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A CLASSIFICATION OF SOME NONCOMMUTATIVE TORI

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ABSTRACT. A detailed description of the isomorphism classes of rational nocommutative tori is given using a classification of rational antisymmetric bicharacters on $\mathbf{Z}^d$. A canonical form for such a torus is presented. An illustration of the extent to which K_0 can fail to distinguish the isomorphism classes of these tori is also given.

Isomorphism classes of the canonical smooth subalgebras of the C^*-algebras associated with an arbitrary antisymmetric bicharacter ρ on $\mathbf{Z}^3$ are in a one-to-one correspondence with the isomorphism classes of ρ. The same is true for the C^*-algebras themselves except in some cases where the possibility exists that (at most) two different bicharacters on $\mathbf{Z}^3$ may yield isomorphic C^*-algebras.

1. In the following G denotes the abelian group $\mathbf{Z}^d, d \in \mathbf{N}, d \geq 2$. For $a, b \in \mathbf{Z}$ let (a) denote the ideal generated by a, write $a|b$ if $(b) \subseteq (a)$ and write (a, b) for the greatest common divisor of a and b. The quotient ring $\mathbf{Z}/(a)$ is denoted $\mathbf{Z}_a$. Let T denote the group $\{z \in \mathbf{C} \mid |z| = 1\}$. If $f : X \to T$ is a map, $\overline{f} : X \to T$ is the map defined by $\overline{f}(x) = \overline{f(x)}$ $(x \in X)$. If A is an abelian group, the (group of) characters of $A, \mathrm{Hom}(A, T)$ is also written $\hat{A}$.

To each $\rho \in \mathrm{Hom}(G \wedge G, T)$ associate a C^*-algebra A_ρ, the universal C^*-algebra generated by d unitaries $u_1, \ldots, u_d$ subject to the relations $u_j u_i = \rho(e_i \wedge e_j) u_i u_j$ ($\{e_1, \ldots, e_d\}$ the standard basis of G). Denote by A_ρ^∞ the dense subalgebra of smooth elements with respect to the canonical action of T^d (see [**3**] and references therein). The algebra A_ρ has a trace τ which induces a homomorphism $\tau_* : K_0(A_\rho) \to \mathbf{R}$. This homomorphism does not depend on the trace chosen [**5**]. If ρ is rational, that is, range of ρ is a finite subgroup of T, call A_ρ a rational d-torus. For $c \in [0, 1]$ and $\rho : \Lambda^2 \mathbf{Z}^2 \to T$ defined by $\rho(e_1 \wedge e_2) = \exp(2\pi i c)$, denote the C^*-algebra A_ρ by A_c, the usual rotation algebra [**7, 8**].

Given an abelian group H, two elements h_1, h_2 of $\mathrm{Hom}(G \wedge G, H)$ are isomorphic or congruent (written $h_1 \simeq h_2$) if there is an $\alpha \in \mathrm{Aut}(G)$

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with $h_1(\alpha \wedge \alpha) = h_2$. The C^*-algebras associated with congruent characters of $\Lambda^2 G$ are isomorphic via an isomorphism restricting to an isomorphism of their canonical smooth subalgebras.

If the rank of G is two or if ρ is rational, isomorphic C^*-algebras give rise to congruent characters [**7, 8, 4**]. In the rational case more is actually true, namely, a certain functor is invertible [**4, 2**]. In each of the following three cases isomorphic smooth subalgebras yield isomorphic characters: rank$G$ is three and ρ is nondegenerate; rank$G$ is four and ρ is injective; rank$G$ is arbitrary and ρ has generic Diophantine properties [**2, 3**].

