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A THEORETICAL JUSTIFICATION OF THE METHOD OF HARMONIC BALANCE FOR SYSTEMS WITH DISCONTINUITIES

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ABSTRACT. We prove a theorem which provides a rigorous justification of an intuitive method used by electrical engineers to predict the presence or absence of periodic oscillations in nonlinear systems. Although the literature contains some excellent discussions of the conditions under which the method can be rigorously justified, there are some oversights and there is lacking a completely detailed treatment, particularly for unforced discontinuous systems. By applying the theory of topological degree for differential inclusions, we are able to present a unified rigorous justification in full detail, and we can illustrate how our abstract hypotheses match up, point by point, with the standard hypotheses used by engineers.

1. Introduction. In this paper we investigate the existence of periodic solutions to differential inclusions of the form

$$\dot{z}(t) - Az(t) \in F[z(t)], \quad t \in [0, \infty), \tag{1}$$

where $z(t) \in \mathbf{R}^m$, A is a constant $m \times m$ matrix and $F : \mathbf{R}^m \to 2^{\mathbf{R}^m}$. Our principal tool will be the topological degree for upper semi-continuous, compact multivalued maps with compact, convex values (see Cellina-Lasota [**5**], Ma [**11**], Lloyd [**10**]). One of our principal motivations is the *method of harmonic balance*, of which a particular case is the "describing function method" used by electrical engineers. The central idea is to use a finite-dimensional approximation (via Fourier series) to the infinite-dimensional function space on which (1) is defined. Under certain conditions, the existence of a periodic solution to the approximation implies the existence of a periodic solution to

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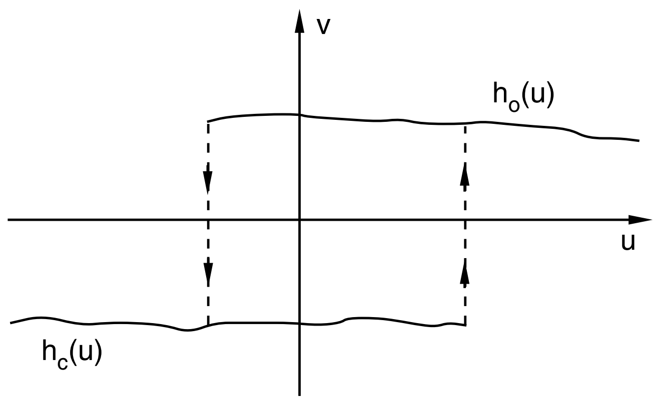

FIGURE 1.

the original problem. For an intuitive description of the method, with numerous examples, see Jordan and Smith [**7**] and Vidyasagar [**20**].

Our work has been motivated by that of Bergen et al. [**2**], Mees [**13**], Mees and Bergen [**14**], and Miller and Michel [**15, 16**]. In [**2**] and [**14**] there is a thorough discussion of computational procedures and error bounds, with a brief discussion of rigorous justification for unforced continuous problems; in [**15**] and [**16**] a method is presented for dealing with discontinuous forced problems via continuous approximations; in [**13**] Mees presents an excellent overview of the state-of-the-art with a brief mention that the theory of degree for relations provides a method for dealing with discontinuous systems of a certain type. However, there are some minor problems. In some of the cited papers it is claimed that problems involving hysteresis can be directly treated, and in [**13**] the same comment is made about toggles. In a problem with hysteresis, there are two difficulties which make the treatment more difficult. The first difficulty is that a problem with hysteresis has a *memory with threshold* (see Figure 1)—for a value of $u \in (\alpha, \beta)$,

the system remembers whether it has decreased from β or increased from α. The phenomena cannot be modelled by single-valued continuous approximations nor by set-valued functions (multifunctions). If, for example, one tries to use multifunctions to model $\dot{y} = f(y)$ by using $\dot{y} \in F(y)$, with $F(y) =$ "the interval from $h_c(y)$ to $h_o(y)$," then, in addition to the problem of memory, one has the possibility of nonphysical solutions—for example $\dot{y}$ might always be the midpoint of $F(y)$, which gives a trajectory in the interior of the region delineated by the hysteresis curves. If one attempts to circumvent the "interior solutions" problem by defining $F(y)$ to be the two isolated points $h_c(y), h_o(y)$, then one is faced with a dearth of positive results—and a wealth of counter examples—for differential inclusions $\dot{y} \in F(y)$ when $F(y)$ fails to be either convex-valued or acyclic, as in this case.

In another paper [**12**] we will show how our approach can be used to obtain theorems on periodic oscillations in systems with relay hysteresis. Skar and Miller [**18**] have earlier studied this problem, obtaining similar results by a different method. There is an extensive Russian language literature on hysteresis, in particular we refer the reader to the book of Krasnosel'skii and Pokrovskii [**8**] and the papers of Braverman, Meerkov and Pyatnitskii [**3, 4**].

In this paper we present a complete, detailed, rigorous justification of the harmonic balance method, using fixed point theory and degree theory for differential inclusions. This yields a unified theory which covers any discontinuities which can be modelled by convex-valued multifunctions, and offers an alternative to the approximation procedure of Michel-Miller for the case when the discontinuous terms are limits of continuous functions. The connection between their approach and ours is clear—the graph of a convex-valued multifunction, under certain reasonable hypotheses, can be approximated by the graphs of a sequence of single-valued continuous functions (in the Hausdorff metric). However, this sequence of approximations can be extremely difficult to determine in complicated high-dimensional problems. The theory of differential inclusions allows us to bypass this problem.

We mention that, in addition to the well-known application of differential inclusions in optimization and control, they are becoming an important tool in many other areas. As examples, we point out the areas of robotics [**17**] and variable structure systems [**1, 19**].

Finally, we point out that one major limitation of degree theory lies in its inability to directly prove non-existence results. If the degree of a mapping is zero, it could be due to the fact that there are several solutions whose degrees add to zero. In fact, the only nonexistence results presently available are for a class of scalar problems for which one can use Bergen's graphical approach on the describing function diagram.

