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OSCILLATIONS OF DIFFERENCE EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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ABSTRACT. We obtain sufficient conditions for the oscillation of all solutions of some difference equations with positive and negative coefficients. Our results include the following: Consider the difference equation

$$(1) \qquad A_{n+1} - A_n + pA_{n-k} - qA_{n-l} = 0, \quad n = 0, 1, 2, \dots,$$

where p and q are nonnegative real numbers and k and l are nonnegative integers such that

$$p > q \geq 0, \quad k \geq l \geq 0, \quad q(k-l) \leq 1$$

and

$$p - q > \frac{k^k}{(k+1)^{k+1}} \quad \text{if } k \geq 1$$

$$p - q \geq 1 \quad \text{if } k = 0.$$

Then every solution of Equation (1) oscillates. Extensions to equations with variable coefficients were also obtained.

1. Introduction and preliminaries. Recently, Györi and Ladas [**5**], Ladas [**7**] and Erbe and Zhang [**3**] investigated the oscillatory behavior of solutions of difference equations of the form

$$(1) \qquad A_{n+1} - A_n + \sum_{j=0}^{m} P_j(n) A_{n-j} = 0, \quad n = 0, 1, 2, \dots,$$

with positive coefficients $P_j(n)$. Our aim in this paper is to obtain oscillation results for some difference equations with positive and negative coefficients.

Let $\mathbf{N} = \{0, 1, 2, \dots\}$ be the set of natural numbers and $\triangle$ denote the forward difference operator defined by $\triangle A_n = A_{n+1} - A_n$. Consider the linear difference equation with positive and negative coefficients

$$(2) \qquad \triangle A_n + P(n) A_{n-k} - Q(n) A_{n-l} = 0, \quad n = 0, 1, 2, \dots,$$

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where

(3) $$P, Q : \mathbf{N} \to \mathbf{R}^+ \qquad \text{and} \qquad k, l \in \mathbf{N},$$

Let

(4) $$p = \liminf_{n\to\infty} P(n) \qquad \text{and} \qquad q = \limsup_{n\to\infty} Q(n).$$

With Equation (2) we associate its "limiting" equation

(5) $$\triangle B_n + pB_{n-k} - qB_{n-l} = 0, \quad n = 0, 1, 2, \dots.$$

In §2 we will obtain sufficient conditions in terms of p, q, k and l for the oscillation of all solutions of Equation (5). In §3, we will establish sufficient conditions for the oscillation of all solutions of Equation (2) in terms of the oscillation of all solutions of the limiting Equation (5).

As usual, a solution $\{A_n\}$ of Equation (2) is said to *oscillate* if, for every $N > 0$, there exists an $n \geq \mathbf{N}$ such that $A_n A_{n+1} \leq 0$. Otherwise the solution is called *nonoscillatory.*

The difference equations in this paper are of arbitrary order. For second order linear difference equations, the reader is referred to Hooker, Kwong and Patula [**6**] and Mingarelli [**8**] and the references cited therein.

Our results have been motivated by the study of differential equations with piecewise constant arguments. See, for example, Aftabizadeh, Wiener and Xu [**1**] and Cooke and Wiener [**2**] and the references cited therein. In turn, the results of this paper have applications to the oscillation of all solutions of some equations with piecewise constant arguments including equations with positive and negative coefficients of the form

(6) $$\dot{y}(t) + py([t-k]) - qy([t-l]) = 0, \quad t \geq 0,$$

where $[\cdot]$ denotes the greatest integer function and

(7) $$p, q \in \mathbf{R}^+ \qquad \text{and} \quad k, l \in \mathbf{N}.$$

Let $m = \max\{k, l)$. By a *solution* of Equation (6) we mean a function y which is defined on the set $\{-m, \dots, 0\} \cup [0, \infty)$ and satisfies the following properties:

(i) y is continuous on $[0,\infty)$.

(ii) The derivative $\dot{y}(t)$ exists at each point $t \in [0,\infty)$ with the possible exception of the points $t \in \mathbf{N}$ where finite one-sided derivatives exist.