Let ρ be a character of $G \wedge G$. Since G is finitely generated, range(ρ) is isomorphic to $T \oplus F_r$, with T a finitely generated torsion subgroup of T and F_r a free rank r subgroup of $T, r \in \mathbf{N}_0$. Fix such an identification for each isomorphism class of ρ and refer to F_r as the free part of range(ρ). There is an $n \in \mathbf{N}$ with $T = \{\exp(2\pi i n^{-1} p) \mid p \in \mathbf{Z}\} \simeq \mathbf{Z}_n$. Call r the rank of ρ ($=$ rank(ρ)) and n the torsion order of ρ ($=$ ord(ρ)). Note rank(ρ) $= 0$ if and only if ρ^n is the trivial character for $n = \mathrm{ord}(\rho)$ if and only if ρ is rational.

Since $G \wedge G$ is free, there is a homomorphism $\phi : G \wedge G \to \mathbf{R}$ with $\rho = \exp(2\pi i \phi)$. If $\phi' \in \mathrm{Hom}(G \wedge G, \mathbf{R})$ and $\rho' = \exp(2\pi i \phi')$, then $\rho \simeq \rho'$ if and only if there is an $\alpha \in \mathrm{Aut}(G)$ with $\phi(\alpha \wedge \alpha) = \phi' \mod \mathbf{Z}$. If $\{\exp(2\pi i \theta_j) \mid j = 1, \dots, r\}$ is a basis of F_r and $\theta_0 = n^{-1}$, there are $\varphi_i, \varphi_i' \in \mathrm{Hom}(G \wedge G, \mathbf{Z})$, $i = 0, 1, \dots, r$, with $\rho = \exp(2\pi i(\theta_0 \varphi_0 + \cdots + \theta_r \varphi_r))$ and $\rho' = \exp(2\pi i(\theta_0 \varphi_0' + \cdots + \theta_r \varphi_r'))$. It follows that $\rho \simeq \rho'$ if and only if there is an $\alpha \in \mathrm{Aut}(G)$ with $\varphi_i(\alpha \wedge \alpha) = \varphi_i'$, $i = 1, \dots, r$, and $\pi\varphi_0(\alpha \wedge \alpha) = \pi\varphi_0'$, where $\pi : \mathbf{Z} \to \mathbf{Z}_n$ is the canonical quotient map.

Thus the isomorphism class of a character is determined by the simultaneous isomorphism class of certain finite sets $\{\pi\varphi_0, \varphi_1, \dots, \varphi_r\}$ with $\varphi_i \in \mathrm{Hom}(G \wedge G, \mathbf{Z})$ and $\pi\varphi_0 \in \mathrm{Hom}(G \wedge G, \mathbf{Z}_n)$. The isomorphism class of a single element of $\mathrm{Hom}(G \wedge G, \mathbf{Z})$ or $\mathrm{Hom}(G \wedge G, \mathbf{Z}_n)$ is known ([**6**] and [**1**] respectively).

2. Given $\rho \in \mathrm{Hom}(G \wedge G, T)$, ρ is rational if and only if there is a $\varphi \in \mathrm{Hom}(G \wedge G, \mathbf{Q})$ with $\rho = \exp(2\pi i \varphi)$. Let $L(\rho) = \{g \in G \mid \rho(g \wedge G) = 1\} = \{g \in G \mid \varphi(g \wedge G) \in \mathbf{Z}\}$, a rank d free submodule of G. The invariants of the finitely generated torsion $\mathbf{Z}$-module $G/L(\rho)$

completely describe the isomorphism class of $G/L(\rho)$ and are given by a sequence of decreasing ideals of $\mathbf{Z}$.

The sequence of invariants is of the form $(p_{2s}) = (p_{2s-1}) \supseteq \cdots \supseteq (p_2) = (p_1)$. To see this, let (n) denote the smallest ideal occuring as an invariant. Then $n = \operatorname{ord}(\rho), n\varphi \in \operatorname{Hom}(G \wedge G, \mathbf{Z})$ and there is an $\alpha \in \operatorname{Aut}(G), a_1, \ldots, a_k \in \mathbf{N}_0$ with $a_1|a_2|\cdots|a_k, 2k = d$ or $2k = d-1$ and $n\varphi(\alpha \wedge \alpha) = \sum_{i=1}^k a_i e_{2i-1} \wedge e_{2i}$ [**6**]. If $\delta_i = (a_i, n)$, we have $\delta_1 = 1$ (since range(ρ) is $\mathbf{Z}_n$) and $\delta_i|\delta_{i+1}, 1 \leq i \leq k-1$. Let $n_j = n\delta_j^{-1}$ and $s = \max\{j \mid \delta_j < n\}$. Then

$$G/L(\rho) = \mathbf{Z}_{n_1} \times \mathbf{Z}_{n_1} \times \cdots \times \mathbf{Z}_{n_s} \times \mathbf{Z}_{n_s}.$$