2. The approximation method. To make our presentation more readable, we first give some background material. Consider the case of a system with a scalar state $x(t)$ which satisfies an integral equation of the form

$$(*)\qquad x(t) = \int_0^t g(t-\tau)n[x(\tau)]\,d\tau, \quad t \in [0,\infty),$$

with $g(\cdot), n[\cdot]$ well-behaved, and $n[\cdot]$ assumed odd, $g(\cdot)$ even for simplicity. If we seek T-periodic (T unknown) solutions of $(*)$ which (for simplicity) also satisfy $x(t+T/2) = -x(t), x(0) = 0$, then such solutions will have Fourier expansions of the form $\sum_{k=0}^{\infty} a_{2k+1}\sin(2k+1)\omega t, \omega = 2\pi/T$. If we approximate $x(t)$ by $a_1 \sin\omega t$, then since $n[\cdot]$ is odd, $n[a_1\sin\omega t]$ has a Fourier expansion $\sum_{k=0}^{\infty} b_{2k+1}\sin(2k+1)\omega t$. The *describing function* is $\eta(a_1,\omega) \equiv b_1(a_1,\omega)/a_1$. Now the idea is to equate the leading terms of the Fourier expansions (the first harmonic) of the left and right sides of $(*)$, obtaining an implicit relationship to be solved for (a_1,ω). The first harmonic of the left side is obviously $a_1\sin\omega t$. Since the right side is in convolution form, its first harmonic can be written $\hat{g}(i\omega)\eta(a_1,\omega)a_1\sin\omega t$, where $\hat{g}$ is the Fourier transform of $g, \hat{g}(i\omega) = (1/2\pi)\int_0^{2\pi} e^{-i\omega t}g(t)\,dt$. Equating these, we get

$$(\mathrm{H})\qquad -1 + \hat{g}(i\omega)\eta(a_1,\omega) = 0.$$

This is the *principle of harmonic balance* of order 1.

The equation (H) is often written $\eta(a_1,\omega) = -1/\hat{g}(i\omega)$. In many simple cases, η does not depend on ω, so one can graph the two curves $\eta(a_1), -1/\hat{g}(i\omega)$ in the complex plane. The points of intersection give solutions of (H). This is the basis for the now standard graphical procedures (cf. Bergen et al. [**2**]).

Under suitable conditions, if (H) has a solution $(\overline{a},\overline{\omega})$, then $(*)$ will have a periodic solution (in general of different period). One situation in which the use of (H) is plausible is when the linear operator $\mathcal{G}\,[y](t) = \int_0^t g(t-\tau)y(\tau)\,d\tau$ attenuates higher harmonics, i.e., it represents a low-pass filter. In mathematical terms, this means that

$\mathcal{G}[y](t)$ is very well approximated by $\mathcal{G}$ applied to the first harmonic of $y(\cdot)$. For a full treatment of the ideas presented above, we refer to Vidyasagar [**20**, pp. 96–112] and Jordan and Smith [**7**, pp. 111–123].

The replacement of a function by one or more initial terms from its Fourier expansion can be represented by a projection. If we use the complex Fourier series $y(t) = a_0 + \sum_{k=1}^{\infty}(a_k e^{i\omega t} + b_k e^{-i\omega kt})$ then we define, for example,

$$P_1 : y(\cdot) \mapsto a_0 + a_1 e^{i\omega t} + b_1 e^{-i\omega t},$$

and P_n can be defined analogously. The harmonic balance principle can then be efficiently summarized by the statement that, under certain conditions, the equation $(*)$ $x = \mathcal{G}[n(x)]$ will have a periodic solution if the equation

$$\text{(H)} \qquad P_1 x = P_1 \mathcal{G}[P_1 n(P_1 x)] = P_1 \mathcal{G}[n(P_1 x)]$$

has a periodic solution. The elimination of a P_1 was based on the fact that the convolution operator $\mathcal{G}$ will commute with the Fourier projection $P_n, n \geq 1$.

We can now formulate our problem in abstract terms. After this, we can precisely state our hypotheses within this abstract setting. Let $W^{1,2}([0, 2\pi], \mathbf{R}^m)$ be the usual Sobolev space of functions from $[0, 2\pi]$ into $\mathbf{R}^m$, and let Z be the subspace of functions $x(\cdot)$ which satisfy the boundary conditions $x(0) = x(2\pi)$, extended to $\mathbf{R}$ by 2π-periodicity. We note the well-known fact that $W^{1,2}([0, 2\pi], \mathbf{R}^m)$ is the set of absolutely continuous functions with derivatives in $L^2([0, 2\pi], \mathbf{R}^m)$. We will denote by X the Hilbert space of functions in $L^2([0, 2\pi], \mathbf{R}^m)$ extended to $\mathbf{R}$ by 2π-periodicity.

For each $x(\cdot) \in X$ the associated Fourier expansion

$$x(t) \to a_0 + \sum_1^{\infty}(a_k e^{ikt} + b_k e^{-ikt}),$$

converges in norm to $x(\cdot)$, and we can define the projection operator

$$P_n : X \to \operatorname{span}_{\mathbf{C}}\{1, e^{it}, e^{-it}, \ldots, e^{int}, e^{-int}\},$$
$$x(t) \mapsto a_0 + \sum_1^{n}(a_k e^{ikt} + b_k e^{-ikt}) \equiv P_n[x](t).$$

Of course the actual period $T = 2\pi/\omega$ of the sought-after solution of (1) is unknown. In order to use degree theory on the space X, we scale (1) so as to fix the period at 2π, which will introduce the unknown actual period as a parameter in the equation. If we define $x(t) = z(Tt/2\pi)$, then $z(\cdot)$ is a T-periodic solution of (1) if and only if $x(\cdot)$ is a 2π-periodic solution of

$$\omega\dot{x}(t) - Ax(t) \in \mathcal{F}[x](t), \quad 0 \le t < \infty, \tag{2}$$

where $\mathcal{F}$ denotes the Nemytskii operator, defined as all measurable selections $y(\cdot)$ from $\{F[x(t)] \mid 0 \le t < \infty\}$ which satisfy $y(t) = y(t+2\pi)$ for almost all $t \in [0, 2\pi]$. As usual, a solution of (2) is a function $x \in Z$ which satisfies (2) a.e. for some $\omega > 0$. We will assume that the operator $[\omega \frac{d}{dt} - A] : Z \to X$ is invertible, i.e., we avoid resonance. Then we can consider $\mathcal{G}_\omega : X \to Z$, the inverse of the previous operator, which can be explicitly written as follows:

$$\begin{gathered}\mathcal{G}_\omega[y](t) = \int_0^{2\pi} G_\omega(t-s)y(s)\,ds, \\ G_\omega(t-s) = \frac{1}{\omega}[I - e^{2\pi A/\omega}]^{-1} \begin{cases} e^{A(t-s)/\omega}, & 0 \le s < t, \\ e^{A(2\pi+t-s)/\omega}, & t \le s \le 2\pi. \end{cases}\end{gathered} \tag{3}$$

