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NODAL OSCILLATION AND WEAK OSCILLATION OF ELLIPTIC EQUATIONS OF ORDER 2m

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1. Introduction. Let L and M_0 be differential operators defined by

$$(1.1)\qquad Lu = \sum_{|\alpha|=0}^{m} \sum_{|\beta|=0}^{m} (-1)^{|\alpha|} D^{\alpha}[A_{\alpha\beta}(x) D^{\beta} u], \quad x \in \Omega \subseteq \mathbf{R}^n,$$

and

$$(1.2)\qquad M_0 v = (-1)^m \sum_{|\alpha|=|\beta|=m} D^{\alpha}[a_{\alpha\beta}(x) D^{\beta} v] + a_0(x) v,$$

where the coefficient functions $A_{\alpha\beta}$ and $a_{\alpha\beta}$, $|\alpha| \le m$, $|\beta| \le m$, are real-valued, satisfy the symmetry conditions

$$(1.3)\qquad A_{\alpha\beta} = A_{\beta\alpha}(x), \quad x \in \Omega,\ |\alpha| \le m,\ |\beta| \le m,$$

$$(1.4)\qquad a_{\alpha\beta}(x) = a_{\beta\alpha}(x), \quad |\alpha| = |\beta| = m,\ x \in \Omega,$$

and are sufficiently smooth on the unbounded open set Ω. (The multi-index notation employed here is that used in [**1, 2** and **6**].) In this paper the sign of $a_0(x)$ is *unrestricted*, unless the contrary is stated.

HYPOTHESIS 1.1. Throughout this paper, G will denote a nonempty open subset of Ω. (We will occasionally need to consider the special case where $G = \Omega$.)

DEFINITION 1.2. If G is bounded and satisfies the hypotheses of [**2**, Lemma 9.1], and if the differential equation

$$(1.5)\qquad Lu = 0$$

has a nontrivial solution u in $H_m^0(G) \cap C^{2m}(G)$, then G is called a *nodal domain* for L. We will say that (1.5) is *nodally oscillatory* in Ω iff, for every $r > 0$, the region $\Omega \cap \{x \in \mathbf{R}^n : |x| > r\}$ contains a nodal domain for L.

DEFINITION 1.3. We will say that (1.5) is *weakly oscillatory* in Ω iff (1.5) has at least one nontrivial C^{2m} solution which is *oscillatory* in Ω in the following sense: the set $\{x \in \Omega : u(x) \neq 0\}$ is unbounded and is expressible (see [**5**, Theorem 4.44]) as the union of a countably infinite collection, $\{G_s \mid s \in \mathbf{Z}_+\}$, $\mathbf{Z}_+ := \{0, 1, 2, \dots\}$, of mutually disjoint, connected, bounded, open sets such that:

(i) $\|u\|_{m,G_s} < \infty$, where the norm is defined as in [**2**];

(ii) each G_s is *regular* in the sense that appropriate versions of Courant's minimum principle [**13**, Lemma 2.3] and a monotonicity principle for eigenvalues [**8**] are valid for L on G_s;

(iii) given $r > 0$, there exists at least one G_s contained in the set $\{x \in \Omega : |x| > r\}$.

(Note that in the case where $n = 1$, each G_s in this definition is a bounded open interval whose endpoints are zeros of u.)

REMARK 1.4. Let N_{2m} denote the set of all nodally oscillatory equations of the form (1.5) and let W_{2m} denote the set of all weakly oscillatory equations of the form (1.5). It is known (see [**11**] and [**16**]) that if $2m \geq 4$, then $W_{2m} \neq N_{2m}$. It is also known (see [**10**, Theorem 4.3] and [**16**, Theorem 3.6]) that if

$$(1.6) \quad n = 1,\ m \geq 2,\ a_0(x) < 0,\ (-1)^m a_{\alpha,\alpha}(x) > 0,\ |\alpha| = m,\ x \in \Omega;$$

if Ω is an interval of the form $(r_0, \infty) := J \subseteq (0, \infty)$; if the principal part of the differential operator M_0 has a Pólya-Levin-Trench representation (in the sense of [**10**]); and if the differential equation

$$(1.7) \qquad M_0 v = 0$$

has at least one nontrivial oscillatory solution, then we can find a nontrivial solution v_0 of (1.7) and distinct points r_1, r_2 in J such that

$$v_0^{(k)}(r_1) = v_0^{(k)}(r_2) = 0, \quad 0 \leq k \leq m - 1.$$

In §2 of the present paper (see Theorem 2.4) we will extend the result just described to the case where M_0 is uniformly strongly elliptic and n is any positive integer. In §3, by using Theorems 2.4, 3.5 and 3.6 to compare L with a special case of M_0, we will obtain a criterion for nodal oscillation of (1.5) (see Theorem 3.10). That criterion is an extension of earlier results for equations of the form (1.7) (see [**7, 10, 19** and **20**]), and it complements known results [**18**] for equations of the form (1.5). Our proof of Theorem 3.6 depends on Theorem 3.5, which is a modification of the general form of Gårding's inequality [**2**, Theorem 7.6].