If $d = 2k+1$ define $C(\rho) = 0$. If $d = 2k$ let $C(\rho)$ be the element of $\mathbf{Z}_n$ given by $C(\pi(n\varphi)) = \pi(\operatorname{Pf}(n\varphi)\delta_1^{-1}\cdots\delta_{k-1}^{-1})$ where $\pi : \mathbf{Z} \to \mathbf{Z}_n$ [**1**]. Since $\pm\operatorname{Pf}(n\varphi) = \operatorname{Pf}(n\varphi(\alpha\wedge\alpha)) = a_1\cdots a_k$ and $(a_j\delta_j^{-1}, n_j) = 1$ we have $(a_j\delta_j^{-1}, n_k) = 1, (\operatorname{Pf}(n\varphi)\delta_1^{-1}\cdots\delta_k^{-1}, n_k) = 1$ and $(\operatorname{Pf}(n\varphi)\delta_1^{-1}\cdots\delta_{k-1}^{-1}, n) = \delta_k = n_1 n_k^{-1}$. In other words, if a is a representative of $C(\rho)$ then $(a, n) = n_1 n_k^{-1}$. Note that $C(\rho)$ may be rewritten as $\pi(\operatorname{Pf}(\varphi)n^k\delta_1^{-1}\cdots\delta_k^{-1}\delta_k) = \pi(nn_k^{-1}\operatorname{Pf}(\varphi)\prod_{j=1}^k n_j)$. Thus, once n is known, we need only compute the class of $n_k^{-1}\operatorname{Pf}(\varphi)\Pi_{j=1}^k n_j$ mod $\mathbf{Z}$, which involves knowing n_k and the product of all the n_j, not necessarily each n_j. This will be useful in the proof of Corollary 3.

It is clear from §1 that the isomorphism class of ρ is the same as the isomorphism class of $\pi(n\varphi)$. The following theorem is an immediate consequence of results in [**1**] and [**4**].

THEOREM 1. *Assume $\rho, \rho' \in \operatorname{Hom}(G \wedge G, T)$ are rational. Then A_ρ is isomorphic to $A_{\rho'}$ if and only if $G/L(\rho)$ and $G/L(\rho')$ are isomorphic and $C(\rho) = \pm C(\rho')$.*

COROLLARY 2. *Given A_ρ with ρ a rational character of $\Lambda^2\mathbf{Z}^d$, there is a unique sequence of positive integers $n_1, \ldots, n_s$ with $2s \leq d, n_1 = \operatorname{ord}(\rho), 1 < n_s|\cdots|n_1$, and a unique integer $p, 0 < p \leq n_1/2$ with $n_1 n_s^{-1} = (p, n)$ if $2s = d$; $p = n_1 n_s^{-1}$ otherwise, such that A_ρ is isomorphic to $A_{n_1^{-1}} \otimes A_{n_2^{-1}} \otimes \cdots \otimes A_{n_{s-1}^{-1}} \otimes A_{pn_1^{-1}} \otimes C(T^{d-2s})$.*

PROOF. Let $n = \operatorname{ord}(\rho)$. We have shown that there is a unique sequence $n_1, \dots, n_s$ of positive integers such that the invariants of $G/L(\rho)$ are $(n_s) = (n_s) \supseteq \cdots \supseteq (n_1) = (n_1)$ with $n_1 = n$. Define $p = \delta_s$ if $2s < d$. If $2s = d$, let p be the unique integer with $0 < p \leq n_1/2$ and $\pi(p) = \pm C(\rho)$. Note $(p, n) = n_1 n_s^{-1}$. Define $\rho' \in \operatorname{Hom}(G \wedge G, T)$ to be $\exp(2\pi i \varphi')$ with

$$\varphi' = n_1^{-1} e_1 \wedge e_2 + \cdots + n_{s-1}^{-1} e_{2s-3} \wedge e_{2s-2} + p n_1^{-1} e_{2s-1} \wedge e_{2s}.$$

