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PERIODIC PERTURBATIONS OF LINEAR PROBLEMS AT RESONANCE ON CONVEX DOMAINS

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ABSTRACT. We consider Dirichlet problems for semilinear elliptic equations whose nonlinear term is periodic and whose linear part is resonance. We show that such problems have infinitely many positive and infinitely many negative solutions on domains in the plane which are convex. The arguments used do not carry over to dimension greater than three. This work complements some earlier work of ours.

1. Introduction. Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary. As is well known, the principle eigenvalue λ_1 of the Dirichlet problem

$$\begin{aligned} \Delta u + \lambda u = 0, &\quad x \in \Omega, \\ u = 0, &\quad x \in \partial\Omega, \end{aligned} \tag{1}$$

is simple and has an associated eigenfunction ϕ with the properties

$$\phi(x) > 0, \quad x \in \Omega, \quad \frac{\partial \phi(x)}{\partial \nu} < 0, \quad x \in \partial\Omega,$$

where $\partial/\partial\nu$ is the exterior normal derivative to $\partial\Omega$. (We normalize ϕ so that $\phi_{\max} = 1$.)

In this paper we consider the *resonant* nonlinear problem

$$\begin{aligned} \Delta u + \lambda_1 u + g(u) = h(x), &\quad x \in \Omega, \\ u = 0, &\quad x \in \partial\Omega, \end{aligned} \tag{2}$$

where $h : \overline{\Omega} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are Hölder continuous functions and satisfy

$$\int_\Omega h\phi \, dx = 0, \tag{3}$$

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$$g(s+T) = g(s), \quad -\infty < s < \infty, \qquad \int_0^T g(s)\,ds = 0,\ g \neq 0, \tag{4}$$

where T is the period of g.

Problems of this type have been considered in our earlier work [**5**] and [**10**], where we have shown that such problems have infinitely many positive and infinitely many negative solutions in case x is a one-dimensional variable and also in higher dimensions, whenever Ω is an annular domain whose inner and outer radii satisfy certain restrictions. In the case where x is a two-dimensional variable, we were also able to show in [**5**] that the result holds whenever Ω is a disc. Numerical experiments [**5**] indicate that the latter result does not hold for Ω a ball in dimensions greater than 3. For more numerical experiments which support this conjecture, see [**11**].

This paper complements our work [**5**] and [**10**] cited above. We show that our earlier approach and a somewhat more intricate analysis allow us to obtain results about the existence of infinitely many solutions in the case of two space dimensions and for domains Ω which are convex or more generally are such that the eigenfunction ϕ satisfies certain geometric properties. Again our method of proof does not work in higher dimensions.

Our method of attack is to embed problem (2) into the one parameter family of problems

$$\begin{aligned} \Delta u + \lambda u + g(u) = h(x), &\quad x \in \Omega \\ u = 0, &\quad x \in \partial\Omega. \end{aligned} \tag{5}$$

We then employ bifurcation and continuation techniques to study the solution set of (5) and then consider λ_1-sections of this solution set to obtain the desired result.

To make this paper somewhat self-contained, we state the necessary tools from bifurcation theory in the next section.

E. N. Dancer has pointed out to us that, in his paper [**6**], he has obtained results very similar to ours by studying the asymptotic behavior of certain integrals. We thank him for pointing out his paper to us.

2. On bifurcation from infinity. In this section we shall state an abstract result about bifurcation from infinity which we shall need in our discussion. We refer to [**10**] for proofs (see also [**8**] and [**9**]).

Let X be a real Banach space with norm $\|\cdot\|$. We consider the equation

$$u = K(\lambda)u + k(\lambda, u), \quad u \in X, \tag{6}$$

where $K : [a, b] \subset R \to \mathcal{B}(X)$ is a differentiable family of compact linear operators on X and $k : [a, b] \times X \to X$ is a completely continuous mapping satisfying

$$\frac{k(\lambda, u)}{\|u\|} \to 0 \quad \text{as } \|u\| \to \infty, \tag{7}$$

uniformly on $[a, b]$.