Now the problem of finding a solution of (2) can be rewritten as a fixed point problem in the space X. In fact, set $T_\omega = i \circ \mathcal{G}_\omega \circ \mathcal{F} : X \to 2^X$, where i is the natural imbedding of Z into X, and consider the equation

$$0 \in (I - T_\omega)[x]. \tag{4}$$

It is clear that $x \in X$ solves (4) if and only if $x \in Z$ and solves (2).

We note below (Remark 1) that $T_\omega : X \to 2^X$ is an upper semicontinuous, compact set-valued operator with closed convex values.

Our original inclusion (1) has now been replaced by an equivalent *family* of equations (4). The new family is defined on the space X and is parametrized by ω. If we define

$$X_n = P_n X \equiv \{P_n x \mid x \in X\}, \quad X^* = (I - P_n)X, \quad x_n(t) = P_n[x](t),$$

$x^*(t) = (I - P_n)[x](t)$ for $x \in X$, then we can write a finite-dimensional approximation

$$O_n \in (I - P_n T_\omega)[x_n], \quad x_n \in X_n, \quad O_n = P_n[O]. \tag{5}$$

To compare this approximation with the exact problem, we separate (4) into finite and infinite-dimensional parts by applying P_n and $I - P_n$ respectively:

$$\begin{aligned} &\text{(a)} \quad O_n \in P_n(I - T_\omega)[x_n + x^*], \\ &\text{(b)} \quad O^* \in (I - P_n)(I - T_\omega)[x_n + x^*]. \end{aligned} \tag{6}$$

We can then construct the homotopy

$$\begin{aligned} &\text{(a)} \quad O_n \in (I - P_n T_\omega)[x_n] \\ &\qquad\qquad + \lambda\{P_n(I - T_\omega) - (I - P_n T_\omega)P_n\}[x], \\ &\text{(b)} \quad O^* \in (I - P_n)(I - \lambda T_\omega)[x], \end{aligned} \tag{6λ}$$

for $x = x_n + x^*, 0 \le \lambda \le 1$. Unfortunately, the algebraic system (5) does not have isolated solutions because there are n complex m-vectors $a_1, a_2, \ldots, a_n$, one real n-vector a_0 and $\omega \in \mathbf{R}_+$ as unknowns, and only $n \cdot m$ complex equations and n real equations. This is not surprising, since our system (1) is autonomous, and so if $(a_0, a_1, \ldots, a_n)$ satisfies (5), for some $\omega > 0$, then also $(a_0, a_1 e^{i\theta}, \ldots, a_n e^{in\theta})$ satisfies (5) for arbitrary real θ. Therefore we have to fix the time origin; one way to do this is to add to (5) the condition that a nonzero component $a_{i_0 j_0}$ of some a_{i_0}, $1 \le i_0 \le n$, be real, i.e., $\arg a_{i_0 j_0} = 0$. This condition implies the choice of a particular solution of (5).

We take this fact into account by defining the appropriate operators as follows. For any $\lambda \in [0, 1]$, let $V_\lambda : \mathbf{R}_+ \times X_n \times X^* \to \mathbf{R} \times 2^{X_n} \times 2^{X^*}$ be the operator defined by $V_\lambda = (V_\lambda^a, V_\lambda^b)$, where

$$V_\lambda^a : \mathbf{R}_+ \times X_n \times X^* \to \mathbf{R} \times X_n \quad \text{and} \quad V_\lambda^b : \mathbf{R}_+ \times X_n \times X^* \to X^*$$

are given by

$$\begin{aligned} V_\lambda^a[\omega, x_n, x^*] &= (\arg a_{i_0 j_0}(\omega, x_n), (I - P_n T_\omega)[x_n] \\ &\qquad + \lambda\{P_n(I - T_\omega) - (I - P_n T_\omega)P_n\}[x_n + x^*]), \\ V_\lambda^b[\omega, x_n, x^*] &= (I - P_n)(I - \lambda T_\omega)[x_n + x^*]. \end{aligned}$$

Clearly, $\{V_\lambda\}$, for $\lambda \in [0,1]$, is a family (homotopy) of compact vector fields with closed convex values (i.e., a family of compact, closed convex-valued, upper semicontinuous perturbations of the identity), see Remark 1 below. To see this it suffices to add and subtract the identity on $\mathbf{R}_+ \times X$ from V_λ^a, for any $\lambda \in [0,1]$. In the sequel we will denote by (H_λ) the equations

$$(\mathrm{H}_\lambda) \qquad \text{(a)} \quad O \in V_\lambda^a[\omega, x_n, x^*], \qquad \text{(b)} \quad O \in V_\lambda^b[\omega, x_n, x^*].$$

We make the following assumptions. Here, $||x||_2$ and $||x||_\infty$ denote respectively the $L^2([0,2\pi], \mathbf{R}^m)$ and the sup norm.

(A1) $F : \mathbf{R}^m \to 2^{\mathbf{R}^m}$ in (1) is upper semi-continuous and satisfies

$$\sup_{y\in F(x)} |y| \equiv |F(x)| \le \alpha|x| + \beta \qquad \text{for some} \quad \alpha > 0, \quad \text{and} \quad \beta \ge 0.$$

In addition, $0 \in F(0)$ and $F(x)$ is a nonempty convex, compact set for all $x \in \mathbf{R}^m$.

(A2) Let $A_n \subset \mathbf{R}_+ \times X_n$ be the open set satisfying the following assumptions.

