ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 2, Spring 1990

NONCOMMUTATIVE RANDOM VARIABLES AND SPECTRAL PROBLEMS IN FREE PRODUCT C^*-ALGEBRAS

DAN VOICULESCU

ABSTRACT. This paper is a write up of the author's lectures on spectral problems in free product C^*-algebras at GPOTS 1987 in Lawrence, Kansas. We give an overview of our work on these questions [**22, 23, 24, 25, 26**] together with the motivating analogies, examples, explicit computations and open problems.

1. Free products of C^*-Algebras and Hilbert spaces.

1.1. We will work in the category $\mathcal{C}^*$ of unital C^*-algebras and unit-preserving $*$-homomorphisms. This is why the free products, we consider, correspond to what are usually called free products with amalgamation over $\mathbf{C}$.

The free product of C^*-algebras $A_\iota, \iota \in I$, is their categorical direct sum in $\mathcal{C}^*$, i.e., a C^*-algebra A with injections $\alpha_\iota : A_\iota \hookrightarrow A$ such that every collection of $\rho_\iota : A_\iota \to B$ corresponds to precisely one $\rho : A \to B$ such that $\rho_\iota = \rho \circ \alpha_\iota$.

In practice the free product algebra A, denoted $*_{\iota \in I} A_\iota$, can be viewed as the algebraic free product completed with respect to the largest C^*-seminorm which restricts on the A_ι's to their given norms.

EXAMPLE 1.2. If G_ι are discrete groups, $\iota \in I$, and G is their free product $*G_\iota$, then $C^*(G)$ is isomorphic to $*C^*(G_\iota)$.

1.3. Let $(\mathcal{H}_\iota, \xi_\iota)$ be Hilbert spaces with specified unit vectors $\xi_\iota \in \mathcal{H}_\iota, ||\xi_\iota|| = 1$. We define a Hilbert space with specified unit vector $(\mathcal{H}, \xi) = *_{\iota \in I}(\mathcal{H}_\iota, \xi_\iota)$, the free product of the $(\mathcal{H}_\iota, \xi_\iota)$'s. Let

Received by the editors on February 12, 1988.

$\mathcal{H}_\iota^0 = \mathcal{H}_\iota \ominus \mathbf{C}\xi_\iota$ and define

$$\mathcal{H} = \mathbf{C}\xi \oplus \bigoplus_n \bigoplus_{\iota_1 \neq \iota_2 \neq \cdots \neq \iota_n} \mathcal{H}_{\iota_1}^0 \otimes \mathcal{H}_{\iota_2}^0 \otimes \cdots \otimes \mathcal{H}_{\iota_n}^0.$$

EXAMPLE 1.4. If G_ι are discrete groups and $\xi_\iota \in \ell^2(G_\iota), \xi_\iota(g) = \delta_{g,e}$, then

$$*_{\iota \in I}(\ell^2(G_\iota), \xi_\iota) = (\ell^2(G), \xi),$$

where $G = *_{\iota \in I} G_\iota$ and $\xi(g) = \delta_{g,e}$.

EXAMPLE 1.5. If $\mathcal{H}$ is a Hilbert space let

$$\mathcal{T}(\mathcal{H}) = \mathbf{C}1 \oplus \bigoplus_{n \geq 1} \mathcal{H}^{\otimes n}$$

be the Fock-space for Boltzmann statistics. Then we have

$$(\mathcal{T}(\mathcal{H}_1 \oplus \mathcal{H}_2), 1) = (\mathcal{T}(\mathcal{H}_1), 1) * (\mathcal{T}(\mathcal{H}_2), 1).$$

1.6. If $(\mathcal{H}, \xi) = *_{\iota \in I}(\mathcal{H}_\iota, \xi_\iota)$ then there are natural isomorphisms $\mathcal{H} \simeq \mathcal{H}_\iota \otimes \mathcal{H}(\iota)$, where

$$\mathcal{H}(\iota) = \mathbf{C}\xi \oplus \bigoplus_n \bigoplus_{\substack{\iota_1 \neq \iota_2 \neq \cdots \neq \iota_n \\ \iota_1 \neq \iota}} \mathcal{H}_{\iota_1}^0 \otimes \cdots \otimes \mathcal{H}_{\iota_n}^0$$

and where $\xi_\iota \otimes \xi$ corresponds to ξ and $\mathcal{H}_\iota^0 \otimes \xi$ to $\mathcal{H}_\iota^0$.

1.7. Let (A_ι, φ_ι) be C^*-algebras with specified states. The GNS construction yields Hilbert spaces with specified unit vectors $(\mathcal{H}_\iota, \xi_\iota)$ and representations π_ι of A_ι on $\mathcal{H}_\iota$.

If $(\mathcal{H}, \xi) = *_{\iota \in I}(\mathcal{H}_\iota, \xi_\iota)$,the isomorphisms $\mathcal{H} \simeq \mathcal{H}_\iota \otimes \mathcal{H}(\iota)$ yield representations of the A_ι's on $\mathcal{H}$ and hence a representation of $A = *_{\iota \in I} A_\iota$ on $\mathcal{H}$. The state φ of A corresponding to the vector ξ in $\mathcal{H}$ is called the free product of the states $\varphi_\iota, \varphi = *_{\iota \in I} \varphi_\iota$. If the φ_ι's are traces, then φ is also a trace state.

The C^*-algebra A_{red} generated by $\cup_{\iota\in I}\pi_\iota(A_\iota)$ is called the reduced free product of the (A_ι,φ_ι) and we write $(A_{\mathrm{red}},\varphi) = *_{\iota\in I}(A_\iota,\varphi_\iota)$.

1.8. The free product state φ is characterized by the following properties:

$$\varphi(\alpha_{\iota_1}(a_1)\alpha_{\iota_2}(a_2)\cdots\alpha_{\iota_n}(a_n)) = 0$$

whenever $\iota_1 \neq \iota_2 \neq \cdots \neq \iota_n, a_j \in A_{\iota_j}$ and $\varphi_{\iota_j}(a_j) = 0$.

EXAMPLE 1.9. If $G_\iota(\iota\in I)$ are discrete groups and $G = *_{\iota\in I}G_\iota$, let τ_ι and, respectively, τ be the corresponding canonical trace states on $C^*(G_\iota)$, respectively, $C^*(G)$. Then we have $\mathcal{C} = *_{\iota\in I}\mathcal{C}_\iota$.

Slightly abusing notations we shall also denote by τ_ι and τ the canonical traces on $C^*_{\mathrm{red}}(G_\iota)$ and $C^*_{\mathrm{red}}(G)$. We have $(C^*_{\mathrm{red}}(G),\tau) = *_{\iota\in I}(C^*(G_\iota),\tau_\iota)$ and also $(C^*_{\mathrm{red}}(G),\tau) = *_{\iota\in I}(C^*_{\mathrm{red}}(G_\iota),\tau_\iota)$.