In this setting we have

LEMMA 1. *Let $\lambda_1 \in (a, b)$ be such that*

$$\ker(\text{id} - K(\lambda_1)) = \operatorname{span}\phi, \quad \|\phi\| = 1, \tag{8}$$

$$K'(\lambda_1)\phi \notin \operatorname{range}(\text{id} - K(\lambda_1)), \tag{9}$$

and $P \subset X$ is an open cone containing ϕ. Then there exists ϵ_0 and a continuum (i.e., a closed, connected set) $\mathcal{C} \subset [a, b] \times P$ of solutions of (6) *with the property that, for any $0 < \epsilon < \epsilon_0$, we can find a subcontinuum $\mathcal{C}_\epsilon \subset \mathcal{C}$ such that*

$$\mathcal{C}_\epsilon \subset \mathcal{U}_\epsilon := \{(\lambda, u) : |\lambda - \lambda_1| < \epsilon, \ \|u\| > 1/\epsilon\},$$

and $\mathcal{C}_\epsilon$ connects (λ_1, ∞) to $\partial\mathcal{U}_\epsilon$. Moreover, if $\{(\lambda_n, u_n)\} \subset \mathcal{C} \cap \mathcal{U}_\epsilon$ is such that $\|u_n\| \to \infty$, then

$$\lambda_n \to \lambda_1 \ \text{and} \ \frac{u_n}{\|u_n\|} \to \phi. \tag{10}$$

COROLLARY 2. *Let the assumptions of Lemma* 1 *hold and assume that $K(\lambda)$, $k(\lambda, \cdot)$ map X continuously into a Banach space $Y \subset X$*

which is compactly embedded in X and that $K : [a,b] \to \mathcal{B}(X,Y)$, $k : [a,b] \times X \to Y$ are continuous with

$$\frac{k(\lambda,u)}{\|u\|} \to 0, \qquad \text{in } Y, \text{ as } \|u\| \to \infty, \tag{11}$$

uniformly on $[a,b]$. Then, if $\{(\lambda_n,u_n)\} \subset \mathcal{C} \cap \mathcal{U}_{\epsilon_0}$ is such that $\|u_n\| \to \infty$, we get

$$\lambda_n \to \lambda_1 \quad \text{and} \quad \left\| \frac{u_n}{\|u_n\|} - \phi \right\|_Y \to 0. \tag{12}$$

In particular, if $\tilde{P} \subset Y$ is any open cone containing ϕ, then, by decreasing $\epsilon_0 > 0$ if necessary, we obtain that

$$\mathcal{C} \subset [a,b] \times \tilde{P}. \tag{13}$$

3. The semilinear problem. We shall now consider the resonant Dirichlet problem (2) given in the introduction with all terms satisfying the hypotheses stated there. We embed (2) into the one parameter problem

$$\begin{aligned} \Delta u + \lambda u + g(u) &= h(x), \quad x \in \Omega \\ u &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{14}$$

Let $K : C(\overline{\Omega}) \to C(\overline{\Omega})$ denote the operator defined by $Kf = u$ if and only if u solves

$$\begin{aligned} \Delta u &= f, \quad x \in \Omega \\ u &= 0, \quad x \in \partial\Omega. \end{aligned}$$

As is well known, K is a bounded linear operator from $C(\overline{\Omega})$ to $C_0^1(\overline{\Omega})$, and, hence

$$K : C(\overline{\Omega}) \to C(\overline{\Omega})$$

is compact. Further, by regularity theory, we have

$$K : C^\mu(\overline{\Omega}) \to C_0^{2+\mu}(\overline{\Omega}),$$

continuously. Our problem (14) is hence equivalent to the operator equation

$$u = \lambda K u + K(g(u) + h), \tag{15}$$

in the space $X = C_0(\overline{\Omega})$. We may hence apply Lemma 1 and Corollary 2 with $K(\lambda) = \lambda K$ and $k(\lambda, u) = K(g(u) + h)$, and, letting $Y = C_0^1(\overline{\Omega})$, $P = \{u \in X : \int_\Omega u\phi\, dx > 0\}$, $\tilde{P} = \{u \in Y : u > 0$, in Ω, $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega\}$. We hence obtain $\epsilon_0 > 0$ and a continuum $\mathcal{C} \subset R \times \tilde{P}$ of solutions of (15) such that $\mathcal{C} \cap \mathcal{U}_\epsilon \neq \varnothing$, for any $0 < \epsilon \leq \epsilon_0$, and such that if $(\lambda_n, u_n) \in \mathcal{C}$, with $|\lambda_n - \lambda_1| < \epsilon_0$ and $\|u_n\| = \max |u_n| \to \infty$, then

$$\lambda_n \to \lambda_1 \quad \text{and} \quad \frac{u_n}{\max u_n} \to \phi, \quad \text{in } C_0^1(\overline{\Omega}).$$

In fact, regularity theory and arguments as used in Corollary 2 imply

$$\frac{u_n}{\max u_n} \to \phi \quad \text{in } C_0^{2+\mu}(\overline{\Omega}). \tag{16}$$

We may now prove our main result.