(h_1) For all $(\omega, x_n) \in A_n$, there exists $r_1 = r_1(\omega, x_n) > 0$ such that

$$\left[\sum_{|k|>n} |\hat{G}_\omega(ik)|^2\right]^{1/2} \sup_{||x^*||_2<r_1} \quad \sup_{y\in\mathcal{F}\,[x_n+x^*]} ||(I-P_n)y||_2 < r_1,$$

where $\hat{G}_\omega(ik)$ is the Fourier transform of the matrix $G_\omega(t)$ evaluated at ik, $k = \pm(n+1), \pm(n+2), \dots$.

(h_2) For all $(\omega, x_n) \in A_n$, we have $||(I - P_nT_\omega)[x_n]||_2 < \sigma(\omega, x_n)$, where

$$0 < \sigma(\omega, x_n) = \left[\sum_{|k|=0}^{n} |\hat{G}_\omega(ik)|^2\right]^{1/2} \sup_{||x^*||_2<r_1} ||P_n\mathcal{F}\,[x_n+x^*] - P_n\mathcal{F}\,[x_n]||_2$$

and equality holds on the boundary of A_n.

The term $|\hat{G}_\omega(ik)|^2$ is the sum of the squares of the entries, and the last "norm" on the right is defined by the usual convention:

$$\sup\{||z - z_n||_2 \,|\, z \in P_n\mathcal{F}\,[x_n + x^*], \quad z_n \in P_n\mathcal{F}\,[x_n]\}.$$

Note that $(I - P_n T_\omega)[x_n]$ is set-valued. Its norm is defined by the same convention. Let Ω_n be the connected component of A_n containing the solution $(\overline{\omega}, \overline{x}_n)$ of the harmonic balance equation (5) for which $\arg a_{i_0 j_0}(\overline{\omega}, \overline{x}_n) = 0$. Assume that

(A_3) (i) $I - e^{2\pi A/\omega}$ is invertible whenever ω is such that $(\omega, x_n) \in \Omega_n$ for some x_n;

(ii) $(\omega, 0) \notin \overline{\Omega}_n$ for any $\omega \in \mathbf{R}_+$. Moreover, the function $(\omega, x_n) \to a_{i_0 j_0}(\omega, x_n)$ has real part different from zero in Ω_n. (This implies that $\arg a_{i_0 j_0}(\omega, x_n)$ is a continuous function in Ω_n.)

(iii) $\deg(V_0^a, \Omega_n, 0)$ is well-defined and different from zero.

THEOREM. *Under Assumptions* (A1), (A2), (A3), *the inclusion* (4) *will have a solution* (ω, x), *where* $x = x_n + x^*, (\omega, x_n) \in \overline{\Omega}_n$ *and* $||x^*||_2 \leq r_1(\omega, x_n)$.

REMARK 1. The conditions on F in (A1) ensure that the Nemytskii operator $\mathcal{F}$ maps X into 2^X and is closed convex valued and bounded (i.e., the image of a bounded set is a bounded set) ([**9**]). Moreover, the imbedding i of Z into X is compact (i.e., it sends bounded sets into relatively compact sets). Therefore the operator $T_\omega[x] = i \circ \mathcal{G}_\omega \circ \mathcal{F}[x]$ is a closed convex valued upper semi-continuous, compact operator. Hence T_ω in the space X will meet the conditions for using the Schauder fixed point theorem (see Dugundji-Granas [**6**]) and Leray-Schauder degree theory for the set-valued compact vector field V_λ for any $\lambda \in [0, 1]$ (see Cellina-Lasota [**5**]). In the sequel we will omit the imbedding map i in the notation of T_ω, keeping in mind that, for any $x \in X, T_\omega[x] \subset X$ is a set of absolutely continuous, 2π-periodic functions with derivatives in X.

REMARK 2. The assumption (A3)(i) is not essential, since any given system can be "pole-shifted" to an equivalent system satisfying (A3)(i). To see this, note that the eigenvalues of $e^{2\pi A/\omega} - I$ are $(e^{2\pi\lambda/\omega} - 1)$, where λ is an eigenvalue of A. Thus (A3)(i) will hold if $\operatorname{Re}\lambda \neq 0$ for all eigenvalues of A. This can always be achieved by using the equivalent system $\dot{x}(t) - (A - \alpha I)[x](t) \in \mathcal{F}[x](t) + \alpha I x(t)$. For some real α this system will satisfy (A3)(i) and our hypotheses on $\mathcal{F}$ will not be affected. The other hypotheses must be verified in the new system. In fact, in most real-world situations, one has $\operatorname{Re}\lambda_k < 0$ for all λ_k.

REMARK 3. Assumption(A3)(ii) guarantees that the trivial solution is excluded when we apply a fixed point theorem or degree theory and that $\arg a_{i_0 j_0}$ is a continuous function in Ω_n. Let $\Omega = \{(\omega, x_n, x^*) \in \mathbf{R}_+ \times X_n \times X^* | (\omega, x_n) \in \Omega_n, ||x^*||_2 < r_1(\omega, x_n)\}$. If $0 \notin V_\lambda(\partial\Omega)$ for any $\lambda \in [0,1)$ (i.e., the homotopy (H_λ) is admissible in $\overline{\Omega}$), then assumption (A3)(iii) and the homotopy invariance property of the topological degree guarantee that

$$\deg(V_\lambda, \Omega, 0) = \deg(V_0^a, \Omega_n, 0) \neq 0$$

for any $\lambda \in [0,1)$.