Then $G/L(\rho')$ and $G/L(\rho)$ are isomorphic and $C(\rho) = \pm C(\rho')$. Theorem 1 implies A_ρ and $A_{\rho'}$ are isomorphic. Finally, $A_{\rho'}$ is isomorphic to the above tensor product of rotation algebras. □

COROLLARY 3. *Let* $\mathcal{A}_1 = \otimes_1^\ell A_{s_i t_i^{-1}} \otimes C(T^b)$ *and* $\mathcal{A}_2 = \otimes_1^k A_{q_i r_i^{-1}} \otimes C(T^c)$ *with integers* $0 \leq s_i \leq t_i, 0 \leq q_i \leq r_i, 1 = (s_i, t_i) = (q_i, r_i)$ *and* $b, c \in \{0, 1\}$. *Then* $\mathcal{A}_1$ *is isomorphic to* $\mathcal{A}_2$ *if and only if*

(i) $2\ell + b = 2k + c \ (= d)$.

(ii)

$$\mathbf{Z}_{t_1} \times \mathbf{Z}_{t_1} \times \cdots \times \mathbf{Z}_{t_\ell} \times \mathbf{Z}_{t_\ell} \simeq \mathbf{Z}_{r_1} \times \mathbf{Z}_{r_1} \times \cdots \times \mathbf{Z}_{r_k} \times \mathbf{Z}_{r_k}.$$

(iii) If d is even (so $b = c = 0$), then

$$\prod_{i=1}^{k} s_i (t_1, t_2, \dots, t_k)^{-1} = \pm \prod_{i=1}^{k} q_i (r_1, \dots, r_k)^{-1} \mod \mathbf{Z}.$$

PROOF. If $\varphi_1 \in \operatorname{Hom}(\Lambda^2 \mathbf{Z}^{2\ell+b}, \mathbf{Q})$ and $\varphi_2 \in \operatorname{Hom}(\Lambda^2 \mathbf{Z}^{2k+c}, \mathbf{Q})$ are defined as $\varphi_1 = \sum^\ell s_i t_i^{-1} e_{2i-1} \wedge e_{2i}, \varphi_2 = \sum^k q_i r_i^{-1} e_{2i-1} \wedge e_{2i}$, then $\mathcal{A}_j \simeq A_{\rho_j}$ for $\rho_j = \exp(2\pi i \varphi_j)$, $j = 1, 2$. Thus $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic if and only if ρ_1 is isomorphic to ρ_2. This is equivalent to $2\ell + b = 2k + c \ (= d), \mathbf{Z}^d/L(\rho_1)$ and $\mathbf{Z}^d/L(\rho_2)$ are isomorphic (a restatement of condition (ii)) and $C(\rho_1) = \pm C(\rho_2)$. Since $C(\rho_1) = C(\rho_2) = 0$ unless d is even, we need only consider the case when d is even, in other words, when $b = c = 0$. To show $C(\rho_1) = \pm C(\rho_2)$, it suffices to show

$$n_k^{-1} \operatorname{Pf}(\varphi_1) \prod^{k} n_i = \pm n_k^{-1} \operatorname{Pf}(\varphi_2) \prod^{k} n_i \mod \mathbf{Z},$$

where n_i are described in the discussion preceding Theorem 1. However, $n_k = (t_1, \dots, t_k) = (r_1, \dots, r_k)$ (the greatest common divisor of this set of integers) and $\Pi^k n_i = \Pi^k t_1 = \Pi^k r_i$. Thus

$$n_k^{-1} \mathrm{Pf}\,(\varphi_1) \prod^k n_i = \prod^k s_i (t_1, \dots, t_k)^{-1}$$

and

$$n_k^{-1} \mathrm{Pf}\,(\varphi_2) \prod^k n_i = \prod^k q_i (r_1, \dots, r_k)^{-1}. \quad \square$$