THEOREM 3. *Consider the boundary value problem* (2), *where* Ω *is a convex bounded domain in* $\mathbf{R}^2$ *and* λ_1 *is the principle eigenvalue of* (1). *Further, let* g *and* h *be Hölder continuous and satisfy* (3) *and* (4). *Then the problem* (2) *has an infinite number of solutions* $\{u_n\}_{n=1}^\infty \subset C_0^{2+\mu}(\overline{\Omega})$, *with* $u_n > 0$ *in* Ω, $\partial u_n/\partial\nu < 0$ *on* $\partial\Omega$ *and such that* $\max u_n \to \infty$ *and* $u_n/\max u_n \to \phi$ *in* $C_0^{2+\mu}(\overline{\Omega})$, *as* $n \to \infty$. *Also, there exist infinitely many negative solutions with similar properties.*

PROOF. Since most of the arguments to follow are valid in arbitrary dimensions, we proceed with the general case until it becomes necessary to restrict the dimension to the case $n = 2$. We embed (2) into the one parameter problem (14) and use the setup discussed before the statement of the theorem. If $(\lambda, u) \in \mathcal{C}_\epsilon$ we multiply (14) by ϕ and integrate by parts to obtain

$$(\lambda_1 - \lambda) \int_\Omega u\phi\, dx = \int_\Omega g(u)\phi\, dx. \tag{17}$$

Since $u \in P$ it follows that the right-hand side of (17) determines the sign of $\lambda_1 - \lambda$.

Let $\|u\| = \max u$ denote the norm in the space X and, instead of (17), we shall consider

$$\|u\|^2(\lambda_1 - \lambda)\int_\Omega \frac{u}{\|u\|}\phi\,dx = \|u\|\int_\Omega g(u)\phi\,dx \tag{18}$$

and determine the sign of that quantity for large $\|u\|$. Let us now consider a sequence of solutions $\{(u_k, \lambda_k)\} \subset \mathcal{C}_\epsilon$ with

$$\|u_k\| = a_k + kT, \quad 0 \le a_k \le T,\ k \ge 2,$$

where a_k will be chosen appropriately and T is the period of g.

We let

$$v_k = \frac{u_k}{\|u_k\|},$$

and recall that $v_k \to \phi$ in $C_0^{2+\mu}(\overline{\Omega})$.

Since Ω is assumed convex it follows from a result in [**2, 7**] that $\nabla\phi$ only vanishes at a single point, where ϕ assumes its maximum and $D^2\phi$ is negative definite there. Therefore, the same will be true for v_k for all sufficiently large k. (This is the only consequence of convexity which is needed in our discussion, and hence we could replace the convexity assumption by this implication as an assumption, certainly a somewhat less restrictive requirement. Certain types of symmetry conditions on that domain, as used in [**4**], for example, will also be sufficient.) We now use the co-area formula (see [**1**] or [**3**]) and find that

$$\|u_k\|\int_\Omega g(u_k)\phi\,dx = \|u_k\|\int_0^{\|u_k\|} g(t)\int_{u_k=t}\frac{\phi}{|\nabla u_k|}\,dS_t\,dt, \tag{19}$$

where dS_t denotes the Riemannian $n-1$-density on the level sets $\{u_k = t\}$. The latter may be rewritten as

$$\int_0^{\|u_k\|} g(t)\int_{v_k=t/\|u_k\|}\frac{\phi}{|\nabla v_k|}\,dS_t\,dt. \tag{20}$$

If we define

$$f_k(s) = \int_{v_k=s}\frac{\phi}{|\nabla v_k|}\,dS_s,$$

then (20) becomes

$$\int_0^{\|u_k\|} g(t) f_k\left(\frac{t}{\|u_k\|}\right) dt. \tag{21}$$

The latter integral we now write as the sum of the integrals

$$\int_0^{kT} g(t) f_k\left(\frac{t}{\|u_k\|}\right) dt = I_1 \tag{22}$$

and

$$\int_{kT}^{a_k+kT} g(t) f_k\left(\frac{t}{\|u_k\|}\right) dt = I_2. \tag{23}$$

We first consider the integral I_2. Using the periodicity of g we find that (23) may be rewritten as

$$I_2 = \int_0^{a_k} g(t) f_k\left(\frac{t+kT}{a_k+kT}\right) dt. \tag{24}$$

We next observe that each f_k for k sufficiently large will be of class $C^{1+\mu}$ on any given compact subinterval of $[0,1)$ (recall $\phi_{\max} = 1$), and we may conclude that