On the other hand, the equation $0 \in V_1$ is not equivalent to the equation

$$(0,0,0) \in (\arg a_{i_0 j_0}, P_n(I - T_\omega)[x_n + x^*], (I - P_n)(I - T_\omega)[x_n + x^*]),$$

nor is this equivalent to

$$(0,0) \in (\arg a_{i_0 j_0}, (I - T_\omega)[x_n + x^*]).$$

In fact, given any non-singleton set $A \subset X$, we have the proper inclusions $\{0\} \subset A - A$ and $A \subset P_n A + (I - P_n)A$. Therefore we have to make the following observations. Let $\lambda_0 \in [0,1)$ and consider the following sets:

$$\begin{aligned} \mathcal{S} = \{(\omega, x_n, x^*) \in \overline{\Omega} : (\omega, x_n, x^*) \quad & \text{is a solution of} \\ x_n \in \lambda_0 P_n T_\omega[x_n + x^*] + (1-\mu)(P_n T_\omega[x_n] - \lambda_0 P_n T_\omega[x_n]), \\ x^* \in \lambda_0 (I - P_n) T_\omega[x_n + x^*], \quad & \text{for some} \quad \mu \in [0,1]\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}' = \{(\omega, x_n, x^*) \in \overline{\Omega} : (\omega, x_n, x^*) \quad & \text{is a solution of} \\ x \in \mu\lambda_0 T_\omega[x] + (1-\mu)(\lambda_0 P_n T_\omega[x] + \lambda_0 (I - P_n) T_\omega[x]); \\ x = x_n + x^*, \quad \text{for some} \quad & \mu \in [0,1]\}. \end{aligned}$$

It is easy to see that $\mathcal{S}$ and $\mathcal{S}'$ are contained in Ω whenever $\lambda_0 \in [0,1)$ (i.e., the previous homotopies are admissible); thus, we have

$$\deg(\Phi_{\lambda_0}, \Omega, 0) = \deg(V_{\lambda_0}, \Omega, 0) \neq 0$$

for any $\lambda_0 \in [0,1)$, where $\Phi_{\lambda_0} : \overline{\Omega} \to \mathbf{R} \times X$ is given, for any $\lambda_0 \in [0,1)$, by $\Phi_{\lambda_0}(\omega, x) = (\arg a_{i_0 j_0}, x - \lambda_0 T_\omega[x])$. Hence the conclusion of the Theorem will follow from the solution property of the topological degree.

REMARK 4. Assumption (A2)(h_1) is the "low-pass filter" assumption (Bergen et al. [**2**]), i.e., it implies that the linear part of the system as represented by $\mathcal{G}_\omega$ attenuates high frequencies. To explain this in simple terms, we consider the scalar convolution operator

$$\mathcal{G} : f(\cdot) \mapsto \int_0^{2\pi} g(t-s) f(x)\, ds,$$

with g given in X, $f \in X$, $t \in [0, 2\pi]$. If $f(t) \sim f_0 + \sum_1^\infty (f_k e^{ikt} + f_{-k} e^{-ikt})$, then

$$x(t) \equiv \mathcal{G}[f](t) \sim \hat{g}(0) f_0 + \sum_1^\infty \hat{g}(ik) f_k e^{ikt} + \hat{g}(-ik) f_{-k} e^{ikt}], \quad t \in [0, 2\pi],$$

where $\hat{g}(\cdot)$ is the Fourier transform of $g(\cdot)$, $\hat{g}(ik) = (\frac{1}{2}\pi) \int_0^{2\pi} e^{-ikt} g(t) dt$. Thus the high-frequency part of $x(\cdot)$, $(I - P_n)[x]$, can be estimated by

$$||(I - P_n)[x]||_\infty \leq \sum_{|k|>n} |\hat{g}(ik)| \cdot |f_k| \leq \Big\{ \sum_{|k|>n} |\hat{g}(ik)|^2 \Big\}^{1/2} ||f||_2.$$

The connection with (A2)(h_1) is now clear.

REMARK 5. Assumption (A2)(h_2) ensures that our homotopy is an admissible homotopy. Intuitively, it requires the distortion of the low-frequency harmonics $||(I - P_n T_\omega)[x_n]||_2 = ||x_n - P_n T_\omega[x_n]||_2$ to be small in comparison with $\sigma(\omega, x_n)$, where $\sigma(\omega, x_n)$ is the product of two terms, the term $\sum_0^n |G(ik\omega)|^2$, which measures the low-frequency response of the linear part of the system, and the term

$\sup_{||x^*||_2 \le r_1} ||P_n\mathcal{F}[x_n + x^*] - P_n\mathcal{F}[x_n]||_2$, which measures the growth of $P_n\mathcal{F}$ on the ball $B(x_n, r_1)$.

REMARK 6. The region Ω_n, in the scalar case, can often be determined by graphical and numerical techniques (Bergen et al. [**2**]). In general, it can be determined by the fact that equality holds in A2(h_2) for $(\omega, x_n) \in \partial\Omega_n$.

REMARK 7. The determination of the solution $(\overline{\omega}, \overline{x}_n)$ to $V_0^a = 0$ and the verification of (A3)(iii) is a finite-dimensional "algebraic" problem. If $x_n(t) \sim a_0 + \sum_{|k|=1}^n a_k e^{ikt}$, the a_k's unknown, then in many cases we can explicitly compute the Fourier expansion of $\mathcal{G}_\omega\mathcal{F}[x_n] \sim c_0 + \sum_{|k|\ge 1} c_k e^{ikt}$, i.e., we can explicitly determine $c_k(\omega, a_0, a_1, a_{-1}, \dots, a_n, a_{-n})$, *keeping in mind that* c_k *is set-valued.* Then $V_0^a = 0$ reduces to

$$0 \in a_j - c_j(\omega, a_0, a_1, \dots, a_n, a_{-n}), \quad j = 0, \pm 1, \dots, \pm n,$$

with the condition that $\arg a_{i_0 j_0}(\overline{\omega}, \overline{x}_n) = 0$.

REMARK 8. We note that $\sigma(\omega, x_n) > 0$ for all $(\omega, x_n) \in \Omega_n$ by assumption (A2)(h_2). Indeed, if there existed $(\tilde{\omega}, \tilde{x}_n) \in \Omega_n$ such that $\sigma(\tilde{\omega}, \tilde{x}_n) = 0$, then it would follow that, for any $x^* \in B(0, r_1)$ where $r_1 = r_1(\tilde{\omega}, \tilde{x}_n)$, we have $||P_n\{\mathcal{F}[\tilde{x}_n + x^*] - \mathcal{F}[\tilde{x}_n]\}||_2 = 0$. This implies that $P_n T_{\tilde{\omega}}[x_n + x^*] = P_n T_{\tilde{\omega}}[\tilde{x}_n] = \tilde{z} \in X_n$ for any $x^* \in B(0, r_1)$. Two cases arise. First, suppose that $(\tilde{\omega}, \tilde{x}_n)$ is a solution of the harmonic balance equation of order n, with $\arg a_{i_0 j_0} = 0, x_n \in P_n T_\omega[x_n]$. In this case, $\tilde{x}_n = \tilde{z}$, and if we consider the equation $x^* \in (I - P_n)T_{\tilde{\omega}}[\tilde{z} + x^*]$, by A3(iv), there exists a solution $x^* \in B(0, r_1)$. Therefore, $x = \tilde{z} + x^*$ is a $2\pi/\omega$-periodic solution of (1). On the other hand, if $(\tilde{\omega}, \tilde{x}_n)$ is not a solution of $x_n \in P_n T_\omega[x_n], \arg a_{i_0 j_0} = 0$, then there is no $2\pi/\tilde{\omega}$-periodic solution x of (1) with $P_n x = \tilde{x}_n$.