The condition that $b, c \in \{0, 1\}$ is not a restriction, since s_i is allowed to be zero and $A_0 = C(T^2)$. Condition (ii) above implies that $\mathrm{ord}(\rho_1) = \mathrm{ord}(\rho_2)$; thus the least common multiple of the t_i is equal to the least common multiple of the r_i.

Thus, for p a prime, the algebras $\mathcal{A}_1 = \otimes^\ell A_{s_i p^{-1}}$ and $\mathcal{A}_2 = \otimes^\ell A_{q_i p^{-1}}$ are isomorphic if and only if the number of i with $(s_i, p) = p$ is equal to the number of i with $(q_i, p) = p$ and $\Pi^\ell s_i = \pm \Pi^\ell q_i \mod p\mathbf{Z}$.

3. Given arbitrary $\rho \in \mathrm{Hom}(G \wedge G, T)$ with $\rho = \exp(2\pi i \varphi)$ and $\varphi \in \mathrm{Hom}(G \wedge G, \mathbf{R})$, there is an identification of $K_0(A_\rho)$ with $\Lambda^{\mathrm{even}} \mathbf{Z}^d$ mapping the class of the unit to 1 in $\Lambda^0 \mathbf{Z}^d = \mathbf{Z}$ such that the map τ_* becomes the map $\exp_\wedge \varphi$ ([**6**]). An isomorphism $\alpha : K_0(A_\rho) \to K_0(A_{\rho'})$ is trace preserving if $\tau'_* \alpha = \tau_*$ for τ, τ' traces on $A_\rho, A_{\rho'}$ respectively.

THEOREM 4. *Let $\mathcal{A}_1$ be a rational d-torus with trace τ_1. There is a $b \in [0, 1] \cap \mathbf{Q}$ and a trace preserving isomorphism $\alpha : K_0(\mathcal{A}_1) \to K_0(A_b \otimes C(T^{d-2}))$ mapping the class of the unit to the class of the unit.*

PROOF. Since $\mathcal{A}_1$ is a rational d-torus, there is a $b \in [0, 1] \cap \mathbf{Q}$ with $b\mathbf{Z} = \tau_*(\oplus_{r>0} \Lambda^{2r} \mathbf{Z}^d)$. Let $\mathcal{A}_2 = A_b \otimes C(T^{d-2})$ and τ_2 be a trace on $\mathcal{A}_2$. Choose $b_j \in \oplus_{r>0} \Lambda^{2r} \mathbf{Z}^d$ with $\tau_{j*} b_j = b$, $j = 1, 2,$. Since $b\mathbf{Z}$ is free, the one element sets $\{b_j\}$, $j = 1, 2$, can each be extended to a basis of $\oplus_{r>0} \Lambda^{2r} \mathbf{Z}^d$. Writing $b = \ell m^{-1}$ with $(\ell, m) = 1$, it follows that range $(\tau_{j*}) = \mathbf{Z} + b\mathbf{Z} = m^{-1} \mathbf{Z}$ and that the one element set $\{\ell - m b_j\}$ of $\ker(\tau_{j*})$ can be extended to a basis of $\ker(\tau_{j*})$,

$j = 1, 2$. Choosing $u, v \in \mathbf{Z}$ with $mv + \ell u = 1, \tau_{j*}(ub_j + v) = m^{-1}$. Thus $K_0(\mathcal{A}_j) \simeq \ker(\tau_{j*}) \oplus \mathbf{Z}(ub_j + v)$. It follows that there is an isomorphism $\alpha : K_0(\mathcal{A}_1) \to K_0(\mathcal{A}_2)$ with $\alpha(\ell - mb_1) = \ell - mb_2$, $\alpha(\ker(\tau_{1*})) = \ker(\tau_{2*})$ and $\alpha(ub_1 + v) = ub_2 + v$. Since $u(\ell - mb_j) = 1 - mv - mub_j = 1 - m(ub_j + v)$, it is immediate that $\alpha 1 = 1$. □