EXAMPLE 1.10. On $\mathcal{T}(\mathcal{H})$ let $\ell(h), h\in\mathcal{H}$, be the operator $\ell(h)\eta = h\otimes\eta$. Let $C^*(\ell(\mathcal{H}))$ be the C^*-algebra generated by the $\ell(h)$'s and let ω_1 be the state determined by $1\in\mathcal{T}(\mathcal{H})$. If $(e_j)_{j\in J}$ is an orthonormal basis in $\mathcal{H}$ then

$$\sum \ell(e_j)\ell(e_j)^* \leq I \text{ and } I - \sum \ell(e_j)\ell(e_j)^* \neq 0,$$

which shows that $C^*(\ell(\mathcal{H}))$ is a Cuntz-algebra if $\mathcal{H}$ is infinite-dimensional or an extension of a Cuntz-algebra if $\dim\mathcal{H} < \infty$. If $\dim\mathcal{H} = 1, C^*(\ell(\mathcal{H}))$ is the C^*-algebra of a unilateral shift. If $\mathcal{H} = \bigoplus_{\iota\in I}\mathcal{H}_\iota$ then we have

$$(C^*(\ell(\mathcal{H})),\omega_1) = *_{\iota\in I}(C^*(\ell(\mathcal{H}_\iota)),\omega_1).$$

In particular $C^*(\ell(\mathcal{H}))$ is the reduced free product of the $C^*(\ell(\mathbf{C}e_j))$ with $j\in J$.

2. Free families of noncommutative random variables and the analogue of the Gaussian functor. There seems to be a rather far reaching analogy between tensor products and free products, in which independent random variables correspond to what we will call free random variables. To explain this, we begin with some background on random variables, independence and the Gaussian functor.

2.1. Roughly, a random variable is a function $f : \Omega \to \mathbf{C}$ where Ω is endowed with a probability measure ν. Hence, the well-known generalization: a random-variable is an element $f \in \mathcal{A}$, where $\mathcal{A}$ is a unital algebra (possibly non-commutative) endowed with a state, i.e., a functional $\varphi : \mathcal{A} \to \mathbf{C}$ with $\varphi(1) = 1$.

2.2. If f is a random variable in $(\mathcal{A}, \varphi)$, its distribution is the functional $\mu_f : \mathbf{C}[X] \to \mathbf{C}$ given by $\mu_f(1) = 1, \mu_f(X^n) = \varphi(f^n)$. If $\mathcal{A}$ is a Banach algebra and φ continuous, then μ_f extends to an analytic functional on $\mathbf{C}$ so that $\mu_f(h) = \varphi(h(f))$, where h is holomorphic on $\mathbf{C}$. If $\mathcal{A}$ is a C^*-algebra, $f = f^*$ and φ is positive, then μ_f extends to a compactly supported probability measure on $\mathbf{R}$. The numbers $\varphi(f^n)$ are the moments of f.

Two subalgebras $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ are independent if they commute and $\varphi(f_1 f_2) = \varphi(f_1)\varphi(f_2)$ whenever $f_j \in \mathcal{A}_j, j = 1, 2$. Two random variables are independent if the algebras they generate are independent, i.e., if $\varphi(f_1^m f_2^n) = \varphi(f_1^m)\varphi(f_2^n), m, n \geq 0$, and $f_1 f_2 = f_2 f_1$.

Independent subalgebras arise from tensor products: if $(\mathcal{A}_1, \varphi_1), (\mathcal{A}_2, \varphi_2)$ are "non-commutative probability spaces" then, in $(\mathcal{A}_1 \otimes \mathcal{A}_2, \varphi_1 \otimes \varphi_2)$,the subalgebras $\mathcal{A}_1 \otimes 1$ and $1 \otimes \mathcal{A}_2$ are independent.

2.3. The Gaussian functor Γ, which is used in second quantization, is a functor from the category of real Hilbert spaces and contractions to operator algebras with specified trace state and state-preserving unital completely positive maps.

Roughly speaking, Γ associates with $\mathcal{H}$ the von Neumann algebra $L^\infty(\mathcal{H})$ with respect to the Gaussian probability measure on $\mathcal{H}$ (i.e., $ce^{-||x||^2} d\lambda(x)$ with $d\lambda$ Lebesgue measure, if $\dim \mathcal{H} < \infty$), which provides the trace state. Moreover, the Gaussian random variables $f_\xi(h) = \langle h, \xi \rangle$ give a natural linear map $\xi \to f_\xi$ from $\mathcal{H}$ into the L^2-space associated with $\Gamma(\mathcal{H})$. A basic property of Γ is

$$\Gamma(\mathcal{H}_1 \oplus \mathcal{H}_2) = \Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_2).$$

We turn now to the analogues for free products of 2.2 and 2.3.

2.4. Let $1 \in \mathcal{A}_j \subset \mathcal{A}, j = 1, 2$, be subalgebras, where $(\mathcal{A}, \varphi)$ is a non-commutative probability space. We say that $\mathcal{A}_1, \mathcal{A}_2$ are free in

$(\mathcal{A}, \varphi)$ if $\varphi(a_1 a_2 \cdots a_n) = 0$ whenever $a_j \in \mathcal{A}_{i_j}, 1 \leq j \leq n, i_j \in \{1, 2\}$, $i_1 \neq i_2 \neq \cdots \neq i_n$, and $\varphi(a_j) = 0, 1 \leq j \leq n$.

If $\mathcal{A}_1, \mathcal{A}_2$ are free in $(\mathcal{A}, \varphi)$, the restrictions $\varphi|\mathcal{A}_j, j = 1, 2$, completely determine φ on the algebra generated by $\mathcal{A}_1, \mathcal{A}_2$.

Two random variables $f_j \in \mathcal{A}, j = 1, 2$, are free if the algebras they generate are free. It follows that if $\{f_1, f_2\}$ is a free pair, then the distributions of $f_1 + f_2, f_1 f_2$ and, more generally, any polynomial expression in f_1, f_2, depends only on the distributions of f_1, f_2.

2.5. If $\mathcal{A}$ is the free product of $\mathcal{A}_1, \mathcal{A}_2$ or the reduced free product of $(\mathcal{A}_1, \varphi_1), (\mathcal{A}_2, \varphi_2)$ and $\varphi = \varphi_1 * \varphi_2$, then the images of $\mathcal{A}_1, \mathcal{A}_2$ in $\mathcal{A}$ are a free pair of subalgebras. Thus, for instance, with the notations of 1.2, if $G = G_1 * G_2$ then $C^*(G_1)$ and $C^*(G_2)$ are free in $(C^*(G), \mathcal{C})$ and the same holds for the reduced C^*-algebras. Also, in the context of 1.10, if $\mathcal{H}_1, \mathcal{H}_2$ are orthogonal subspaces in $\mathcal{H}$, then $C^*(\ell(\mathcal{H}_1))$ and $C^*(\ell(\mathcal{H}_2))$ are free in $(C^*(\ell(\mathcal{H})), \omega_1)$.