(iii) Equation (6) is satisfied on each interval $[n, n+1)$ for $n \in \mathbf{N}$.

As is customary, a solution of Equation (6) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is called *nonoscillatory*.

With the difference equation (5) one associates its characteristic equation

$$\lambda - 1 + p\lambda^{-k} - q\lambda^{-l} = 0. \tag{8}$$

Our proofs in §§ 2 and 3 make use of the following known results.

LEMMA 1. [**5**]. *Assume that $p, q \in \mathbf{R}$ and $k, l \in \mathbf{N}$. Then the following statements are equivalent*:

(a) *Every solution of Equation* (5) *oscillates.*

(b) *Every solution of Equation* (6) *oscillates.*

(c) *The characteristic equation* (8) *has no positive roots.*

LEMMA 2. [**4, 5**]. *Assume that $p \in \mathbf{R}^+$ and $k \in \mathbf{N}$. Then every solution of*

$$\triangle x_n + p x_{n-k} = 0, \quad n = 0, 1, 2, \ldots,$$

oscillates if and only if

$$p > \frac{k^k}{(k+1)^{k+1}} \quad \text{if} \quad k \geq 1$$
$$p \geq 1 \quad \text{if} \quad k = 0.$$

LEMMA 3. [**5**]. *Consider the difference inequality*

$$\triangle y_n + P(n) y_{n-k} \leq 0, \quad n = 0, 1, 2, \ldots, \tag{9}$$

where

$$P : \mathbf{N} \to \mathbf{R}^+ \quad \textit{and} \quad k \in \mathbf{N}.$$

Let $p = \liminf_{n\to\infty} P(n)$, *and assume that* $p + k \neq 1$ *and that every solution of the limiting equation*

$$\triangle x_n + p x_{n-k} = 0, \quad n = 0, 1, 2, \dots,$$

oscillates. Then (9) *cannot have an eventually positive solution.*

2. Difference equations with positive and negative coefficients. In this section we will obtain sufficient conditions in terms of p, q, k and l for the oscillation of all solutions of Equation (5).

The first result is a necessary condition for the oscillation of all solutions of Equation (5).

LEMMA 4. *Assume that*

$$p, q \in \mathbf{R}^+ \quad \text{and} \quad k, l \in \mathbf{N} \tag{10}$$

and that every solution of Equation (5) *oscillates. Then*

$$p > q \quad \text{and} \quad k \geq l. \tag{11}$$

PROOF. Set

$$F(\lambda) = \lambda - 1 + p\lambda^{-k} - q\lambda^{-l} = 0. \tag{12}$$

By Lemma 1, Equation (12) has no positive roots. As $F(\infty) = \infty$, it follows that

$$F(\lambda) > 0 \quad \text{for} \quad \lambda > 0 \tag{13}$$

and in particular $F(1) = p - q > 0$. Thus $p > q$. We now claim that $k \geq l$. Otherwise $k < l$ and $q > 0$ (we make the convention that if $q = 0$, then $l = 0$). Then $F(0+) = -\infty$ which contradicts (13) and completes the proof. □

THEOREM 1. *Assume that*

$$p > q \geq 0, \quad k \geq l \geq 0, \quad q(k - l) \leq 1 \tag{14}$$

and that

$$(15) \qquad \begin{aligned} &p - q > \frac{k^k}{(k+1)^{k+1}} && \text{if} \quad k \geq 1 \\ &p - q \geq 1 && \text{if} \quad k = 0. \end{aligned}$$

Then every solution of Equation (5) *oscillates.*

Proof. The case $k = l$ reduces to Lemma 2. So suppose $k > l$. Assume, for the sake of contradiction, that Equation (5) has an eventually positive solution $\{B_n\}$. Then there exists $n_0 \in \mathbf{N}$ such that $B_n > 0$ for $n \geq n_0$.