4. If $G = \mathbf{Z}^3$, many considerations are simplified. For example, if $\varphi \in \mathrm{Hom}(\Lambda^2\mathbf{Z}^3, \mathbf{R})$ with $\rho = \exp(2\pi i\varphi)$ then, since $\Lambda^4\mathbf{Z}^3 = 0$, $\tau_*(K_0(A_\rho)) = \exp_\wedge \varphi(\mathbf{Z} \oplus \Lambda^2\mathbf{Z}^3) = \mathbf{Z} + \mathrm{range}(\varphi)$ and $\exp(2\pi i\tau_*(K_0(A_\rho))) = \exp(2\pi i\ \mathrm{range}(\varphi)) = \mathrm{range}(\rho)$. Thus $\mathrm{range}(\rho)$ is easily recovered from $K_0(A_\rho)$ and τ_*, so $\mathrm{range}(\rho) = \mathrm{range}(\rho')$ if there is a trace preserving isomorphism of $K_0(A_\rho)$ to $K_0(A_{\rho'})$.

The canonical identification of $\Lambda^2\mathbf{Z}^3$ with $\mathbf{Z}^3$ allows an element $\rho \in \mathrm{Hom}(\Lambda^2\mathbf{Z}^3, T)$ to be viewed as a character χ_ρ of $\mathbf{Z}^3$. Since the action of $\mathrm{Aut}(\mathbf{Z}^3)$ on $\Lambda^2\mathbf{Z}^3$ becomes an action of $\mathrm{SL}(3, \mathbf{Z})$ on $\mathbf{Z}^3$, we have $\rho \simeq \rho'$ if and only if there is a $\beta \in \mathrm{SL}(3, \mathbf{Z})$ with $\chi_\rho\beta = \chi_{\rho'}$. These classes are easily computed. Recall that $\chi, \chi' \in \hat{\mathbf{Z}}^m$ are isomorphic (write $\chi \simeq \chi'$) if there is a $\beta \in \mathrm{Aut}(\mathbf{Z}^m)$ with $\chi\beta = \chi'$.

The isomorphism classes of characters on $\mathbf{Z}^m$ are determined in [**9**]. In most cases the range of a character completely determines its isomorphism class. For the sake of completeness and of providing an alternate approach, we give the essentials below.

Since the range of a character χ of $\mathbf{Z}^m$ is isomorphic to $\mathbf{Z}^r \oplus \mathbf{Z}_n, r, n \in \mathbf{N}_0$, we may view χ as a map of $\mathbf{Z}^m$ onto $\mathbf{Z}^r \oplus \mathbf{Z}_n$. The submodule $\ker(\chi)$ of $\mathbf{Z}^m$ has invariants $a_1, \ldots, a_s$ with $a_1|a_2|\cdots|a_s$ and $\mathbf{Z}_{a_1} \times \cdots \times \mathbf{Z}_{a_s} \times \mathbf{Z}^{m-s} \simeq \mathbf{Z}^m/\ker\chi \simeq \mathbf{Z}^r \oplus \mathbf{Z}_n$. The torsion subgroups are isomorphic, and, since $\mathbf{Z}_b \times \mathbf{Z}_c \simeq \mathbf{Z}_d$ if and only if $(b, c) = 1$ and $d = bc$, it follows that $a_1 = \cdots = a_{s-1} = 1, a_s = n$. Thus there is a basis $\{e_1, \ldots, e_m\}$ of $\mathbf{Z}^m$ with $\{e_1, \ldots, e_{s-1}, ne_s\}$ a basis of $\ker(\chi)$. Note that $\chi(e_s)$ is a generator of $\mathbf{Z}_n$. It is possible to arrange that $\{\chi(e_{s+1}), \ldots, \chi(e_m)\}$ is any given basis of $\mathbf{Z}^r$, since χ induces an isomorphism of $\mathbf{Z}^m/\ker(\chi)$ modulo its torsion subgroup with $\mathbf{Z}^r$.