PROOF of THEOREM. We will prove the existence by a two-stage argument. First, for each $(\omega, x_n) \in \Omega_n$ and each $0 \le \lambda \le 1$ we will show that we can apply the Schauder fixed point theorem for set-valued maps in the ball $B(0, r_1)$ to get a solution $x_\lambda^*(\omega, x_n)$ of $\mathrm{H}_\lambda(b)$. Thus $\mathrm{H}_\lambda(a)$

becomes an equation in ω and x_n, when we replace x^* by $x^*_\lambda(\omega, x_n)$. We will then show that the original system (4) has a solution by showing that H_λ represents an admissible homotopy in $\overline{\Omega}$ (see Remark 3).

We turn to the first stage. It is clear that, for a given (ω, x_n) in $\Omega_n, \mathrm{H}_\lambda(b)$ represents a fixed-point problem $x^* \in M_\lambda[x^*]$ for the map $M_\lambda : x^* \mapsto \lambda(I - P_n)T_\omega[x^* + x_n]$ on $(I - P_n)X$. If $||x^*||_2 < r_1$, then (recalling that, for a multifunction $H[z], ||H[z]|| = \sup_{y \in H(z)} ||y||$):

$$\begin{aligned}||M_\lambda[x^*]||_2 &\equiv ||\lambda(I - P_n)T_\omega[x^* + x_n]||_2 \le ||\mathcal{G}_\omega(I - P_n)\mathcal{F}[x^* + x_n]||_2 \\ &\le \left\{ \sum_{|k|>n} |\hat{G}_\omega(ik)|^2 \right\}^{1/2} \sup_{||x^*||_2 \le r_1} \sup_{y \in \mathcal{F}[x^*+x_n]} ||(I - P_n)[y]||_2 \\ &< r_1\end{aligned}$$

by (A2)(h_1) and Remark 4, so $M_\lambda : B(0, r_1) \to 2^{B(0,r_1)}$ in $(I - P_n)X$. Now the Schauder fixed-point theorem for multivalued maps is valid for any upper semi-continuous, compact multifunction with convex compact values (Dugundji-Granas [**6**, p. 96]). Assumption (A1) implies that M_λ is such a map for any $\lambda \in [0, 1]$ (Lasry-Robert [**9**, p. 60]). Therefore, there exists a fixed point $x^*_\lambda(\omega, x_n)$ of the map M_λ, for each $(\omega, x_n) \in \Omega_n, 0 \le \lambda \le 1$.

As already noted, the same considerations as above show that the map $V_\lambda : (\omega, x) \mapsto V_\lambda[(\omega, x)]$ is a compact convex valued vector field defined in $\Omega \subset \mathbf{R}_+ \times X$ for any $\lambda \in [0, 1]$, where Ω is the open set defined in Remark 3. Hence the topological degree is defined (see [**5, 11**]). If we can show that, for $0 \le \lambda < 1$, we have $0 \notin V_\lambda[(\omega, x)]$ for $(\omega, x) \in \partial\Omega$, then it will follow from the homotopy invariance property that $0 \ne \deg(V^a_0, \Omega_n, 0_n) = \deg(V_\lambda, \Omega, 0)$ for any $\lambda \in [0, 1)$. Therefore, by Remark 3, there will exist a nontrivial $2\pi/\omega$-periodic solution of (1) in $\overline{\Omega}$ with $\arg a_{i_0 j_0} = 0$. Assume to the contrary that there exists $(\omega, x_n, x^*) \in \partial\Omega$ such that $0 \in V_\lambda[\omega, x_n, x^*]$ for some $\lambda \in [0, 1)$, where x^* stands for $x^*_\lambda(\omega, x_n)$. Then, in particular, we would have selections $y_n \in \mathcal{F}[x_n]$ and $y, \overline{y} \in \mathcal{F}[x]$ such that at least one of the following inequalities holds as equality:

(7) (a) $||x_n - P_n\mathcal{G}_\omega y_n||_2 \le \sigma(\omega, x_n)$, (b) $||x^*||_2 \le r_1(\omega, x_n)$

and for which

(8a) $0_n = (x_n - P_n\mathcal{G}_\omega y_n) + \lambda[(x_n - P_n\mathcal{G}_\omega y) - (x_n - P_n\mathcal{G}_\omega y_n)]$, (8b) $$0^* = (I - P_n)(x - \lambda \mathcal{G}_{\omega} \overline{y}).$$

Equations (8a)–(8b) imply that

$$0 \geq ||x_n - P_n \mathcal{G}_{\omega} y_n||_2 - \lambda ||P_n \mathcal{G}_{\omega}(y - y_n)||_2,$$
$$0 \geq ||x^*||_2 - \lambda ||(I - P_n)\mathcal{G}_{\omega} \overline{y}||_2,$$

respectively. From these inequalities we obtain

(9) $$\begin{aligned} &\text{(a)} \quad 0 \geq ||x_n - P_n \mathcal{G}_{\omega} y_n||_2 - \lambda \sigma(\omega, x_n), \\ &\text{(b)} \quad 0 \geq ||x^*||_2 - \lambda r_1(\omega, x_n), \end{aligned}$$

where the estimates on the right-hand side of (9a) and (9b) are obtained by the usual Fourier expansion techniques, i.e.,

$$||P_n \mathcal{G}_{\omega}(y - y_n)||_2 \leq \left[\sum_{|k|=0}^{n} |\hat{G}_{\omega}(ik)|^2 \right]^{1/2} ||y - y_n||_2,$$
$$||(I - P_n)\mathcal{G}_{\omega} \overline{y}||_2 \leq \left[\sum_{|k| \geq n+1} |\hat{G}_{\omega}(ik)|^2 \right]^{1/2} ||\overline{y}||_2,$$

and, by using (A2)(h_2) and (A2)(h_1), respectively. But $\lambda \in [0, 1)$; hence, from (9a)–(9b), we obtain

$$0 > ||x_n - P_n \mathcal{G}_{\omega} y_n||_2 - \sigma(\omega, x_n), \quad 0 > ||x^*||_2 - r_1(\omega, x_n).$$

Therefore, neither (7a) nor (7b) can reach equality, contradicting the fact that $(\omega, x_n, x^*) \in \partial\Omega$ for some $\lambda \in [0, 1)$.

