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ON PERTURBATIONS OF REFLEXIVE ALGEBRAS

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We denote by $\mathcal{H}$, $\mathcal{L}(\mathcal{H})$, and $\mathcal{K}$ a complex Hilbert space, the algebra of bounded linear operators on $\mathcal{H}$, and the ideal of compact operators on $\mathcal{H}$, respectively. We recall that a subalgebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is said to be *reflexive* if it contains every operator T such that $T\mathcal{M} \subset \mathcal{M}$ whenever $\mathcal{M}$ is closed invariant subspace for $\mathcal{A}$.

In this paper we provide elementary examples that answer in the negative the following two questions.

PROBLEM 1. Suppose that $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is a reflexive algebra. Is then $\mathcal{A} + \mathcal{K}$ norm-closed?

PROBLEM 2. Suppose that $\mathcal{A}_n$, $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ are similar reflexive algebras, $n \geq 0$, and $\lim_{n\to\infty} \operatorname{dist}(\mathcal{A}_n, \mathcal{A}) = 0$. Can we choose invertible operators X_n such that $X_n^{-1}\mathcal{A}X_n = \mathcal{A}_n$ and $\lim_{n\to\infty} ||X_n - I|| = 0$?

The distance mentioned in Problem 2 is, of course, the Pompeiu-Hausdorff distance between the unit balls of $\mathcal{A}_n$ and $\mathcal{A}$.

We note that Problem 1 has an affirmative answer if the invariant subspaces of $\mathcal{A}$ are totally ordered by inclusion (i.e., $\mathcal{A}$ is a nest algebra); see [**6**]. The answer to Problem 1 is negative for algebras with commutative invariant subspace lattice (CSL-algebras); see [**7**]. See also [**1**] and [**11**] for more details about such algebras.

The answer to Problem 2 is positive if $\mathcal{A}_n$ and $\mathcal{A}$ are nest algebras. Problem 2 has a negative answer is $\mathcal{A}$ is a CSL-algebra (see [**5**]), but it is open for algebras acting on finite-dimensional spaces. See [**2, 3, 4, 10** and **12**] for more information about this problem.

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We begin with our example concerning Problem 1; this example is related to that given in [**4**]. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_j : 0 \leq j < \infty\}$, and define operators $T, P_0, S \in \mathcal{L}(\mathcal{H})$ such that

$$P_0 x = (x, e_0)e_0, \qquad x \in \mathcal{H},$$

$$Se_j = e_{j+1}, \qquad j \geq 0,$$

$$T = S + P_0.$$

Next, denote by $\mathcal{A}$ the weakly closed unital algebra generated by T.

PROPOSITION 3. *The algebra $\mathcal{A}$ is reflexive and $\mathcal{A} + \mathcal{K}$ is not closed in the norm topology.*

This result will be proved in several steps. Let us set $\Lambda = \{\lambda \in \mathbf{C} : |\lambda| < 1\} \cup \{1\}$.

LEMMA 4. *The function $f : \Lambda \to \mathcal{H}$ defined by $f(\lambda) = e_0 + \sum_{k=1}^{\infty} \lambda^{k-1}(\lambda - 1)e_k$ is analytic on $\operatorname{int}(\Lambda)$. $lim_{r\uparrow 1} f(r) = f(1)$, and $T^* f(\lambda) = \lambda f(\lambda), \lambda \in \Lambda$.*

PROOF. The analyticity of f is immediate, and so is the relation $||f(r) - f(1)|| = (1-r)(1-r^2)^{-1/2}$, $r \in (0,1)$. Since $T^* = S^* + P_0$, we have $T^* e_0 = e_0$ and $T^* e_j = e_{j-1}$, $j \geq 1$. Thus

$$\begin{aligned} T^* f(\lambda) &= e_0 + \sum_{k=1}^{\infty} \lambda^{k-1}(\lambda - 1)e_{k-1} \\ &= e_0 + (\lambda - 1)e_0 + \lambda \sum_{j=1}^{\infty} \lambda^{j-1}(\lambda - 1)e_j = \lambda f(\lambda), \end{aligned}$$

as claimed. □

Recall that $\operatorname{Alg}\operatorname{Lat}\mathcal{A} = \operatorname{Alg}\operatorname{Lat} T$ is the algebra of all operators $A \in \mathcal{L}(\mathcal{H})$ such that $A\mathcal{M} \subset \mathcal{M}$ for every invariant subspace $\mathcal{M}$ of T.