Set

$$(16) \qquad c_n = B_n - q \sum_{j=l+1}^{k} B_{n-j}, \quad n \geq n_0 + k.$$

Then

$$(17) \qquad \begin{aligned} \triangle c_n &= \triangle B_n - q(B_{n-l} - B_{n-k}) \\ &= -(p-q)B_{n-k} < 0 \qquad \text{for} \quad n \geq n_0 + k. \end{aligned}$$

Thus c_n is a strictly decreasing sequence for $n \geq n_0 + k$. We claim that

$$(18) \qquad L \equiv \lim_{n \to \infty} c_n \in \mathbf{R}.$$

Otherwise, $L = -\infty$ and $\{B_n\}$ must be unbounded. Hence, there exists $n_1 \geq n_0 + k$ such that $B_{n_1} = \max\{B_n : n \leq n_1\}$ and $c_{n_1} < 0$. Then

$$0 > c_{n_1} = B_{n_1} - q \sum_{j=l+1}^{k} B_{n_1 - j} \geq B_{n_1}[1 - q(k-l)] \geq 0,$$

which is a contradiction. Thus (18) holds. It now follows, by taking limits in (17), that $\lim_{n\to\infty} B_n = 0$. Hence $L = 0$. As the sequence $\{c_n\}$ decreases to zero, we conclude that

$$(19) \qquad c_n > 0 \qquad \text{for} \quad n \geq n_0 + k.$$

Also, from (16), we see that $c_n < B_n$ for $n \geq n_0 + k$, and so (17) yields the inequality

$$\triangle c_n + (p-q)c_{n-k} < 0 \qquad \text{for} \quad n \geq n_0 + 2k. \tag{20}$$

But, in view of Lemma 2 and the hypothesis (15), every solution of the difference equation

$$\triangle x_n + (p-q)x_{n-k} = 0$$

oscillates. Then, by Lemma 3, the difference inequality (20) cannot have an eventually positive solution. This contradicts (19) and completes the proof of the theorem. □

A consequence of Lemma 1 and Theorem 1 is the following corollary about Equation (6).

COROLLARY 1. Assume that (14) and (15) hold. Then every solution of Equation (6) oscillates.

3. Variable coefficients. In this section we will establish sufficient conditions for the oscillations of all solutions of Equation (2) in terms of the oscillation of all solutions of the limiting Equation (5).

The next lemma is interesting in its own right.

LEMMA 5. *Assume that* (10) *holds and that*

$$\textit{either} \qquad k > 0 \qquad \textit{or} \qquad p - q > 1. \tag{21}$$

Suppose also that every solution of Equation (5) *oscillates. Then there exists an* $\varepsilon_0 > 0$ *such that, for every* $\varepsilon \in [0, \varepsilon_0]$, *every solution of the equation*

$$\triangle x_n + (p-\varepsilon)x_{n-k} - (q+\varepsilon)x_{n-l} = 0 \tag{22}$$

oscillates.

PROOF. It suffices to show that the characteristic equation of Equation (22),

$$G(\lambda) \equiv \lambda - 1 + (p-\varepsilon)\lambda^{-k} - (q+\varepsilon)\lambda^{-l} = 0,$$

has no positive roots. By Lemma 4, (11) and (13) hold. When $k = 0$, l is also zero and one can see that

$$\varepsilon_0 = \frac{1}{2}(p - q - 1)$$

is a good choice. Next, assume $k > 0$. Then, from (12), we obtain

$$F(\infty) = \infty \quad \text{and} \quad F(0+) = \infty.$$

Hence,

$$m = \min\{F(\lambda) : 0 < \lambda < \infty\}$$

exists and is positive and

$$\lambda - 1 + p\lambda^{-k} - q\lambda^{-l} \geq m \qquad \text{for} \quad \lambda > 0.$$

Set $\delta = (p - q)/3$ and choose $0 < \lambda_1 < \lambda_2$ in such a way that

$$\lambda - 1 + (p - \delta)\lambda^{-k} - (q + \delta)\lambda^{-l} > 0 \qquad \text{for} \quad \lambda \in (0, \lambda_1) \cup (\lambda_2, \infty).$$