2.6. The analogue of the Gaussian functor for free products is a functor Φ from the category of real Hilbert spaces and contractions to the category of unital C^*-algebras with specified trace-state and state-preserving unital completely positive maps.

We define $\Phi(\mathcal{H}) = C^*(s(\mathcal{H}))$ where, for $h \in \mathcal{H}, s(h) = (\ell(h) + \ell(h)^*)/2$ is an operator on $\mathcal{T}(\mathcal{H}_{\mathbf{C}})$ (see 1.5, 1.10), $\mathcal{H}_{\mathbf{C}}$ the complexification of $\mathcal{H}$. The trace-state $\varepsilon_{\mathcal{H}}$ on $\Phi(\mathcal{H})$ is the restriction of the state ω_1 defined by $1 \in \mathcal{T}(\mathcal{H}_{\mathbf{C}})$.

The characteristic property of Φ is

$$(\Phi(\mathcal{H}), \varepsilon_{\mathcal{H}}) = (\Phi(\mathcal{H}_1), \varepsilon_{\mathcal{H}_1}) * (\Phi(\mathcal{H}_2), \varepsilon_{\mathcal{H}_2})$$

if $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

The von Neumann algebra which is the weak closure of $\Phi(\mathcal{H})$ is isomorphic to the type II_1 factor of a free group on $\dim \mathcal{H} (> 1)$ generators.

2.7. The distribution of $s(h)$ is a map $\mathbf{C}[X] \to \mathbf{C}$, which, since $s(h)$ is a bounded self adjoint operator, extends to a compactly supported measure on $\mathbf{R}$, also denoted $\mu_{s(h)}$, so that

$$\varepsilon_{\mathcal{H}}(P(s(h)) = \int P(t) d_{\mu_{s(h)}}(t).$$

Then $d_{\mu_{s(h)}}(t) = \chi(t)dt$, where $\chi(t) = (1-t^2)^{1/2}/\pi$ if $|t| \le 1$ and 0 otherwise. Since the graph of χ is a semiellipse, we see that the analogue of the Gaussian distribution for free products is the semiellipse law. There is a corresponding central limit theorem for free random variables, with limit distribution a semiellipse law.

Also, for the free analogue of the Gaussian functor, there is an analogue of Wick powers with Hermite polynomials replaced by certain Gegenbauer polynomials.

3. Free convolution and dual algebraic structures.

3.1. If μ_S, μ_T are the distributions of free random variables, $S, T \in \mathcal{A}$, then (see 2.4) the distribution μ_{S+T} depends only on μ_S and μ_T. Hence there is an operation $\boxplus$ such that $\mu_S \boxplus \mu_T = \mu_{S+T}$. Similarly, there is an operation $\boxtimes$ such that $\mu_{ST} = \mu_S \boxtimes \mu_T$. Both operations are commutative and associative. In case μ_S, μ_T extend to analytic functionals (respectively, compactly supported probability measures on $\mathbf{R}$), the same holds for $\mu_S \boxplus \mu_T$.

If, instead of free random variables, we would have considered independent random variables, the corresponding operations on the distributions would have been additive and, respectively, multiplicative convolution. That is why we call the operations for the distributions of free random variables, free convolution ($\boxplus$ additive and $\boxtimes$ multiplicative). Note however that free convolution is not bilinear, it is a highly non-linear operation.

3.2. Recall that the usual convolution on a group G naturally arises in the Hopf-algebra approach to group-duality. Roughly, if μ_1, μ_2 are measures on G, they yield functionals on some commutative algebra $\mathcal{F}(G)$ of functions on G. The binary group operation $G \times G \to G$ gives a homomorphism (the Hopf-algebra comultiplication) $\mathcal{F}(G) \to \mathcal{F}(G) \otimes \mathcal{F}(G)$, and the convolution of μ_1 and μ_2 is the composition

$$\mathcal{F}(G) \to \mathcal{F}(G) \otimes \mathcal{F}(G) \xrightarrow{\mu_1 \otimes \mu_2} \mathbf{C}.$$

For free convolution a similar treatment is possible, after defining appropriate group objects, and illustrates the general analogy between free products and tensor products.

3.3. A dual group structure on an algebra $\mathcal{A}$ amounts to giving homomorphisms $\mathcal{A} \to \mathcal{A} * \mathcal{A}, \mathcal{A} \to \mathcal{A}, \mathcal{A} \to \mathbf{C}$ corresponding to the binary (i.e., multiplication), unary (i.e., inverse) and nullary (i.e., neutral element) operations which define a group, with the arrows in all compatibility diagrams pointing in reverse directions. This means that, for every algebra B, we have a natural group structure on $\mathrm{Hom}(A, B)$.

In the state-space $\sum(\mathcal{A}) = \{\varphi : \mathcal{A} \to \mathbf{C} | \varphi \text{ linear }, \varphi(1)\}$, the free convolution $\varphi_1 \circledast \varphi_2$ is defined by composing the dual multiplication with the free product state

$$\mathcal{A} \to \mathcal{A} * \mathcal{A} \xrightarrow{\varphi_1 * \varphi_2} \mathbf{C}.$$

Though we adopted here the algebraic context for simplicity, these considerations fit also in the Banach algebra, C^*-algebra and, as required by many examples, projective limits of C^*-algebras contexts.

Example 3.4. (a) On $\mathbf{C}[X]$ we define a dual group structure by the unital homomorphisms $\delta : \mathbf{C}[X] \to \mathbf{C}[X] * \mathbf{C}[X], j : \mathbf{C}[X] \to \mathbf{C}[X], \varepsilon : \mathbf{C}[x] \to \mathbf{C}$ where $\delta(X) = \sigma_1(X) + \sigma_2(X), \sigma_k$ being the identifications of $\mathbf{C}[X]$ with the first and respectively second copy of $\mathbf{C}[X]$ in $\mathbf{C}[X] * \mathbf{C}[X], j(X) = -X$ and $\varepsilon(X) = 0$. The states $\sum(\mathbf{C}[X])$ are precisely the distributions of random-variables and the free convolution on$\sum(\mathbf{C}[X])$ is precisely the additive free convolution $\boxplus$.

(b) On $\mathbf{C}[X]$ there is a dual semigroup structure defined by $\delta(X) = \sigma_1(X)\sigma_2(X)$ for which free convolution is $\boxtimes$.

(c) On the C^*-algebra $C(\mathbf{T})$ there is a dual group structure $\delta(u) = \sigma_1(u)\sigma_2(u), j(u) = u^{-1}, \varepsilon(u) = 1$ where u is the isomorphism of $\mathbf{T}$ with $\{z \in \mathbf{C} | \ |z| = 1\}$. The free convolution is $\boxtimes$ on probability measures on $\mathbf{T}$.