3. How Theorem is applied. The harmonic balance technique applied to a specific problem will, in general, require a considerable amount of numerical approximation and computation. There is an excellent discussion, complete with flow chart, in Mees [**13**]. It is not our intention to present a completely worked-out example since the computational literature is extensive, rather we present the set-up of a specific artificially created problem to illustrate the use of a multifunction to simplify the treatment of a discontinuity, and to show the connection between our hypotheses and the usual assumptions used to intuitively justify the method.

The procedure for applying our theorem is as follows:

(1) Set up and solve the harmonic balance equations.

(2) Check that the local degree of the solution(s) (ω_0, x_n^0) obtained in (1) is nonzero. This is just the computation of the determinant of a Jacobian.

(3) (a) Bound $\sum_{|k|\geq 2} |\hat{G}_\omega(ik)|^2$ from above for ω restricted to an (unknown) interval (a,b) containing ω_0; then

(b) estimate $\sup ||(I-P_n)y||_2, y \in \mathcal{F}[x_n + x^*], ||x^*|| < r_1$, from above for fixed (but arbitrary) x_n, and r_1 unspecified. Combine the estimates in (a), (b) to check A2(h_1). This will, in general, give an inequality constraint $r_1(x_1,\omega) \geq \rho(|x_1|, a, b)$ for some function ρ. We tentatively choose $r_1 = \rho$.

(4) (a) Bound $\sum_{|k|=0}^{n} |\hat{G}_\omega(ik)|^2$ below for $\omega \in (a,b)$, and then

(b) bound $\sup_{||x^*||<r_1} ||P_1\mathcal{F}[x_n + x^*] - P_1\mathcal{F}[x_n]||_2$ below for x_n fixed but arbitrary. It is easy to obtain a bound by making a specific choice of x^*, but we need a "tight" bound, which, in general, requires extensive computation. Combine (a) and (b) to get a tight lower bound on $\sigma(x_n,\omega)$, call it σ_{num}, then require $||(I - P_n T_\omega)[x_n]||_2 < \sigma_{\mathrm{num}}$. This will (one hopes) define an open set Ω_n, after some trial and error with (a,b), such that $||(I - P_n T_\omega)[x_n]||_2 = \sigma_{\mathrm{num}}$ on $\partial\Omega_n$.

To illustrate the procedure outlined above, we consider the discontinuous differential equation

$$\begin{aligned} &\ddot{y} + 16y = 2 \qquad \text{when} \quad \dot{y} \geq 0, \\ &\ddot{y} + 2\dot{y} + 17y = -2 \qquad \text{when} \quad \dot{y} < 0. \end{aligned}$$

We convert this to the differential inclusion

$$\dot{z}(t) - Az(t) \in F(t,z), \quad z = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -16 & -1 \end{bmatrix}, \tag{10}$$

$$F = \begin{bmatrix} 0 \\ F^{(2)} \end{bmatrix}, \quad F^{(2)}(t,z) = \begin{cases} \{\dot{y} + 2\}, & \dot{y} > 0, \\ \{-y - \dot{y} - 2\}, & \dot{y} < 0, \\ [-y-2, 2] & \dot{y} = 0. \end{cases}$$

Of course, if $y < -4$, we must write $[2, -y-2]$ rather than as written. The choice of A was not made arbitrarily. The idea is to keep the coefficients of y and $\dot{y}$ in $F^{(2)}$ as small as possible, in order to assist in the estimate described in 3(b) above.

This inclusion has trajectories in the phase plane which are circles in the upper half plane and spirals in the lower half plane. In addition, $y(t) \equiv k \in [-2/15, 1/8]$ is a solution of the inclusion (but not of the original equation). Note that (10) has the unique (up to a time shift) nonconstant periodic solution.

$$y(t) \cong \begin{cases} (0.6493)\cos 4t + 1/8, & -\pi/4 \le t \le 0, \\ (0.9194)e^{-t}\cos(4t - .2449) - 2/17, & 0 \le t \le \pi/4, \end{cases}$$

with frequency $\omega_0 = 4$.

We assume a solution to the normalization (2) of the system (10) of the form

$$z_0(t) = \begin{bmatrix} z_0^{(1)} & (t) \\ z_0^{(2)} & (t) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{it} + \begin{bmatrix} \overline{c}_1 \\ \overline{c}_2 \end{bmatrix} e^{i^t} = \begin{bmatrix} a_1 \cos(t+\varphi_1) \\ a_2 \cos(t+\varphi_2) \end{bmatrix},$$

where $\overline{c}_i$ denotes the complex conjugate of c_i, and $a_i = 2|c_i|, \varphi_i = \arg c_i$. Because $F[z_0](t)$ is only set-valued for t in a set of measure zero (a situation which always occurs with only finitely many jump discontinuities), there is essentially only one Nemytskii selection, whose Fourier coefficients are easily computed:

$$f+1 = \begin{bmatrix} 0 \\ f_1^{(2)} \end{bmatrix}, \quad f_1^{(2)} = \frac{1}{2\pi}\int_{-\varphi_2}^{2\pi-\varphi_2} e^{-it} F^{(2)}[z_0](t)\,dt$$
$$= -\frac{33}{4}c_1 + \left(-\frac{1}{2} + \frac{4}{\pi|c_2|}\right)c_2.$$