$$f_k \to f$$

in C^1 on any compact subinterval of $[0,1)$, where f is given by

$$f(s) = \int_{\phi=s} \frac{\phi}{|\nabla\phi|}\, dS_s.$$

It follows also from the nondegeneracy of v_k and ϕ at their maxima that f is continuous and that $f_k \to f$ in $C^0[0,1]$. From these observations it follows that we may pass to the limit in (24) and conclude that

$$I_2 \to \int_0^a g(z)\, dz\, f(1), \tag{25}$$

where a has been preassigned in $[0,T]$ and the sequence $\{a_k\}$ was chosen so that $a_k \to a$.

We next consider the integrals I_1. We first integrate by parts and obtain

$$(26)\qquad \begin{aligned} I_1 &= -\int_0^{kT} G(t)\frac{1}{a_k+kT} f_k'\left(\frac{t}{a_k+kT}\right) dt \\ &= -\int_0^{\frac{kT}{a_k+kT}} G(s(a_k+kT)) f_k'(s)\, ds, \end{aligned}$$

where

$$G(s) = \int_0^s g(t)\, dt$$

is periodic.

As $k \to \infty$, I_1 has the same limit as

$$(27)\qquad \begin{aligned} J_1 &= -\int_0^{kT} G(t)\frac{1}{a_k+kT} f'\left(\frac{t}{a_k+kT}\right) dt \\ &= -\int_0^{\frac{kT}{a_k+kT}} G(s(a_k+kT)) f'(s)\, ds, \end{aligned}$$

provided we can show

$$(28)\qquad \lim_{k\to\infty} \int_0^{\frac{kT}{a_k+kT}} |f_k'(s) - f'(s)|\, ds = 0.$$

Since, in case $n = 2$, the convergence of the sequence $\{f_k\}$ in the norm of $H^{1,1}$ is not obvious, we prove (28) in the appendix.

We next use the periodicity of G to rewrite (27) as

$$(29)\qquad -\frac{1}{T}\int_0^T G(t) \sum_{j=1}^{k} \frac{T}{a_k+kT} f'\left(\frac{t+(j-1)T}{a_k+kT}\right) dt.$$

Letting $k \to \infty$ in (29) obtains

$$-\frac{1}{T}\int_0^T G(t) \int_0^1 f'(\tau)\, d\tau\, dt,$$

which equals

$$-\frac{1}{T}\int_0^T G(t)(f(1) - f(0))\, dt.$$

We hence have that

$$
\begin{aligned}
I_1 + I_2 &\to f(1)\left\{\int_0^a g(s)\,ds - \frac{1}{T}\int_0^T G(s)\,ds\right\} \\
&= f(1)[G(a) - \overline{G}],
\end{aligned} \tag{30}
$$

where $\overline{G}$ is the mean value of G, since $f(0) = 0$. We, therefore, may, once we know that $f(1) \neq 0$, determine the sign of $\lambda - \lambda_1$ by examining the sign of $G(a) - \overline{G}$. On the other hand, as a varies from 0 to T, the function $G(a) - \overline{G}$ will change sign and we will consequently find infinitely many positive solutions of (2) on the continuum $\mathcal{C}_\epsilon$.

To determine whether $f(1) \neq 0$ we proceed as follows. Since $\phi(x) > 0$, $x \in \Omega$, it suffices to consider the function

$$
q(s) = \int_{\phi = s} \frac{1}{|\nabla \phi|}\,dS_s,
$$

since $f(s) = sq(s)$. That $q(1) > 0$, for $n = 2$, and $q(1) = 0$, for $n \geq 3$, follows immediately from the fact that $D^2\phi$ is negative definite, where $\phi(x) = 1$. But the result, for $n = 2$, also holds without this assumption.

We introduce the notation

$$
A(s) = \int_{\phi = s} dS_s
$$

and

$$
V(s) = \int_{\phi \geq s} dx.
$$

The *isoperimetric* inequality (see [**1**]) states that

$$
A^2(s) \geq n^2 \omega_n^{\frac{2}{n}} V^{2-2/n}(s),
$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$. Hence, we obtain

$$\begin{aligned} n^2\omega_n^{2/n}V^{2-2/n}(s) &\le A^2(s) \\ &= \left(\int_{\phi=s} dS_s\right)^2 \\ &\le \int_{\phi=s} \frac{1}{|\nabla\phi|}\,dS_s \int_{\phi=s} |\nabla\phi|\,dS_s \\ &= q(s)\int_{\phi>s} -\nabla\cdot\nabla\phi\,dx \\ &= \lambda_1 q(s)\int_{\phi>s}\phi\,dx \\ &\le \lambda_1 q(s)\int_{\phi>s}\phi_{\max}\,dx \\ &\le \lambda_1 q(s)V(s). \end{aligned} \tag{31}$$

From (31) it follows that

$$n^2\omega_n^{2/n}V^{1-2/n}(s) \le \lambda_1 q(s), \tag{32}$$

which, in case $n=2$, implies

$$\frac{4\pi}{\lambda_1} \le q(s). \tag{33}$$

Inequality (33) implies that $q(1) > 0$, which implies the desired result.