(d) The free real line $\mathbf{R}_{\text{free}}$ (Our initial terminology was non-commutative analogs of various classical groups. Following a suggestion of A. Ramsey we replaced non-commutative by free.) is the inverse limit of C^*-algebras $C(K)$ where K are compact subsets of $\mathbf{R}$ and the operations are essentially as in (a). The free convolution is $\boxplus$ on compactly supported measures on $\mathbf{R}$.

Using projective limits of C^*-algebras and of real C^*-algebras, there are "free" analogues for most matrix Lie groups.

3.5. A dual group structure on $\mathcal{A}$ corresponds to a natural group structure on $\mathrm{Hom}(\mathcal{A}, B)$. If, roughly speaking, $\mathrm{Hom}(\mathcal{A}, B)$ is some kind of Lie group (possibly infinite-dimensional) then there is a natural Lie algebra $\mathrm{Lie}\,\mathrm{Hom}(\mathcal{A}, B)$. An object $\mathrm{Lie}\,\mathcal{A}$ such that

$$\mathrm{Hom}(\mathrm{Lie}\,A\,, B) \simeq \mathrm{Lie}\,\mathrm{Hom}(A\,, B)$$

is what should be called the Lie algebra of the dual group. Thus we are led to consider dual Lie algebras $\mathcal{L}$. The dual abelian group structure is given by dual operations (δ, j, ε) and there is a dual bracket $b : \mathcal{L} \to \mathcal{L} * \mathcal{L}$ with the usual compatibility diagrams among these with reverted arrows. The vector space structure (over $K = \mathbf{R}$ or $\mathbf{C}$) is given by $\alpha_\lambda : \mathcal{L} \to \mathcal{L}\,(\lambda \in K)$ such that

$$\alpha_1 = \mathrm{id}_{\mathcal{L}}\,, \qquad \alpha_0 = i \circ \varepsilon$$

$$\alpha_\lambda \circ \alpha_\mu = \alpha_{\lambda\mu}, \qquad \alpha_{s+t} = d \circ (\alpha_s * \alpha_t) \circ \delta,$$

where i is the inclusion $\mathbf{C} \mapsto \mathcal{L}$ and $d : \mathcal{L} * \mathcal{L} \to \mathcal{L}$ maps each of the $\mathcal{L}$'s identically to $\mathcal{L}$. The compatibility conditions are: the α_λ's are morphisms of the dual group, i.e.,

$$\begin{aligned} &\delta \circ \alpha_\lambda = (\alpha_\lambda * \alpha_\lambda) \circ \delta \\ &j \circ \alpha_\lambda = \alpha_\lambda \circ j, \qquad \varepsilon \circ \alpha_\lambda = \varepsilon; \end{aligned}$$

and the compatibility with the dual bracket,

$$(\alpha_\lambda * \alpha_\mu) \circ b = b \circ \alpha_{\lambda\mu}.$$

Note that, for dual Lie algebras, it makes sense to consider dual adjoint actions $\mathcal{L} \to \mathcal{L} * \mathcal{A}$. Also connecting the dual group and dual Lie algebra there are dual exponential maps $\mathcal{A} \to \mathcal{L}$.

EXAMPLE 3.6. For the dual group $C(\mathbf{T})$ (3.4(c)) we have $\mathrm{Hom}(C(\mathbf{T}), B) \simeq U(B)$ the unitary group of B. The Lie algebra of $U(B)$ can be identified with B_h (the Hermitian part of B) with the bracket $i(a_1a_2 - a_2a_1)$. Therefore the dual Lie algebra of the dual group $C(\mathbf{T})$ is

$\mathbf{R}_{\text{free}}$ (3.4d)) with its dual bracket $b : \mathbf{R}_{\text{free}} \to \mathbf{R}_{\text{free}} * \mathbf{R}_{\text{free}}$ such that, on the subalgebra $\mathbf{C}[X] \subset \mathbf{R}_{\text{free}}$ (X is identified with the identical function $\mathbf{R} \to \mathbf{R} \subset \mathbf{C}$ in the projective limit defining $\mathbf{R}_{\text{free}}$), we have

$$b(X) = i(\sigma_1(X)\sigma_2(X) - \sigma_2(X)\sigma_2(X)).$$

The dual multiplications by scalars are given by $\alpha_\lambda(X) = \lambda X$. The dual exponential map $e : C(\mathbf{T}) \to \mathbf{R}_{\text{free}}$ is such that

$$e(u) = \exp(iX).$$

The dual adjoint action is trivial, i.e., $Ad : \mathcal{L} \to \mathcal{A} * \mathcal{L}$ maps $\mathcal{L}$ identically onto the $\mathcal{L}$ factor in the free product.

In a similar way, there are dual Lie algebras for the dual groups which are the free analogs of the classical matrix groups.

4. Computing free convolutions and spectral problems on free groups.

4.1. Let $G = \mathbf{Z} * \cdots * \mathbf{Z}$ be a free group on n generators and let $u_j, 1 \leq j \leq n$, be the unitaries on $\ell^2(G)$ corresponding to left translation by the n generators and consider $\xi \in \ell^2(G)$, the function $\xi(g) = \delta_{g,e}$. Further, let T_j be operators of the form $T_j = \varphi_j(u_j)$, where the φ_j are real-valued functions, so that $T_j = T_j^*$. Then, $T = T_1 + \cdots + T_n$ is a self-adjoint operator on $\ell^2(G)$, which is actually in the von Neumann algebra $L(G)$ generated by the left regular representation of G. On $L(G)$ the state defined by ξ is a faithful trace-state τ. Computations of norms, spectra or of the trace of the spectral measure for particular operators T, were performed in connection with various problems in the work of several authors: H. Kesten, P. Cartier, J. Cohen, S. Sawyer, C.A. Akemann-P.A. Ostrand, A. Figa-Talamanca, M. Picardello, W. Woess, T. Steger, K. Aomoto, T. Pytlik, J.H. Anderson-B. Blackadar-U. Haagerup and others. We will explain how free convolution provides a general approach to these questions.

4.2. It is sufficient to compute the measure μ on $\mathbf{R}$ obtained by applying the trace τ to the spectral measure of T. Note that the spectrum of T is the support of μ. We have

$$\int t^n d\mu(t) = \tau(T^n) = \langle T^n\xi, \xi\rangle.$$

Thus the measure μ can be obtained by solving a moment problem, if we are able to compute the moments $\langle T^n\xi, \xi\rangle$.

4.3. In $L(G)$ consider the von Neumann subalgebras $\mathcal{A}_j, 1 \le j \le n$, generated by u_j. Then the $\mathcal{A}_j$ are free in $(L(G), \tau)$ in the sense of 2.4, and hence the T_j's are free n-tuples of non-commutative random variables. The measure μ is the distribution of T and

$$\mu = \mu_1 \boxplus \mu_2 \boxplus \cdots \boxplus \mu_n,$$

where μ_j is the distribution of T_j. Since $T_j = \varphi_j(\mu_j), \mu_j$ is the image of Haar measure on $\{z \in \mathbf{C}|\ |z| = 1\}$ via φ_j. Remark also that in fact we could have taken, more generally, $G = G_1 * G_2 * \cdots * G_n$ and T_j self-adjoint convolution operators in $L(G_j)$ viewed as subalgebras of $L(G)$, and assume we know their distributions μ_j.