If $n = 1, \ker(\chi)$ is complemented and thus $\mathrm{range}(\chi)$ is a complete invariant for the isomorphism class of χ. In fact, $\mathrm{range}(\chi)$ is a complete invariant whenever $r < m - 1$ or, equivalently whenever $s > 1$. In this case it is straightforward to see that there is a $\beta \in \mathrm{SL}(m, \mathbf{Z})$ with

$\beta(\ker(\chi)) = \ker(\chi)$ and $\beta(e_i) = e_i$ for $i \neq s-1, s$ such that $\chi(\beta(e_s))$ is the generator 1 of $\mathbf{Z}_n$.

If $r = m-1$ and $n > 1$, we need the element $t(\chi) = \{\pm\chi(e_1)\}$ (where $\chi(e_1)$ is the generator of $\mathbf{Z}_n$ described above, since $s = 1$), defined in [**10**]. In this case range(χ) and $t(\chi)$ are complete invariants for the isomorphism class of χ. We shall quickly show $t(\chi)$ is an invariant. For $\beta \in \mathrm{Aut}(\mathbf{Z}^m)$ we have just seen that there is a basis $\{f_1, \ldots, f_m\}$ of $\mathbf{Z}^m$ with nf_1 a basis of $\ker(\chi\beta)$. Choose $\gamma \in \mathrm{Aut}(\mathbf{Z}^m)$ with $\gamma(e_i) = \beta(f_i)$. If $\gamma(e_1) = \sum_{j=1}^m \gamma_{1j} e_j$, $0 = \chi(n(\gamma(e_1))) = \sum_{j=2}^m n\gamma_{1j}\chi(e_j)$ and $n\gamma_{1j} = \gamma_{1j} = 0$ for $j \geq 2$. Since $\det\gamma = \pm 1$, it follows that $\gamma_{11} = \pm 1$ and $\chi(\beta(f_1)) = \chi(\gamma(e_1)) = \pm\chi(e_1)$.

If $r < m-1$ or $r = m-1$ and $n = 1$ it is easily verified that $\chi\beta = \chi'$ for a $\beta \in \mathrm{Aut}(\mathbf{Z}^m)$ if and only if $\chi\beta' = \chi'$ for a $\beta' \in \mathrm{SL}(m, \mathbf{Z})$, $\chi, \chi' \in \hat{\mathbf{Z}}^m$. If m is odd then $\chi \simeq \chi'$ if and only if there is a $\beta \in \mathrm{SL}(m, \mathbf{Z})$ with $\chi\beta = \chi'$ or $\overline{\chi}\beta = \chi'$.

THEOREM 5. *Assume* $\rho_1, \rho_2 \in \mathrm{Hom}(\Lambda^2\mathbf{Z}^3, T)$. *If* A_{ρ_1} *and* A_{ρ_2} *are isomorphic then* $\rho_1 \simeq \rho_2$ *or* $\overline{\rho}_1 \simeq \rho_2$.

