shall write in the form of column vectors:

$$\mathrm{BL}\left(\begin{pmatrix} X \\ Y \end{pmatrix}, \begin{pmatrix} X \\ Y \end{pmatrix}\right) = \begin{pmatrix} \mathrm{BL}(X,X) & \mathrm{BL}(Y,X) \\ \mathrm{BL}(X,Y) & \mathrm{BL}(Y,Y) \end{pmatrix}. \tag{0.1}$$

We begin by looking at "generalized inverses": we say [**3, 4**] that $T \in \mathrm{BL}(X, Y)$ is *regular*, or *relatively Fredholm*, if there is $T' \in \mathrm{BL}(Y, X)$ for which

$$T = TT'T, \tag{0.2}$$

and that $T \in \mathrm{BL}(X, Y)$ is *decomposably regular*, or *relatively Weyl*, if (0.2) can be arranged with invertible T'. Specializing to the case $Y = X$ and then generalizing, we shall say that an element $a \in A$ of a normed algebra A, or more generally an additive category A, is "regular" if

$$a \in aAa, \tag{0.3}$$

and "decomposably regular" if

$$a \in aA^{-1}a, \tag{0.4}$$

where we write A^{-1} for the group (or groupoid) of invertible elements of A.

THEOREM 1. *If X and Y are normed spaces and $T \in \mathrm{BL}(X,Y)$ is a bounded linear operator, then there is equivalence*

$$T \textit{ regular } \Leftrightarrow \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \textit{ regular.} \tag{1.1}$$

and one-way implication

$$T \textit{ decomposably regular } \Rightarrow \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \textit{ decomposably regular.} \tag{1.2}$$

PROOF. For (1.1) we have the implications

$$\begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \begin{pmatrix} U & T' \\ W & V \end{pmatrix} \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \Rightarrow T = TT'T$$
$$\Rightarrow \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \begin{pmatrix} 0 & T' \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}.$$

For (1.2) suppose that $T = TT'T$ with $T''T' = I = T'T''$. Then

$$\begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \begin{pmatrix} 0 & T' \\ T'' & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}$$
$$\text{with } \begin{pmatrix} 0 & T' \\ T'' & 0 \end{pmatrix} \begin{pmatrix} 0 & T' \\ T'' & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \ \square$$

The implication (1.2) cannot in general be reversed: for if X and Y are finite dimensional and $T \in \mathrm{BL}(X,Y)$ is arbitrary, then (consider the index)

$$T \textit{ decomposably regular } \Leftrightarrow \dim(X) = \dim(Y). \tag{1.3}$$

Thus if X and Y are of finite but unequal dimensions the operator $\left(\begin{smallmatrix} 0 & 0 \\ T & 0 \end{smallmatrix}\right)$ will be decomposably regular, but not the operator T.

In general, if it is possible for an operator $T \in \mathrm{BL}(X,Y)$ to be decomposably regular, then the normed spaces must be isomorphic, so that there exists an invertible operator $S \in \mathrm{BL}(Y,X)$. To test for the decomposable regularity of $T \in \mathrm{BL}(X,Y)$ we look instead at the operator $\begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}$ on $X \times Y$:

THEOREM 2. *If $T \in \mathrm{BL}(X,Y)$ is arbitrary and $S \in \mathrm{BL}(Y,X)$ is invertible then there is equivalence*

$$T \text{ regular } \Leftrightarrow \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix} \text{ regular} \tag{2.1}$$

and equivalence

$$T \text{ decomposably regular } \Leftrightarrow \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix} \text{ decomposably regular.} \tag{2.2}$$

PROOF. For (2.1) we have implications

$$\begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix} = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}\begin{pmatrix} U & T' \\ W & V \end{pmatrix}\begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix} \Rightarrow T = TT'T$$
$$\Rightarrow \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix} = \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}\begin{pmatrix} 0 & T' \\ S' & 0 \end{pmatrix}\begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix},$$

provided $S = SS'S$, and, in particular, if $S' = S^{-1}$ is the inverse of S. This also gives forward implication in (2.2), since if $S' = S^{-1}$ then

$$T'' = (T')^{-1} \Rightarrow \begin{pmatrix} 0 & S \\ T'' & 0 \end{pmatrix} = \begin{pmatrix} 0 & T' \\ S' & 0 \end{pmatrix}^{-1}.$$