Let

$$\eta = \max\{\lambda^{-k} + \lambda^{-l} : \lambda_1 \leq \lambda \leq \lambda_2\}$$

and set

$$\varepsilon_0 = \min\left\{\delta, \frac{m}{2\eta}\right\}.$$

Now let $0 \leq \varepsilon \leq \varepsilon_0$. Then, for $\lambda \in (0, \lambda_1) \cup (\lambda_2, \infty)$,

$$G(\lambda) \geq \lambda - 1 + (p - \delta)\lambda^{-k} - (q + \delta)\lambda^{-l} > 0,$$

while, for $\lambda_1 \leq \lambda \leq \lambda_2$,

$$G(\lambda) = \lambda - 1 + p\lambda^{-k} - q\lambda^{-l} - \varepsilon(\lambda^{-k} + \lambda^{-l}) \geq m - \varepsilon_0\eta \geq m - \frac{m}{2} > 0.$$

The proof is complete. □

The next lemma will be needed in our proof of Theorem 2.

LEMMA 6. *Let $a, b \in \mathbf{R}^+$ and $l, k, n_0 \in \mathbf{N}$ be such that*

$$a + b > 0 \qquad \textit{and} \qquad k > l.$$

Assume that the inequality

$$a\sum_{j=l+1}^{k} y_{n-j} + b\sum_{j=n}^{\infty} y_{j-k} \le y_n, \quad n \ge n_0$$

has a positive solution $\{y_n\}_{n=n_0-k}^{\infty}$ *such that*

$$y_{n_0} < y_{n_0-j} \quad \text{for} \quad j = 1, 2, \dots, k. \tag{23}$$

Then the equation

$$a\sum_{j=l+1}^{k} x_{n-j} + b\sum_{j=n}^{\infty} x_{j-k} = x_n, \qquad n \ge n_0, \tag{24}$$

has a positive solution $\{x_n\}_{n=n_0-k}^{\infty}$. *Furthermore,*

$$0 < x_n \le y_n, \qquad n \ge n_0 - k.$$

PROOF. Define the set of nonnegative sequences

$$X = \big\{x = \{x_n\}_{n=n_0}^{\infty} : \ 0 \le x_n \le y_n \qquad \text{for} \quad n \ge n_0\big\}.$$

For every $x \in X$ define the sequence $\tilde{x} = \{\tilde{x}_n\}_{n=n_0-k}^{\infty}$ by

$$\tilde{x}_n = \begin{cases} x_n, & n > n_0 \\ x_{n_0} + y_n - y_{n_0}, & n \in \{n_0 - k, \dots, n_0\}. \end{cases}$$

Clearly

$$0 \le \tilde{x}_n \le y_n \qquad \text{for} \quad n \ge n_0 - k,$$

and, in view of (23),

$$\tilde{x}_n > 0 \qquad \text{for} \quad n \in \{n_0 - k, \dots, n_0\}. \tag{25}$$

Define the mapping T on X as follows: For every $x = \{x_n\} \in X$, let the n-th term of the sequence Tx be

$$a\sum_{j=l+1}^{k} \tilde{x}_{n-j} + b\sum_{j=n}^{\infty} \tilde{x}_{j-k}.$$

Then one can see that $T : X \to X$. Furthermore, T is monotone in the sense that if $x^{(1)}, x^{(2)} \in X$ and $x^{(1)} \le x^{(2)}$ (that is, $x_n^{(1)} \le x_n^{(2)}$ for $n \ge n_0$) then $Tx^{(1)} \le Tx^{(2)}$.

Set

$$x^{(0)} = \{y_n\}_{n=n_0}^{\infty} \quad \text{and} \quad x^{(m)} = Tx^{(m-1)} \quad \text{for} \quad m = 1, 2, \ldots.$$