2. Definitions and results for $\mathbf{M_0}$.

DEFINITION 2.1. Following [**8, 9, 10** and **12**], we will say that G has *bounded thickness* iff we can find a positive number t and a line Γ such that every line Γ' parallel to Γ has the property that every maximal connected subset of $\Gamma' \cap G$ has diameter not greater than t. The infimum of the set of all such t is called the *thickness* of G.

For example, the bounded spherical shell $\{x \in \mathbf{R}^n : r_1 < |x| < r_2\}$, where $0 < r_1 < r_2 < \infty$, has thickness $2\left(r_2^2 - r_1^2\right)^{1/2}$, and so does the unbounded cylindrical shell

$$\left\{(y_1, \dots, y_{n+1}) \in \mathbf{R}^{n+1} : r_1 < \left[\sum_{k=1}^{n} y_k^2\right]^{1/2} < r_2\right\}.$$

We now recall a version of Poincaré's inequality that was proved in [**8**] and is a generalization of [**2**, Lemma 7.3].

LEMMA 2.2. *If G has thickness $t \in (0,\infty)$ and the set $\Gamma' \cap G$ in Definition 2.1 has at most k maximal connected subsets, where k is some positive integer, then, for every ϕ in $C_0^\infty(G)$ and every j in $\{0, 1, \dots, m-1\}$, we have*

$$|\phi|_{j,G} \le c_0 (kt)^{m-j} |\phi|_{m,G}, \tag{2.2.1}$$

where the seminorms are as in [**2**] *and the positive constant c_0 is a rational function of m and n only.*

REMARK 2.3. Motivated by the well-known formula

$$(-1)^m \int_G \phi \Delta^m \phi \, dx = \sum_{|\alpha|=m} \int_G \left[\frac{m!}{\alpha!}\right] |D^\alpha \phi|^2 \, dx, \tag{2.3.1}$$

which is valid for every real-valued ϕ in $C_0^\infty(G)$, we define the *weighted* seminorm $|\cdot|_{m,G,w}$ by

$$|u|_{m,G,w} = \left[\sum_{|\alpha|=m} \left[\frac{m!}{\alpha!}\right] \int_G |D^\alpha u|^2 \, dx\right]^{1/2}. \tag{2.3.2}$$

Note that, if

$$c_3 := \max\{m!/\alpha! : |\alpha| = m\}, \tag{2.3.3}$$

then

$$|u|_{m,G} \le |u|_{m,G,w} \le c_3^{1/2} |u|_{m,G}. \tag{2.3.4}$$

We also define the *modified* ellipticity constant $E(M_0; G)$:

$$E(M_0; G) = \inf \left\{ \left[\sum_{|\alpha|=|\beta|=m} \int_G a_{\alpha\beta} D^\alpha \phi D^\beta \phi\right] |\phi|_{m,G,w}^{-2} : 0 \ne \phi \in C_0^\infty(G) \right\}. \tag{2.3.5}$$

Note that (2.3.5) implies

$$E(M_0; G) \ge E(M_0; \Omega). \tag{2.3.6}$$

We will impose the *modified ellipticity condition*

$$0 < E(M_0; G) < \infty. \tag{2.3.7}$$

It is also convenient at this point to define the quadratic form

$$f_G[\phi] := \int_G \left[\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^\alpha \phi D^\beta \phi + a_0(x) \phi^2\right] dx \tag{2.3.8}$$

and the eigenvalue

$$\mu_0(M_0; G) := \inf \{ f_G[\phi] \|\phi\|_{0,G}^{-2} : 0 \neq \phi \in C_0^\infty(G)\}. \tag{2.3.9}$$

We are now in a position to state and prove our first major result, Theorem 2.4, which is a generalization of known one-dimensional results due to Leighton and Nehari [**16**, Theorem 3.6] and the author [**10**, Theorem 4.3]. These two known results and our new result give sufficient conditions under which weak oscillation implies nodal oscillation.

We note that (2.4.1), one of the hypotheses of Theorem 2.4, is satisfied if the coefficient $a_0(x)$ is negative and dominates the principal part of M_0. We also note that (2.4.1) is a generalization, to the n-dimensional case, of the sign hypotheses that were imposed on the coefficients $a_{\alpha,\alpha}(x)$, $|\alpha| = m$, and $a_0(x)$ in the one-dimensional case (see 1.6) above and [**16, 10**]). We also note that our proof of Theorem 2.4 uses ideas quite different from those employed in the one-dimensional cases considered in [**16**, Theorem 3.6] (for $m = 2$) and [**10**, Theorem 4.3] (for $m \geq 3$).