The equation of harmonic balance for $n = 1$ is

$$(11) \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{it} + \begin{bmatrix} \overline{c}_1 \\ \overline{c}_2 \end{bmatrix} e^{-it} = \hat{G}_\omega(i) \begin{bmatrix} 0 \\ f_1^{(2)} \end{bmatrix} e^{it} + \hat{G}_\omega(-i) \begin{bmatrix} 0 \\ \overline{f}_1^{(2)} \end{bmatrix} e^{-it},$$

and this is clearly equivalent to $[c_1, c_1]^{\text{tr}} = \hat{G}_\omega(i)[0, f_1^{(2)}]^{\text{tr}}$, where superscript "tr" denotes the transpose. By definition, $\hat{G}_\omega(ik) = \int_0^{2\pi} e^{-iks} G_\omega(s)\,ds$, where G_ω is given just below (3),

$$\hat{G}_\omega(ik) = -[A - ik\omega]^{-1} = \frac{1}{(16 - k^2\omega^2) + ik\omega} \begin{bmatrix} 1 + ik\omega & 1 \\ -16 & ik\omega \end{bmatrix}.$$

Then the equation of harmonic balance becomes

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{f_1^{(2)}}{(16-\omega^2)+i\omega} \begin{bmatrix} 1 \\ i\omega \end{bmatrix}. \tag{12}$$

Thus $c_2 = i\omega c_1$ and

$$\Big[\Big(\frac{97}{4}-\omega^2\Big)+i\Big(\frac{3\omega}{2}-\frac{4}{\pi|c_1|}\Big)\Big]c_1 = 0. \tag{13}$$

This yields the solution, unique up to a rotation of c_1 and c_2,

$$\varpi \cong 4.924, \quad c_1^0 \cong 0.1724, \quad c_2^0 = -i\varpi c_1^0. \tag{14}$$

As already noted, the rotational symmetry is no surprise, since the original system is autonomous (t is determined only up to an additive constant, thus e^{ikt} is determined only up to a phase angle).

We require that the degree of the mapping defined by (13) be nonzero at the solution (14). The equation for c_1 can be written in the form

$$(|c_1|,\theta) \to \left(-\frac{3\omega}{2}+\frac{4}{\pi|c_1|}\right), \quad \theta = \arg c_1. \tag{15}$$

The topological degree in A3(iii) can be now evaluated as the Jacobian of system (15) with respect to $(|c_1|,\theta)$ with $\omega = \omega_0$, at the point $|c_1| \cong 0.1724$. It is easy to check that the Jacobian is different from 0. (Note that we do not need to compute the degree for the mapping in c_2 because the identity $c_2 = i\omega c_1$ eliminates c_2 from the mapping.)

In order to keep things simple, we restrict ω to (3.9, 5), using our approximate and true frequencies as a guide (a luxury not available in real world applications).

We note that $|\hat{G}_\omega(ik)|^2$ can be taken as the sum of the squares of the moduli of the entries of the second column, since $F[x] = [0, F^{(2)}[x]]^{\mathrm{tr}}$. Thus, for (A2)($\mathrm{h}_1$),

$$\sum_{|k|\geq 2} |\hat{G}_\omega(ik)|^2 < 0.18406.$$

The remaining term in (A2)(h_1) can be estimated as follows: for $y \in \mathcal{F}[x_1+x^*]$, $||(I-P_1)y||_2 \leq ||x_1||_2 + ||x^*||_2 + 2\sqrt{2\pi} < ||x_1||_2 +$

$2\sqrt{2\pi}+r_1$ for $||x^*||_2 < r_1$. At first glance, it might appear that $(I-P_1)$ annihilates the x_1 term that appears in the definition of $F^{(2)}[z]$, but this is not necessarily the case since the selection $y(t)$ might jump among the formulas defining $F^{(2)}[x_1+x^*](t)$, depending on the sign of $[x_1^{(2)}+x^{*(2)}](t)$. Assumption (A2)(h_1) now reduces to $1.13+(.226)||x_1||_2 < r_1$ for $(\omega, x_1)\in\Omega_1, 3.9<\omega<5$.

We now turn to (A2)(h_2) and the final step 4 of our program. To estimate $\sigma(\omega, x_1)$ from below, we first note that

$$\left\{\sum_{|k|=0}^{1}|\hat{G}_\omega(ik)|^2\right\}^{1/2} \geq 0.306 \qquad \text{for} \quad 3.9<\omega<5.$$

We next turn to the term

$$\sup_{||x^*||_2<r_1} ||P_1\mathcal{F}[x_1+x^*]-P_1\mathcal{F}[x_1]||_2. \tag{13}$$

Given $x_1(t), x^*(t)$, the difference $y(t)-z(t)$, for $y(t)\in F[x_1+x^*](t)$ and $z(t)\in\mathcal{F}[x_1](t)$, will be of the form $[0,\omega]^{\mathrm{tr}}$, where $\omega(t)$ is defined a.e. as

(14)

$$\begin{array}{lllll}
x^{*(2)}(t) & \text{for} & x_1^{(2)}(t)+x^{*(2)}(t)>0 & & \\
 & & \text{and} & x_1^{(2)}(t)>0, & \\
x_1^{(1)}(t)+2x_1^{(2)}(t)+x^{*(2)}(t)+4 & \text{for} & ''>0 & \text{and} & ''<0, \\
x_1^{(1)}(t)-2x_1^{(2)}(t)-x^{*(1)}(t)-x^{*(2)}(t)-4 & \text{for} & ''<0 & \text{and} & ''>0, \\
x^{*(1)}(t)-x^{*(2)}(t) & \text{for} & ''<0 & \text{and} & ''<0,
\end{array}$$

At this stage extensive calculations (best done by computer) are required as one tries to discover a reasonable region Ω_1 for which (13) has a "tight" lower bound. This requires the production of a computer code which will calculate the first term of the Fourier expansion $\mathcal{F}[x_1+x^*]$, using (14), for a selection of functions x_1 near the solution of the harmonic balance equation, with $||x^*||<r_1, r_1 = 1.13+(.226)||x_1||$. The idea is to search among possible functions x_1 for

a large value of this coefficient, since that will increase the computed supremum in (13).

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