PROOF. It is enough to show that if A_{ρ_1} and A_{ρ_2} are isomorphic then $\chi_1 \simeq \chi_2$ (where χ_j is the character defined by $\rho_j, j = 1, 2$). Let τ_j be a trace on $A_{\rho_j}, j = 1, 2$. The isomorphism of algebras induces a trace preserving isomorphism $\alpha : K_0(A_{\rho_1}) \to K_0(A_{\rho_2})$ mapping 1 to 1. It is clear from the above discussion that it suffices to prove $t(\chi_1) = t(\chi_2)$ if $\mathrm{rank}(\rho_1) = 2$ and $\mathrm{ord}(\rho_1) = n > 1$. It is also clear that if $t(\chi_j) = \{\pm w_j \mod n\mathbf{Z}\}$ then there is a basis $\{\exp(2\pi i\theta_k) |\, k = 1, 2\}$ of the free part of range(ρ_1), a basis $\{e_k(j) \wedge e_i(j) |\, k < i \leq 3\}$ of $\Lambda^2\mathbf{Z}^3$ and $\varphi_j \in \mathrm{Hom}(\Lambda^2\mathbf{Z}^3, \mathbf{R})$ defined by $\varphi_j(e_1(j) \wedge e_2(j)) = w_j n^{-1}, \varphi_j(e_2(j) \wedge e_3(j)) = \theta_1, \varphi_j(e_3(j) \wedge e_1(j)) = \theta_2$ such that $\rho_j = \exp(2\pi i\varphi_j), j = 1, 2$. Identify $K_0(A_{\rho_j})$ with $\mathbf{Z} \oplus \Lambda^2\mathbf{Z}^3$ in such a way that $\tau_{j*} = 1 \oplus \varphi_j, j = 1, 2$.

Since α preserves the trace, $\alpha(\ker(\pi\tau_{1*})) = \ker(\pi\tau_{2*})$ for $\pi : \mathbf{R} \to F$ (a group homomorphism), in particular for $\pi : \mathbf{R} \to \mathbf{R}/\mathbf{Q}$, the canonical quotient map. Thus α is an isomorphism (mapping 1 to 1) of the rank two abelian groups $\ker(\pi\tau_{j*}) = \mathbf{Z} \oplus \mathbf{Z}\, e_1(j) \wedge e_2(j), j = 1, 2$, and $\alpha(e_1(1) \wedge e_2(1)) = n \pm e_2(2) \wedge e_2(2)$ for some $n \in \mathbf{Z}$. It follows that

$$\begin{aligned}\exp(2\pi i w_2 n^{-1}) &= \exp(2\pi i \tau_{2*}(e_1(2)\wedge e_2(2)))\\ &= \exp(\pm 2\pi i \tau_{2*}(n \pm e_1(2)\wedge e_2(2)))\\ &= \exp(\pm 2\pi i \tau_{2*}\alpha(e_1(1)\wedge e_2(1)))\\ &= \exp(\pm 2\pi i \tau_{1*}(e_1(2)\wedge e_2(2)))\\ &= \exp(\pm 2\pi i w_1 n^{-1}) \text{ and } t(\chi_1) = t(\chi_2). \quad \square\end{aligned}$$

Using the remark preceding Theorem 5 we may also conclude that if $\rho, \rho' \in \mathrm{Hom}(\Lambda^2 \mathbf{Z}^3, T)$ and $\mathrm{rank}(\rho) \le 1$ or $\mathrm{rank}(\rho) = 2$ with $\mathrm{ord}(\rho) = 1$, then A_ρ is isomorphic to A_ρ, if and only if $\rho \simeq \rho'$.

COROLLARY 6. *Assume* $\rho, \rho' \in \mathrm{Hom}(\Lambda^2 \mathbf{Z}^3, T)$. *Then* A_ρ^∞ *is isomorphic to* $A_{\rho'}^\infty$ *if and only if* $\rho \simeq \rho'$.

PROOF. If ρ is nondegenerate this is contained in [**2**]. Since ρ is nondegenerate if and only if $\mathrm{rank}(\rho) \ge 2$, consider ρ with $\mathrm{rank}(\rho) \le 1$. An isomorphism of A_ρ^∞ with $A_{\rho'}^\infty$ extends to an isomorphism of A_ρ with $A_{\rho'}$ [**2**], so the comment preceding the Corollary shows $\rho \simeq \rho'$. $\square$

It would be of interest to know if A_ρ is isomorphic to $A_{\overline{\rho}}$ for some $\rho \in \mathrm{Hom}(\Lambda^2 \mathbf{Z}^3, T)$ with $\rho \not\simeq \overline{\rho}$. If so, it provides an interesting example of a C^*-algebra with nonisomorphic differential structures. It is known that the dimension range fails to distinguish the algebras A_ρ and $A_{\overline{\rho}}$ for such ρ [**2**].

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