Thus the problem reduces to the computation of the operation $\boxplus$.

4.4. The computation of $\boxplus$ is based on the following simple fact.

FACT. *Let $C^*(\ell(\mathbf{C}^2))$ be the C^*-algebra acting on $\mathcal{T}(\mathbf{C}^2)$* (see 1.10) *with the state ω_1. Let $T = \ell(e_1)^* + \sum_{k\ge 0} \alpha_k \ell(e_1)^k, S = \ell(e_2)^* + \sum_{k\ge 0} \beta_k \ell(e_2)^k$ and $R = \ell(e_1)^* + \sum_{k\ge 0}(\alpha_k + \beta_k)\ell(e_1)^k$, where e_1, e_2 is the canonical basis of $\mathbf{C}^2$. Then we have*

$$\langle (T+S)^n 1, 1\rangle = \langle R^n 1, 1\rangle$$

and hence $\mu_{T+S} = \mu_R$. Since T and S are a free pair of random variables this means

$$\mu_T \boxplus \mu_S = \mu_R.$$

Thus, for the distributions of random variables of the form $\ell(e_1)^* + \sum_{k\ge 0} \alpha_{k+1}\ell(e_1)^k$ in $(C^*(\ell(\mathbf{C})), \omega_1)$ additive free convolution is linearized by the coefficients α_k. Hence the computation of $\boxplus$ amounts to finding, for every distribution μ, a random variable of the special form $\ell(e_1)^* + \sum_{k\ge 0} \alpha_{k+1}\ell(e_1)^k$ with distribution the given distribution μ. Note that $\mathcal{T}(\mathbf{C}) \simeq H^2$ (the Hardy space) and $\ell(e_1)^* + \sum_{k\ge 0} \alpha_{k+1}\ell(e_1)^k$ is precisely the Toeplitz operator with symbol $z^{-1} + \sum_{k\ge 0} \alpha_{k+1} z^k$ or, equivalently, the Toeplitz matrix

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \cdots \\ 0 & 0 & 0 & 1 & \alpha_1 & \alpha_2 & \alpha_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

4.5. The following theorem, which computes $\boxplus$, is obtained using the idea outlined in 4.4.

THEOREM. *If μ is the distribution of a random variable let $G_\mu(z)$ be the generating series*

$$G_\mu(z) = z^{-1} + \sum_{n\geq 1} \mu(X^n) z^{-n-1}.$$

Let further $K(z)$ be such that

$$K(G_\mu(z)) = G_\mu(K(z)) = z$$

and define

$$\mathcal{R}_\mu(z) = \sum_{k\geq 0} R_{k+1}(\mu) z^k$$

by

$$\mathcal{R}_\mu(z) + z^{-1} = K(z).$$

If μ_1, μ_2, μ_3 are distributions of random variables and $\mu_3 = \mu_1 \boxplus \mu_2$, then we have

$$\mathcal{R}_{\mu_3} = \mathcal{R}_{\mu_1} + \mathcal{R}_{\mu_2}.$$

Remark that if μ is a measure or an analytic functional, then $G_\mu(z)$ is precisely the Cauchy transform

$$G_\mu(z) = \int \frac{d\mu(\zeta)}{z - \zeta}.$$

The map $\mu \to \mathcal{R}_\mu$ linearizes additive free convolution as does the logarithm of the Fourier transform $\log \hat{\mu}$ for usual additive convolution.

EXAMPLE 4.6. Returning to the context of 4.1, let $G = \mathbf{Z} * \cdots * \mathbf{Z}$ and $T_j = u_j + u_j^{-1}$. The operator $T = (u_1 + u_1^{-1}) + \cdots + (u_n + u_n^{-1})$ is up to constants the discrete Laplacian on G. If ν is the distribution of T_j then $\mu = \nu \boxplus \cdots \boxplus \nu$. Since T_j is given by the image via the function $z + z^{-1}$ of Haar measure $d\lambda$ on $\{z \in \mathbf{C} | \, |z| = 1\}$, we have

$$\begin{aligned} G_\nu(z) &= \int_{|\zeta|=1} (z - \zeta - \zeta^{-1})^{-1} d\lambda(\zeta) \\ &= (2\pi i)^{-1} \int_{|\zeta|=1} (z - \zeta - \zeta^{-1})^{-1} \zeta^{-1} d\zeta \\ &= -(2\pi i)^{-1} \int_{|\zeta|=1} (\zeta - \zeta_1)^{-1} (\zeta - \zeta_2)^{-1} d\zeta \\ &= (\zeta_1 - \zeta_2)^{-1} = (z^2 - 4)^{-1/2}, \end{aligned}$$

where $\zeta_k = \frac{1}{2}(z \pm \sqrt{z^2 - 4})$ are the roots of $\zeta^2 - z\zeta + 1 = 0$. (Note that $\zeta_1 \zeta_2 = 1$.)

Then $G_\nu(K(z)) = z$ gives $K^2 - 4 = z^{-2}$ and hence $K = (z^{-2} + 4)^{1/2}$ and $\mathcal{R}_\nu = (z^{-2} + 4)^{1/2} - z^{-1}$ so that $\mathcal{R}_\mu = n\mathcal{R}_\nu$. It follows for μ that

$$K = z^{-1} + \mathcal{R}_\mu = z^{-1} + n\mathcal{R}_\nu = n(z^{-2} + 4)^{1/2} - (n-1)z^{-1}.$$

Hence G_μ is obtained by solving

$$n(G^{-2} + 4)^{1/2} - (n-1)G^{-1} = z.$$

This gives

$$G_\mu(z) = \frac{(n-1)z - n\sqrt{z^2 - 4(2n-1)}}{4n^2 - z^2}.$$

The choice of the branch of the square root is determined by the fact that $G_\mu(z) = z^{-1} + \cdots$ at ∞. Since this branch of $G_\mu(z)$ in the upper half plane has no poles on the real line and is algebraic, the theory of the moment problem gives that μ is absolutely continuous with respect to Lebesgue measure and has density

$$-\frac{1}{\pi} \operatorname{Im} G_\mu(t) \quad \text{for } t \in \mathbf{R}.$$

4.7. As for usual convolution, there are semigroups and infinitely divisible measures for free convolution. Let $\mathcal{P}$ denote the compactly supported probability measures on $\mathbf{R}$. We call $\mu \in \mathcal{P}$ F-infinitely divisible (F stands for free) if, for every $n \in \mathbf{N}$, there is $\mu_{1/n} \in \mathcal{P}$ such that $\underbrace{\mu_{1/n} \boxplus \cdots \boxplus \mu_{1/n}}_{n-times} = \mu$. A family $(\mu_t)_{t\geq 0} \subset \mathcal{P}$ is an F-convolution semigroup if $\mu_{t+s} = \mu_t \boxplus \mu_s$ and μ_t depends continuously on t. F-convolution semigroups are in 1-1 correspondence with F-infinitely divisible measures. Every F-convolution semigroup is of the form $\mathcal{R}_{\mu_t} = t\mathcal{R}_{\mu_1}$ where μ_1 is F-infinitely divisible.