LEMMA 5. *Fix* $A \in \operatorname{Alg}\operatorname{Lat}\mathcal{A}$, *and define* $u : \Lambda \to \mathbf{C}$ *by* $u(\lambda) = (Ae_0, f(\overline{\lambda})), \lambda \in \Lambda$. *Then* U *is analytic and bounded on* $\operatorname{int}(\Lambda)$, *and* $\lim_{r\uparrow 1} u(r) = u(1)$. *Moreover, if* $u(\lambda) = \sum_{n=0}^{\infty} u_n \lambda^n$ *is the power series expansion of* u, *then*

$$\begin{aligned}(Ae_i, e_j) &= 0, && \text{if } j < i,\\ &= u_{j-1}, && \text{if } j \geq i \geq 1,\\ &= u(1) - \sum_{k=0}^{j-1} u_k, && \text{if } j \geq i = 0.\end{aligned}$$

PROOF. The analyticity of u and the relation $\lim_{r\uparrow 1} u(r) = u(1)$ follow immediately from Lemma 4. To show that u is bounded, we verify that $\overline{u(\overline{\lambda})}$ is an eigenvalue of A^* with eigenvector $f(\lambda)$. Indeed, since $A^* \in \operatorname{Alg}\operatorname{Lat} T^*$, each $f(\lambda)$ is an eigenvector of A^*, and the formula for the corresponding eigenvalue follows because $(f(\lambda), e_0) = 1$. In order to determine the matrix entries of A we use now the relations

$$A^* e_0 = A^* f(1) = \overline{u(1)} e_0,$$

and $A^* f(\lambda) = \overline{u(\overline{\lambda})} f(\lambda)$, $|\lambda| < 1$. The latter equation can be rewritten as

$$\begin{aligned}&\sum_{k=0}^{\infty} \lambda^k (A^* e_k - A^* e_{k+1})\\ &\qquad = \left(\sum_{k=0}^{\infty} \overline{u}_k \lambda^k\right)\left(\sum_{k=0}^{\infty} \lambda^k (e_k - e_{k+1})\right), \quad |\lambda| < 1.\end{aligned}$$

or, equivalently,

$$A^* e_k - A^* e_{k+1} = \sum_{j=0}^{k} \overline{u_j} (e_{k-j} - e_{k-j+1}).$$

These equations now yield

$$\begin{aligned} A^* e_k &= A^* e_0 - \sum_{p=0}^{k-1} (A^* e_p - A^* e_{p+1}) \\ &= \overline{u(1)} e_0 - \sum_{p=0}^{k-1} \sum_{j=0}^{p} \overline{u_j} (e_{p-j} - e_{p-j+1}) \\ &= \overline{u(1)} e_0 - \sum_{j=0}^{k-1} \overline{u_j} \sum_{p=j}^{k-1} (e_{p-j} - e_{p-j+1}) \\ &= \overline{u(1)} e_0 - \sum_{j=0}^{k-1} \overline{u_j} (e_0 - e_{k-j}) \\ &= \left(\overline{u(1)} - \sum_{j=0}^{k-1} \overline{u_j} \right) e_0 + \sum_{j=1}^{k} \overline{u_{k-j}} e_j . \end{aligned}$$

These relations immediately imply the formulas for (Ae_i, e_j). □

COROLLARY 6. *Let A and u be as in Lemma 5.*

(i) *If A is compact then $A = 0$.*

(ii) $||A|| \le \sup\{|u(\lambda)| : |\lambda| < 1\} + \left(\sum_{i=0}^{\infty} |u(1) - \sum_{k=0}^{i-1} u_k|^2 \right)^{1/2}$.