Conversely if $\begin{pmatrix} U & T' \\ W & V \end{pmatrix}$ is a generalized inverse of $\begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}$ then $S = SWS$ and $TUS = 0 = SVT$, so that if S is invertible, then

$$W = S' = S^{-1} \quad \text{and} \quad TU = 0 = VT. \tag{2.3}$$

At the same time there is implication [**1**, Problem 71]

$$\begin{pmatrix} U & T' \\ S' & V \end{pmatrix} \text{ invertible } \Rightarrow \begin{pmatrix} I & SV \\ S'U & S'T' \end{pmatrix} \text{ invertible} \tag{2.4}$$
$$\Rightarrow T' - USV \text{ invertible.}$$

By (2.3) the invertible operator $T' - USV$ is another generalized inverse for T. □

It is profitable to review the arguments [**5**] for (2.4), which remain valid in categories more general than BL [**3**]. For the first implication we observe

$$\begin{pmatrix} I & SV \\ S'U & S'T' \end{pmatrix} = \begin{pmatrix} 0 & S \\ S' & 0 \end{pmatrix} \begin{pmatrix} U & T' \\ S' & V \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} 0 & S \\ S' & 0 \end{pmatrix} = \begin{pmatrix} 0 & S \\ S' & 0 \end{pmatrix}^{-1}, \tag{2.5}$$

while, for the second

$$\begin{pmatrix} I & SV \\ 0 & S'(T' - USV) \end{pmatrix} = \begin{pmatrix} I & 0 \\ -S'U & I \end{pmatrix} \begin{pmatrix} I & SV \\ S'U & S'T' \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} I & 0 \\ S'U & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ -S'U & I \end{pmatrix}^{-1} \tag{2.6}$$

The derivation of (2.3) is also valid in more general categories. From its proof it is clear that the equivalence (2.1) holds whenever S is regular; the same argument shows that (2.2) remains valid if "decomposably regular" is replaced by "invertible": if $T \in \mathrm{BL}(X,Y)$ and if $S \in \mathrm{BL}(Y,X)$ is invertible, then

$$T \text{ invertible } \Leftrightarrow \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix} \text{ invertible.} \tag{2.7}$$

It is clear that (2.7) holds in more general categories than BL; thus, by Atkinson's theorem,

$$T \text{ Fredholm } \Leftrightarrow \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix} \text{ Fredholm.} \tag{2.8}$$

More subtle is that (2.8) holds with "Fredholm" replaced by "Weyl," in the sense of having an invertible essential inverse:

THEOREM 3. *If $T \in \mathrm{BL}(X,Y)$ is arbitrary and $S \in \mathrm{BL}(Y,X)$ is Weyl then there is the equivalence*

$$T \textit{ Weyl } \Leftrightarrow \begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix} \textit{ Weyl.} \tag{3.1}$$

PROOF. If $T' \in \mathrm{BL}\,(Y, X)$ is an essential inverse for T which has an inverse $T'' \in \mathrm{BL}\,(X, Y)$ then

$$T''T' = I = T'T'' \quad \text{and} \quad I - T'T, I - TT' \quad \text{are finite rank,} \tag{3.2}$$

and if also S' is an invertible essential inverse for S, then, by (2.7), $\begin{pmatrix} 0 & T' \\ S' & 0 \end{pmatrix}$ is an invertible essential inverse for $\begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}$. Conversely if $\begin{pmatrix} U & T' \\ W & V \end{pmatrix}$ is an invertible essential inverse for $\begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}$ then, by the proof of Theorem 2, first

$$W - S', TU, VT \quad \text{are finite rank} \tag{3.3}$$

and second

$$T' - USV \quad \text{is invertible,} \tag{3.4}$$

so that $T' - USV$ is an invertible essential inverse for T. □

For bounded operators (rather than for more general categories) we can generalize (3.1), and supplement (2.8): if $T \in \mathrm{BL}\,(X, Y)$ and $S \in \mathrm{BL}\,(Y, X)$ are both Fredholm, then

$$\text{index}\left(\begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}\right) = \text{index}(T) + \text{index}(S). \tag{3.5}$$

This is familiar with the direct sum $\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} : \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} Y \\ X \end{pmatrix}$ in place of $\begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}$: but now

$$\begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} \text{ with } \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}^{-1}. \tag{3.6}$$

We have been unable to show, as might be suggested by Theorem 3, that (2.2) extends to decomposably regular S: this would imply in particular two way implication in (1.2) when the spaces X and Y are isomorphic.

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