If $(\mu_t)_{t\geq 0}$ is an F-convolution semigroup and $G(t,z) = G_{\mu_t \boxplus \nu}(z)$ where $\nu \in \mathcal{P}$, then $G(t,z)$ satisfies the complex quasilinear equation

$$\frac{\partial G}{\partial t}(z,t) + \frac{\partial G}{\partial z}(z,t)\varphi(G(z,t)) = 0$$

where $\varphi(z) = \mathcal{R}_{\mu_1}(z)$.

Note that, in this context, the analogue of the heat equation, which corresponds to convolution by the normal distribution, is given by the equation

$$\frac{\partial G}{\partial t} + \alpha G\frac{\partial G}{\partial z} = 0$$

since $\mathcal{R}_\mu(z) = \alpha z$ if μ is a centered semiellipse law.

4.8. F-infinitely divisible measures are characterized by the following theorem.

THEOREM. *Let $\varphi(z) = \sum_{n\geq 0} a_{n+1}z^n$ be a power series. The following conditions are then equivalent*:

(i) *$\varphi(z) = \mathcal{R}_\mu(z)$ where $\mu \in \mathcal{P}$ is F-infinitely divisible.*

(ii) *φ is the Taylor series of a holomorphic function in some neighborhood of $(\mathbf{C}\backslash\mathbf{R}) \cup \{0\}$ such that $\varphi(\overline{z}) = \overline{\varphi(z)}$ and $\operatorname{Im} z > 0 \Rightarrow \operatorname{Im}\varphi(z) \geq 0$.*

The proof of the theorem is based on the theory of the Navenlinna-Pick problem and on a careful study of univalence for solutions of the differential equation satisfied by the Cauchy-transform of the μ_t's.

4.9. We pass now to multiplicative free convolution $\boxtimes$. Note that

$$(\mu_1 \boxtimes \mu_2)(X) = \mu_1(X)\mu_2(X)$$

if $\mu_j \in \sum(\mathbf{C}[X])$ (see 3.4(a)) are distributions of random variables. Hence the set of distributions of random variables with non-zero first moment.

$$\sum{}^* = \{\mu \in \sum(\mathbf{C}[X]) | \mu(X) \neq 0\}$$

is a semigroup for $\boxtimes$.

THEOREM. *If $\mu \in \sum^*$ let $\Psi_\mu(z)$ be the generating series*

$$\Psi_\mu(z) = \sum_{n \geq 1} \mu(X^n) z^n.$$

Let further $\chi(z)$ be such that

$$\chi(\Psi_\mu(z)) = z$$

and define

$$S_\mu(z) = \chi(z) z^{-1}(1+z).$$

If $\mu_1, \mu_2 \in \sum^*$ and $\mu_3 = \mu_1 \boxtimes \mu_2$ then we have

$$S_{\mu_3}(z) = S_{\mu_1}(z) S_{\mu_2}(z).$$

Thus the map $\mu \rightarrow S_\mu$ plays in the free context the same role as the Mellin transform and gives a way for computing free convolution.

The proof of the preceding theorem rests on the following idea, roughly: $(\sum^*, \boxtimes)$ is an infinite-dimensional Lie group whose Lie algebra is $(\sum, \boxplus)$ with the trivial bracket, and the computation of $\boxplus$ is done via the exponential map $\sum \rightarrow \sum^*$, also called free-exponential (not to be confused with the dual exponential).

Note also, the following concerning $\boxtimes$:

(a) If μ_1, μ_2 are probability measures on $\{z \in \mathbf{C} | \ |z| = 1\}$ then so is $\mu_1 \boxtimes \mu_2$.

(b) If μ_1, μ_2 are compactly supported probability measures on $\{t \in \mathbf{R} \mid t > 0\}$ then so is $\mu_1 \boxtimes \mu_2$.

Assertion (a) is related to 3.4(c) whereas (b) follows from the fact that if $\mathcal{A}$ is a C^*-algebra with trace-state τ and $\{a, b\}$ a free pair in $(\mathcal{A}, \tau)$ with $a \geq 0$ and $b \geq 0$, then we have for the distributions $\mu_{a,b} = \mu_{a^{1/2}ba^{1/2}}$ and $a^{1/2}ba^{1/2} \geq 0$.

EXAMPLE 4.10. Let e, f be a free pair of self-adjoint idempotents in $(\mathcal{A}, \tau)$ where τ is a trace-state on the C^*-algebra $\mathcal{A}$. Then, by the above, the trace of the spectral measure of efe is the distribution of efe and equals $\mu_3 \boxtimes \mu_f$. If $\alpha = \tau(e), \beta = \tau(f)$ then $\mu_e = (1-\alpha)\delta_0 + \alpha\delta_1$ and $\mu_f = (1-\beta)\delta_0 + \beta\delta_1$ so that our problem is the computation of $((1-\alpha)\delta_0 + \alpha\delta_1) \boxtimes ((1-\beta)\delta_0 + \beta\delta_1)$, where δ_t is the Dirac measure at t. A straightforward application of 4.9 and of the solution of the moment problem gives

$$\mu_{efe} = \Big(-\frac{1}{\pi}\operatorname{Im} G\Big)\lambda + c_0\delta_0 + c_1\delta_1,$$

where

$$\begin{aligned} G(z) &= \frac{1}{z} + \frac{z - (\alpha+\beta) + \sqrt{(z-a)(z-b)}}{2(1-z)z} \\ a, b &= \alpha + \beta - 2\alpha\beta \pm \sqrt{4(\alpha - \alpha^2)(\beta - \beta^2)} \\ \lambda &= \text{ Lebesgue measure on } \mathbf{R}\,, \\ c_0 &= 1 - \min(\alpha, \beta), \quad c_1 = \max(\alpha + \beta - 1, 0). \end{aligned}$$

In particular $\tau(e \wedge f) = c_1 = \max\{\alpha + \beta - 1, 0\}$.

4.11. In the study of $\boxtimes$ an important role is played by the equation of a semigroup. If $(\mu_t)_{t \geq 0}$ is a semigroup for $\boxtimes$, and if

$$\Psi(z, t) = \Psi_{\mu_t \boxtimes \nu}(z),$$

then we have

$$\frac{\partial}{\partial t}\Psi(z, t) + \varphi(\Psi(z, t)) z \frac{\partial}{\partial z}\Psi(z, t) = 0,$$

where $\varphi(z) = \log S_{\mu_1}(z)$.