It follows by induction that the sequence $\{x^{(m)}\}$ of elements of X is such that

$$0 \le x_n^{(m+1)} \le x_n^{(m)} \le y_n \qquad \text{for} \quad n \ge n_0.$$

Hence,

$$x_n \equiv \lim_{m\to\infty} x_n^{(m)}, \ n \ge n_0,$$

exists and $x = \{x_n\}_{n=n_0}^{\infty}$ belongs to X. Also $x = Tx$ and so $\tilde{x}$ is a solution of Equation (24). It remains to show that

$$\tilde{x}_n > 0 \qquad \text{for} \quad n \ge n_0 - k. \tag{26}$$

In view of (25), if (26) were false there would exist some $m > n_0$ such that

$$x_m = 0 \quad \text{and} \quad x_n > 0 \quad \text{for} \quad n \in \{n_0 - k, \ldots, m-1\}.$$

Then, from (24),

$$\begin{aligned} 0 = x_m &= a \sum_{j=l+1}^{k} \tilde{x}_{m-j} + b \sum_{j=m}^{\infty} \tilde{x}_{j-k} \\ &\ge a\tilde{x}_{m-l-1} + b\tilde{x}_{m-k} > 0, \end{aligned}$$

which is a contradiction. □

Theorem 2. *Consider the difference equation* (2) *and assume that* (3), (4) *and* (21) *hold and that*

$$q(k-l) < 1. \tag{27}$$

Suppose that every solution of Equation (5) *oscillates. Then every solution of Equation* (2) *also oscillates.*

PROOF. The case $k = l$ reduces to Lemma 3. So suppose $k > l$. Assume, for the sake of contradiction, that Equation (2) has an eventually positive solution $\{A_n\}$. Then, there exists $n_0 \in \mathbf{N}$ such that $A_n > 0$ for $n \geq n_0$. Choose $\varepsilon > 0$ such that $\varepsilon \in (0, \varepsilon_0]$,

$$p - q - 2\varepsilon > 0 \qquad \text{and} \qquad (q+\varepsilon)(k-l) \leq 1, \tag{28}$$

where ε_0 is as described in Lemma 5.

Then, there exists $n_1 \geq n_0$ such that

$$\Delta A_n + (p-\varepsilon)A_{n-k} - (q+\varepsilon)A_{n-l} \leq 0, \quad n \geq n_1.$$

Set

$$y_n = A_n - (q+\varepsilon)\sum_{j=l+1}^{k} A_{n-j}, \quad n \geq n_1. \tag{29}$$

Then

$$\begin{aligned} \triangle y_n &= \triangle A_n - (q+\varepsilon)(A_{n-l} - A_{n-k}) \\ &\leq -(p-q-2\varepsilon)A_{n-k} < 0 \qquad \text{for} \quad n \geq n_1 + k. \end{aligned} \tag{30}$$

As in the proof of Theorem 1, one can show that, in view of (28),

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} y_n = 0 \tag{31}$$

and that

$$y_n > 0 \quad \text{for} \quad n \geq n_1.$$

From (30), summing up from n to ∞, we obtain

$$-y_n + (p-q-2\varepsilon)\sum_{j=0}^{\infty} A_{n-k+j} \leq 0,$$

and, by using (29), we find

$$(q+\varepsilon)\sum_{j=l+1}^{k} A_{n-j} + (p-q-2\varepsilon)\sum_{j=n}^{\infty} A_{j-k} \leq A_n, \quad n \geq n_1.$$

In view of (31), there exists $n_2 \geq n_1$ such that

$$A_{n_2} < A_{n_2 - j} \qquad \text{for} \quad j = 1, 2, \dots, k.$$

By applying Lemma 6 we conclude that the equation

$$(32) \qquad (q+\varepsilon) \sum_{j=l+1}^{k} x_{n-j} + (p-q-2\varepsilon) \sum_{j=n}^{\infty} x_{j-k} = x_n, \quad n \geq n_2,$$

has a positive solution $x = \{x_n\}_{n=n_2-k}^{\infty}$. It follows from Equation (32) that

$$\begin{aligned} \triangle x_n &= (q+\varepsilon)(x_{n-l} - x_{n-k}) - (p-q-2\varepsilon) x_{n-k} \\ &= (q+\varepsilon) x_{n-l} - (p-\varepsilon) x_{n-k}, \end{aligned}$$

that is, Equation (22) has a positive solution. But this contradicts the conclusion of Lemma 5. □

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