PROOF. (i). If A is compact then we must have $u_k = \lim_{n\to\infty}(Ae_n, e_{n+k}) = 0$ for every k. We conclude that $u = 0$, and hence all the entries in the matrix of A are zero.

(ii) We have

$$\begin{aligned} ||A|| &\le ||AP_0|| + ||A(I - P_0)|| \\ &= ||AP_0|| + ||ASS^*|| \\ &\le ||AP_0|| + ||AS||. \end{aligned}$$

Clearly, AS is a Toeplitz operator with symbol $\lambda u(\lambda)$, so that

$$||AS|| = \sup\{|\lambda u(\lambda)| : |\lambda| < 1\} = \sup\{|u(\lambda)| : |\lambda| < 1\},$$

while AP_0 is a rank-one operator with norm $\left(\sum_{i=0}^{\infty} |u(1) - \sum_{k=0}^{i-1} u_k|^2\right)^{1/2}$. The corollary follows. □

LEMMA 7. *Every operator in* $\operatorname{Alg}\operatorname{Lat} T$ *is the weak limit of a sequence of operators of the form* $p(T)$, *with* p *a polynomial. In particular,* $\mathcal{A}$ *is a reflexive algebra.*

PROOF. Let A and u be as in Lemma 5 and consider the polynomials.

$$u_n(\lambda) = \sum_{k=0}^{n} \left(1 - \frac{k}{n}\right) u_k \lambda^k,$$

and the operators $A_n = u_n(T), n \geq 0$. Clearly

$$\begin{aligned}(A_n e_i, e_j) &= 0, && \text{if } j < i,\\ &= u_{j-i}^n && \text{if } j \geq i \geq 1.\\ &= u_n(1) - \sum_{k=0}^{j-1} u_k^n, && \text{if } j \geq i = 0.\end{aligned}$$

where $u_k^n = (1 - k/n)u_k$ if $k \leq n$, and $u_k^n = 0$ if $k > n$. We have $\lim_{n\to\infty} u_k^n = u_k, k \geq 0$. Moreover, since $\sum_{i=0}^{\infty} |u(1) - \sum_{k=0}^{i-1} u_k|^2 < \infty$, it follows that $u(1) = \sum_{k=0}^{\infty} u_k$. Consequently, the Cesàro sums $u_n(1)$ converge to $u(1)$ as $n \to \infty$. Thus we conclude that $\lim_{n\to\infty}(A_n e_i, e_j) = (Ae_i, e_j)$ for all i and j. The lemma will follow once we prove that $\sup_n ||A_n|| < \infty$. First, it is a well-known consequence of the positivity of the Féjer kernel that

$$\sup\{|u_n(\lambda)| : n \geq 0,\ |\lambda| < 1\} \leq \sup\{|u(\lambda)| : |\lambda| < 1\}.$$

Thus, by virtue of Corollary 6(ii), it suffices to show that

$$\sup\left\{\left(\sum_{i=0}^{\infty}\left|u_n(1) - \sum_{k=0}^{i-1} u_k^n\right|^2\right)^{1/2} : n \geq 0\right\} < \infty.$$

Set

$$\alpha_i = u(1) - \sum_{k=0}^{i-1} u_k, \quad \alpha_i^n = u_n(1) - \sum_{k=0}^{i-1} u_k^n, \quad i, n \geq 0.$$

Then $\alpha_i^n = 0$ for $i \geq n$, and, for $i < n$,

$$\begin{aligned}\alpha_i^n &= \sum_{k=i}^{n} u_k^n = \sum_{k=i}^{n}\left(1-\frac{k}{n}\right)(\alpha_k - \alpha_{k+1}) \\ &= \left(1-\frac{i}{n}\right)\alpha_i - \frac{1}{n}\sum_{k=i+1}^{n}\alpha_k.\end{aligned}$$