4.12. In connection with 4.11 we would like to draw attention to the fact that, for multiplicative free convolution, the analogue of Theorem 4.8 is open.

PROBLEM (a) Characterize the probability measures on $\{z \in \mathbf{C} \mid |z| = 1\}$ which are infinitely divisible under $\boxtimes$.

(b) Characterize the compactly supported probability measures on $\{t \in \mathbf{R} \mid t > 0\}$ which are infinitely divisible under $\boxtimes$.

5. Free products with amalgamation. A large part of the material in the previous sections can be extended to the context of free products with amalgamation. Roughly speaking this will mean replacing the complex field **C**by an algebra B. Since this extension is rather technical our brief presentation will be somewhat vague about details and sketchy.

5.1. The general idea being to replace **C** by some unital algebra B (over **C**) the natural category is that of algebras over B, i.e., $\mathcal{A} \supset B$. The free product with amalgamation over B

$$\mathcal{A}_1 *_B \mathcal{A}_2$$

is defined as the categorical direct sum.

States will be linear maps $\mathcal{A} \xrightarrow{\varphi} B$ which are $B - B$-bimodule maps and such that $\varphi|B = \mathrm{id}_B$. For C^*-algebras we require, additionally, that φ be completely positive.

Let (A, φ) be an algebra over B with specified state and let $B \subset \mathcal{A}_j \subset \mathcal{A}, j = 1, 2$, be subalgebras. Then the pair $\mathcal{A}_1, \mathcal{A}_2$ is free in $(\mathcal{A}, \varphi)$ if

$$\varphi(a_1 a_2 \cdots a_n) = 0$$

whenever $a_j \in \mathcal{A}_{i_j}$ with $i_1 \neq i_2 \neq \cdots \neq i_n$ and $\varphi(a_j) = 0$ for $1 \leq j \leq n$. A pair of random variables $a_1, a_2 \in \mathcal{A}$ is called free if the subalgebras of $\mathcal{A}$ generated by $B \cup \{a_1\}$ and $B \cup \{a_2\}$ form a free pair of subalgebras in $(\mathcal{A}, \varphi)$.

EXAMPLES 5.2. (a) Let $G_j \supset H, j = 1, 2$, be groups and let $G = G_1 *_H G_2$. If $B = C^*_{\mathrm{red}}(H), \mathcal{A} = C^*_{\mathrm{red}}(G)$ and $\mathcal{A}_j = C^*_{\mathrm{red}}(G_j)$ we have $B \subset \mathcal{A}_j \subset \mathcal{A}$ and there is a canonical conditional expectation φ of $\mathcal{A}$ onto B. The pair $\mathcal{A}_1, \mathcal{A}_2$ is free in $(\mathcal{A}, \varphi)$.

(b) Let G_1, G_2 be groups and $G = G_1 * G_2$. Let $\mathcal{A} = C^*_{\text{red}}(G) \otimes B, \mathcal{A}_j = bfC^*_{\text{red}}(G_j) \otimes B$ where the tensor products are spatial tensor products, and let $\varphi = \tau \otimes \text{id}_B$ where τ is the canonical trace on $C^*_{\text{red}}(G)$. Then $\mathcal{A}_1, \mathcal{A}_2$ is a free pair of subalgebras in $(\mathcal{A}, \varphi)$.

5.3. In the context of algebras over B, let (A, φ) be an algebra with specified state, and let $a \in A$ be a random variable. Quantities of the form $\varphi(ab_1ab_2 \cdots b_{n-1}a), b_j \in B$, are called moments of a. Correspondingly the distribution of a is a state $\mu_a : B\langle X\rangle \to B$ defined by $\mu_a = \varphi \circ \chi_a$, where $B\langle X\rangle$ is the algebra freely generated by B and an indeterminate X and $\chi_a : B\langle X\rangle \to \mathcal{A}$ is the unique homomorphism such that $\chi_a(b) = b$ for $b \in B$ and $\chi_a(X) = a$.

Moments of the form $\varphi(abab \cdots ba) = \mu_a(XbX \cdots bX)$ are called symmetric moments. The $B - B$-submodule of $B\langle X\rangle$ generated by the symmetric monomials $XbX \cdots bX$ and 1 will be denoted by $SB\langle X\rangle$. The restriction S_{μ_a} of μ_a to $SB\langle X\rangle$ is called the symmetric distribution of a. The set of states of $B\langle X\rangle$ is denoted by $\sum_B$, and the set of $B - B$-bimodule maps $SB\langle X\rangle \to B$, the identities on B, is denoted by $S\sum_B$.

5.4. As in the scalar case, and also the "B-valued" situation, if $B \subset \mathcal{A}_j \subset \mathcal{A}, j = 1, 2$, is a free pair of subalgebras in $(\mathcal{A}, \varphi)$ then φ is completely determined by its restrictions $\varphi|\mathcal{A}_j, j = 1, 2$. As in the scalar case there are operations $\boxplus, \boxtimes$ on $\sum_B$ so that $\mu_{a_1} \boxplus \mu_{a_2} = \mu_{a_1+a_2}$, $\mu_{a_1} \boxtimes \mu_{a_2} = \mu_{a_1a_2}$ if a_1, a_2 is a pair of free random variables in $(\mathcal{A}, \varphi)$. An important feature of these operations is that the symmetric parts of $\mu_1 \boxplus \mu_2$ and $\mu_1 \boxtimes \mu_2$ depend only on the symmetric parts of μ_1 and μ_2. Hence there are also operations $\boxplus$ and $\boxtimes$ on $S\sum_B$ and the canonical map $\sum_B \to S\sum_B$ is a homomorphism both for $\boxplus$ and $\boxtimes$.

5.5. There is an analogue of Theorem 4.5 for the operation $\boxplus$ in the B-valued case on symmetric distributions. In order to avoid complications with formal power series we will assume we are working with Banach algebras and continuous states. Roughly, the theorem computing $\boxplus$ looks just like Theorem 4.5 with complex functions on $\mathbf{C}$ replaced by B-valued functions on B.

THEOREM. *If μ is the symmetric distribution of a random variable a in $(\mathcal{A}, \varphi)$ let*

$$G_\mu(b^{-1}) = \varphi((b-a)^{-1}) = b^{-1} + \sum_{n \geq 1} \mu(b^{-1}(Xb^{-1})^n)$$

defined for $b \in B$ invertible and $||b^{-1}||$ sufficiently small. Let further L be such that

$$G_\mu(L(b)) = L(G_\mu(b)) = b$$

and define $\mathcal{R}_\mu(b)$ by

$$(L(b))^{-1} = b^{-1} + \mathcal{R}_\mu(b)$$

if b is invertible and $||b||$ sufficiently small. If μ_1, μ_2, μ_3 are symmetric distributions of random variables and $\mu_3 = \mu_1 \boxplus \mu_2$ then we have

$$\mathcal{R}_{\mu_3} = \mathcal{R}_{\mu_1} + \mathcal{R}_{\mu_2}.$$

5.6. The operation $\boxtimes$ is not commutative in general. It can be computed using differential equations.

5.7. The framework of dual algebraic structures has a natural extension to the case of free products with amalgamation over B.