Since $f(1)=0$, for $n \ge 3$ and convex domains, we can only conclude in this case that

$$\|u\|^2(\lambda_1-\lambda)\to 0,$$

as $\|u\|\to\infty$, as an estimate for the order of convergence as $\lambda\to\lambda_1$.

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Appendix

Here we shall give a proof of (28).

If $n \geq 3$ it follows immediately that $f_k \to f$ in the norm of $H^{1,1}[0,1]$, since the measure of the level sets decreases fast enough (see (32)). We next consider the case that $n = 2$. If we can show that

$$\ln(a_k + kT)(1-s)|f_k'(s) - f'(s)| \to 0 \tag{34}$$

uniformly for $s \in [0,1]$, as $k \to \infty$, then (28) results from the computations to follow.

Choose $\epsilon > 0$ small. Then, for all large enough k, we obtain, from (34),

$$\begin{aligned}\int_0^{\frac{kT}{a_k+kT}} |f_k'(s) - f'(s)|\, ds &\leq \frac{1}{\ln(a_k+kT)} \frac{\epsilon}{2} \int_0^{\frac{kT}{a_k+kT}} \frac{1}{1-s}\, ds \\ &= \frac{\epsilon}{2}\left(1 - \frac{\ln(a_k)}{\ln(a_k+kT)}\right) < \epsilon.\end{aligned}$$

Hence, it suffices to prove (34). Without loss, we may consider the functions

$$q_k(s) = \int_{v_k=s} \frac{1}{|\nabla v_k|}\, dS_s, \quad q(s) = \int_{\phi=s} \frac{1}{|\nabla \phi|}\, dS_s,$$

instead of the functions f_k, and f. Then

$$(1-s)q_k'(s) = \int_{v_k=s} \frac{1-v_k}{|\nabla v_k|^3}(\Delta v_k - 2\partial_\nu^2 v_k)\, dS_s, \tag{35}$$

with

$$\partial_\nu^2 v_k = \nu^T D^2 v_k \nu$$

and

$$\nu = -\frac{\nabla v_k}{|\nabla v_k|},$$

the outward normal. Then $(1-s)q_k'(s)$ converges uniformly on $[0,1]$ to $(1-s)q'(s)$, since $v_k \to \phi$ in $C_0^{2+\mu}$. (Note that $(1-v_k)/|\nabla v_k|^2$ is well behaved as $s \to 1$ since $D^2 v_k$ is negative definite where $v_k = 1$.)

If we can show that

$$\ln(\|u_k\|)(v_k - \phi) \to 0, \tag{36}$$

as $k \to \infty$ in $C_0^{2+\mu}$, then (34) will follow in a similar manner.

To prove (36), we write $u_k = r_k\phi + w_k$, with $\int_\Omega \phi w_k \, dx = 0$. Then (5) becomes

$$\Delta w_k + \lambda_1 w_k = h - g(u_k) + (\lambda_k - \lambda_1)u_k. \tag{37}$$

It follows from (18) that $\|u_k\|(\lambda_k - \lambda_1)$ is bounded and, hence, that the right-hand side of (37) is bounded in C^μ. Thus, the sequence $\{w_k\}$ is bounded in $C_0^{2+\mu}$ by, say c, since w_k belongs to the orthogonal complement of $\operatorname{span}\phi$. Thus, if x_0 is such that $\phi(x_0) = 1$, it follows that

$$r_k - c \le r_k\phi(x_0) + w_k(x_0) \le \|u_k\| \le r_k + \|w_k\| \le r_k + c.$$

Thus, $|\|u_k\| - r_k| \le c$. Hence,

$$\ln(\|u_k\|)(v_k - \phi) = \frac{\ln(\|u_k\|)}{\|u_k\|}(w_k + (r_k - \|u_k\|)\phi) \to 0$$

in $C_0^{2+\mu}$ as $k \to \infty$.

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