A famous result of Hardy (cf. [**8**]), showing that the Cesàro operator is bounded with norm 2 in ℓ^2, implies that

$$\left(\sum_{i=0}^{n}\left|\frac{1}{n-i}\sum_{k=i+1}^{n}\alpha_k\right|^2\right)^{1/2} \leq 2\left(\sum_{k=0}^{n}|\alpha_k|^2\right)^{1/2}.$$

We deduce that

$$\begin{aligned}\left(\sum_{i=0}^{\infty}|\alpha_i^n|^2\right)^{1/2} &\leq \left(\sum_{i=0}^{n}\left|\left(1-\frac{i}{n}\right)\alpha_i\right|^2\right)^{1/2} + \left(\sum_{i=0}^{n-1}\left|\frac{1}{n}\sum_{k=i+1}^{n}\alpha_k\right|^2\right)^{1/2} \\ &\leq \left(\sum_{i=0}^{n}|\alpha_i|^2\right)^{1/2} + \left(\sum_{i=0}^{n-1}\left|\frac{1}{n-i}\sum_{k=i+1}^{n}\alpha_k\right|^2\right)^{1/2} \\ &\leq 3\left(\sum_{i=0}^{\infty}|\alpha_i|^2\right)^{1/2},\end{aligned}$$

and this concludes the proof of the lemma. □

Let $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathcal{K}$ denote the quotient map. The proof of Proposition 3 follows immediately from Lemma 7 and the next observation.

LEMMA 8. *The algebra $\mathcal{A}$ contains no nonzero compact operators, and $\pi|\mathcal{A}$ is not bounded below.*

PROOF. That $\mathcal{A} \cap \mathcal{K} = \{0\}$ follows from Corollary 6(i). To see that $\pi|\mathcal{A}$ is not bounded below we note that $||\pi(T^n)|| = ||\pi(S^n)|| = 1$, while $||T^n|| = \sqrt{n+1},\ n \geq 0$. □

We note that a somewhat more detailed analysis of $\mathcal{A}$ shows that the weak and ultraweak topologies coincide on this algebra.

We proceed now to our example concerning Problem 2. Let $\mathcal{H}$ be, as before, a Hilbert space with orthonormal basis $\{e_n : 0 \le n < \infty\}$ and define operators $R, U_n, R_n \in \mathcal{L}(\mathcal{H})$ such that

$$Re_j = 2^{-j} e_j, \qquad j \ge 0,$$
$$U_n e_n = e_{n+1}, \qquad U_n e_{n+1} = e_n, \qquad U_n e_j = e_j, \qquad n \ne j \ne n+1,$$

and $R_n = U_n^{-1} R U_n, n \ge 0$. (Note that $U_n^{-1} = U_n$.) Define three-dimensional algebras $\mathcal{A}, \mathcal{A}_n \subset \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ by

$$\mathcal{A} = \left\{ \begin{bmatrix} \lambda I & \gamma R \\ 0 & \mu I \end{bmatrix} : \lambda, \mu, \gamma \in \mathbf{C} \right\},$$
$$\mathcal{A}_n = \left\{ \begin{bmatrix} \lambda I & \gamma R_n \\ 0 & \mu I \end{bmatrix} : \lambda, \mu, \gamma \in \mathbf{C} \right\}, \qquad n \ge 0.$$

Recall that, for two subspaces $\mathcal{M}, \mathcal{N}$ of a normed space $\mathcal{X}$, we have $\operatorname{dist}(\mathcal{M}, \mathcal{N}) \le \varepsilon$ if and only if, for every vector x in the open unit ball of $\mathcal{M}$ [respectively, $\mathcal{N}$], there is a vector y in the open unit ball of $\mathcal{N}$ [respectively, $\mathcal{M}$] such that $||x - y|| < \varepsilon$.