5.8. Theorem 5.5 is a quite general device for computations of spectra in reduced free products. As an example we shall sketch the application to convolution operators on free groups. Let $G = \mathbf{Z} * \mathbf{Z}$ and let ρ be the regular representation of G on $\ell^2(G)$ and

$$T = \sum c_g \rho(g) \in C^*_{\mathrm{red}}(G)$$

where the sum is finite. The computation of spectra of such operators is equivalent to deciding whether such an operator is invertible. By a simple matrix trick the invertibility of T is equivalent to the invertibility of an operator of the form

$$\begin{aligned} Y = {} & \alpha_{-1} \otimes \rho(g_1^{-1}) + \alpha_1 \otimes \rho(g_1) + \nu \otimes 1 \\ & + \beta_{-1} \otimes \rho(g_2^{-1}) + \beta_1 \otimes \rho(g_1) \in M_n \otimes C^*_{\mathrm{red}}(G), \end{aligned}$$

where M_n denotes the $n \times n$ complex matrices and g_1, g_2 are the generators. Replacing Y by

$$\begin{pmatrix} 0 & Y^* \\ Y & 0 \end{pmatrix},$$

we may actually assume that Y is self-adjoint at the expense of replacing $n \times n$ matrices by $2n \times 2n$ matrices. If Y is self-adjoint, the invertibility of Y is equivalent to the fact that $\varphi((zI - Y)^{-1})$, where $\varphi : M_n \otimes C^*_{\text{red}}(G) \to M_n$ is as in 5.2(b), can be analytically continued from the upper half-plane to a neighborhood of 0. In turn this can be decided using Theorem 5.5 which computes $\varphi((b \otimes I - Y)^{-1})$, where $b \in M_n$ (this is the function G for Y).

NOTES. §1 is based on [**22**]. For Cuntz algebras and their extensions, in connection with 1.10, see [**9, 16, 17, 10**].

§2 is based on material in [**22**]. For the usual Gaussian functor in 2.3 see, for instance, [**21**].

§3 is based on [**22**] and [**25**]. For the analogues of the classical matrix groups see [**25**] (the C^*-algebra for the analogue of the unitary group, without the dual group structure, first appears in [**6**]).

§4 is based on [**22, 23, 24**]. For other work related to spectral computations referred to in 4.1 see [**1, 3, 4, 7, 8, 11, 12, 13, 14, 18, 19, 20**]. Example 4.6 has been computed with other methods by several authors, the first such computation seems to be in [**14**]. For complete details in the computation in 4.10 see [**24**] (for a different approach see [**3**]). A reference for the moment problem is [**2**].

§5 is based on [**26**]. For free products of Hilbert modules over C^*-algebras see [**22**]. Concerning 5.8 we would like to mention a different general approach to computations of spectra of convolution operators on free groups in [**4**].

REFERENCES

1. C.A. Akemann and P.A. Ostrand, *Computing norms in group C^*-algebras,* Amer. J. Math. **98** (1976), 1015–1047.

2. J.H. Akhiezer, *The Classical Moment Problem,* Moscow, 1961.

3. J.H. Anderson, B. Blackadar and U. Haagerup, *Minimal projections in Choi's algebra*, to appear.

4. K.Aomoto, *Spectral theory on a free group and algebraic curves,* J. Fac. Sci. Univ. Tokyo, Sect. I.A., Math. **31** (1984), 297–318.

5. W.B. Arveson, *The harmonic analysis of automorphism groups,* in *Operator Algebras and Applications* (ed. R.V. Kadison), Proc. Symp. Pure Math. **38**, Amer. Math. Soc., Providence, 1981.

6. L.G. Brown, *Ext of certain free product C^*-algebras,* J. Operator Theory **6** (1981), 135–141.

7. P.Cartier, *Harmonic analysis on trees,* Proc. Symposia in Pure Math., Vol. 26, Amer. Math. Soc. 1973, 419–424.

8. J.M. Cohen, *Radial Functions on free products,* J. Funct. Anal. **59** (1984), 167–174.

9. J. Cuntz, *Simple C^*-algebras generated by isometries,* Comm. Math. Phys. **57** (1977), 173–185.

10. D.E. Evans, *On O_n*, Publ. Res. Inst. Math. Sci. **16** (1980), 915–927.

11. J. Farant and M.A. Picardello, *The Planchard measure for symmetric graphs,* Ann. Mat. Pura Appl. (IV), vol. CXXXVIII (1984), 151–155.

12. A. Figa-Talamanca and T. Steger, *Harmonic analysis for anisotropic random walks on homogeneous trees,* preprint.

13. A. Figa-Talamanca and M.A. Picardello, *Harmonic Analysis on Free Groups,* Lecture Notes in Pure and Applied Math., Vol. 87, Marcel Dekker, New York, 1983.

14. H. Kesten, *Symmetric random walks on groups,* Trans. Amer. Math. Soc. **92** (1959), 336–359.

15. A.M. Mantero and A. Zappa, *The Poisson transform on free groups and uniformly bounded representations,* J. Funct. Anal. **51** (1983), 372–399.

16. W. Paschke and N. Salinas, *Matrix algebras over O_n,* Michigan Math. J. **26** (1979), 3–12.

17. M. Pimsner and S. Popa, *The Ext-groups of some C^*-algebras considered by J. Cuntz,* Rev. Rommaine Math. Pures Appl. **23** (1978), 1069–1076.

18. M. Picardello and W.Woess, *Random walks on amalgams,* Monatsh. Math. **100** (1985), 21–33.

19. T. Pytlik, *Radial functions on free groups and a decomposition of the regular representation into irreducible components,* J. Reine Angew. Math. **326** (1981), 124–135.

20. S. Sawyer, *Isotropic random walks in a tree,* Zeitschr. f. Wahrsch. **42** (1978), 279–292.

21. B. Simon, *The $P(\varphi)_2$ Euclidean* (*Quantum*) *Field Theory,* Princeton University Press, Princeton, 1974.

22. D. Voiculescu, *Symmetries of some reduced free product C^*-algebras,* in *Operator Algebras and Their Connections with Topology and Ergodic Theory,* Lecture Notes in Math., Vol. 1132, Springer-Verlag, 1985, 556–588.

23. ———, *Addition of certain non commuting random variables,* J. Funct. Anal., **66** (1986), 323–346.

24. ———, *Multiplication of certain non-commuting random variables,* J. Operator Theory, to appear.

25. ———, *Dual algebraic structures on operator algebras related to free products,* J. Operator Theory **17** (1987), 85–98.

26. ———, *Operations on certain non-commutative operator-valued random variables,* (preliminary version) INCREST Preprint No. 42/1986.

Department of Mathematics, University of California, Berkeley, CA 94720