PROPOSITION 9. *The algebras $\mathcal{A}_n$ and $\mathcal{A}$ are similar, reflexive, and* $\lim_{n\to\infty} \operatorname{dist}(\mathcal{A}_n, \mathcal{A}) = 0$. *However, if $X_n \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ are invertible operators such that $\mathcal{A}_n = X_n^{-1} \mathcal{A} X_n$, then* $\lim_{n\to\infty} \inf ||X_n - I|| > 0$.

PROOF. Clearly $\mathcal{A}_n = (U_n \oplus U_n)^{-1} \mathcal{A} (U_n \oplus U_n)$ so that $\mathcal{A}_n$ and $\mathcal{A}$ are indeed similar. The equality $\lim_{n\to\infty} \operatorname{dist}(\mathcal{A}_n, \mathcal{A}) = 0$ is an immediate consequence of the fact that $\lim_{n\to\infty} ||R_n - R|| = 0$. The reflexivity of $\mathcal{A}$ (and $\mathcal{A}_n$) follows easily from [**9**], but is also easy to verify directly. Indeed, if $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \operatorname{Alg} \operatorname{Lat} \mathcal{A}$, clearly $C = 0$ and $A, D \in \operatorname{Alg} \operatorname{Lat}(I)$ so that $A = \lambda I$, $D = \mu I$ for some scalars λ and μ. Thus $\left[\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right] \in \operatorname{Alg} \operatorname{Lat} \mathcal{A}$. Using invariant subspaces of the forms $\{\alpha Rx \oplus \beta x : \alpha, \beta \in \mathbf{C}\}$, we see that, for each $x \in \mathcal{H}$, there is a $\gamma_x \in \mathbf{C}$ such that $Bx = \gamma_x Rx$. Linearity of B now implies that $\gamma_x = \gamma$ does not depend on x.

We will conclude the proof of the proposition assuming the following result, which we prove later.

LEMMA 10. *Assume that* $X_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}$ *is an operator such that* $X_n\mathcal{A}_n = \mathcal{A}X_n$ *and* $D_n \neq 0$. *Then there exists a scalar* γ_n *such that* $RD_n = \gamma_n A_n R_n$.

Assume that there exist operators $X_n = \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}$ such that $X_n\mathcal{A}_n = \mathcal{A}X_n$ and $\lim_{n\to\infty} \|X_n - I\| = 0$. Clearly then $D_n \neq 0$ eventually, so we can choose γ_n as in Lemma 10. Denote by $[a_{ij}^n]_{i,j=0}^\infty$ and $[d_{ij}^n]_{i,j=0}^\infty$ the matrices of A_n and D_n, respectively, in the basis $\{e_i : i \geq 0\}$. It is immediate that $d_{00}^n = \gamma_n a_{00}^n$ and $2^{-n} d_{nn}^n = 2^{-n-1}\gamma_n a_{nn}^n$. Thus $\gamma_n = d_{00}^n / a_{00}^n = 2d_{nn}^n / a_{nn}^n$, and the last equality implies that

$$1 = \lim_{n\to\infty} \frac{d_{00}^n}{a_{00}^n} = 2 \lim_{n\to\infty} \frac{d_{nn}^n}{a_{nn}^n} = 2,$$

which is simply not true. This contradiction concludes the proof of the proposition. □

We conclude the paper with a proof of Lemma 10. The relation $X_n\mathcal{A}_n = \mathcal{A}X_n$ implies the existence of scalars $\lambda_n, \mu_n, \gamma_n$ such that

$$\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \begin{bmatrix} \lambda_n I & \gamma_n R_n \\ 0 & \mu_n I \end{bmatrix} = \begin{bmatrix} 0 & R \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix}.$$

Thus we have $\mu_n D_n = 0$ and $\gamma_n A_n R_n + \mu_n B_n = RD_n$. Since $D_n \neq 0$, we deduce that $\mu_n = 0$, and therefore $RD_n = \gamma_n A_n R_n$, as desired. □

Let us note that Lemma 10 can also be deduced from a more general result proved in [**12**].

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