THEOREM 2.4. *Suppose that the coefficient $a_0(x)$ is bounded below on any bounded, regular* (*see Definition* 1.3 *condition* (ii)), *open set $G \subseteq \Omega$, and that the negative part of $a_0(x)$ is so large that, for any ϕ in $C_0^\infty(G)$, we have*

$$\int_G \phi M_0 \phi \, dx \leq 0. \tag{2.4.1}$$

If (1.7) *is weakly oscillatory in Ω, then* (1.7) *is also nodally oscillatory in Ω.*

PROOF. Let $\{G_s : s \in \mathbf{Z}_+\}$ be the collection whose existence is guaranteed by Definition 1.3. Since a bounded set necessarily has bounded thickness, we see that, given any s in $\mathbf{Z}_+$, we can find t in $(0, \infty)$ such that the bounded, open set G_s (which we will sometimes denote by $G_{s,t}$) has thickness t. From (2.3.8), integration by parts, and

(2.4.1), we deduce that

$$
\begin{aligned}
&\inf\left\{f_{G_s}[\phi] : \phi \in C_0^\infty(G_s),\ \|\phi\|_{0,G_s} = 1\right\} \\
&\qquad = \inf\left\{\int_{G_s}\left[\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}D^\alpha\phi D^\beta\phi + a_0\phi^2\right]dx : \right. \\
&\qquad\qquad\qquad\qquad \left. \phi \in C_0^\infty(G_s),\ \|\phi\|_{0,G_s} = 1\right\} \\
&\qquad = \inf\left\{\int_{G_s}\phi M_0\phi\,dx : \phi \in C_0^\infty(G_s),\ \|\phi\|_{0,G_s} = 1\right\} \le 0.
\end{aligned}
\tag{2.4.2}
$$

Using (2.4.2), Lemma 2.2 and the proof of [**13**, Lemma 2.3], we see that if

$$
c(G_{s,t}) := \inf\{a_0(x) : x \in G_{s,t}\}, \tag{2.4.3}
$$

then

$$
0 \ge \mu_0(M_0; G_{s,t}) \ge c_0^{-2}(kt)^{-2m}E(M_0; G_{s,t}) + c(G_{s,t}). \tag{2.4.4}
$$

It is also clear from (2.3.9) that the eigenvalue $\mu_0(M_0; G_{s,t})$ is nonincreasing with respect to t, and it can be shown that $\mu_0(M_0; G_{s,t})$ is continuous in t. Furthermore, the argument given in [**8**] shows that

$$
\lim_{t\to 0+}\left[c_0^{-2}(kt)^{-2m}E(M_0; G_{s,t}) + c(G_{s,t})\right] = +\infty. \tag{2.4.5}
$$

From (2.4.4), (2.4.5) and the monotonicity and continuity of $\mu_0(M_0; G_{s,t})$ with respect to t, we deduce that we can find t_0 (in the interval $(0, t]$) and an open set $G'_s := G_{s,t_0} \subseteq G_{s,t}$ such that $\mu_0(M_0; G'_s) = 0$. It follows from [**13**, Lemma 2.3] that equation $M_0 v_s = 0$ has a nontrivial solution in $H_m^0(G'_s) \cap C^{2m}(\overline{G'_s})$.

Thus, we have proved that, given any s in $\mathbf{Z}_+$, one can find a set G'_s (contained in G_s and belonging to the family $\{G_s : s \in \mathbf{Z}_+\}$) and a corresponding function v_s (belonging to $H_m^0(G'_s) \cap C^{2m}(\overline{G'_s})$) such that $M_0 v_s = 0$.

But, by Definition 1.3, given any $r > 0$, one can find s in $\mathbf{Z}_+$ such that $G_s \subset \{x \in \Omega : |x| > r\}$. From this fact and the preceding paragraph, we deduce that M_0 is nodally oscillatory in Ω. □

3. Results for L.

REMARK 3.1. Define the set $\Lambda(L,\Omega)$ as follows:

$$\Lambda(L,\Omega) = \left\{ \sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x)\xi^{\alpha+\beta}|\xi|^{-2m} : 0 \neq \xi \in R^n,\ x \in \Omega \right\}.$$

We will suppose that L is uniformly strongly elliptic in the following sense: there exist constants E_0 and E_1 such that

$$0 < E_0 := \inf \Lambda(L,\Omega) \leq \sup \Lambda(L,\Omega) := E_1 < +\infty. \tag{3.1.1}$$

REMARK 3.2. To prepare the way for our comparison theorem on nodal oscillation, we make the following observations.

Using integration by parts and the symmetry condition (1.3), we can easily show that if G satisfies Hypothesis 1.1, then, for every real-valued ϕ in $C_0^\infty(G)$, we have
(3.2.1)

$$\begin{aligned}\int_G \phi L\phi\, dx = \sum_{|\alpha|=|\beta|=m} \int_G A_{\alpha\beta}(x) D^\alpha\phi D^\beta\phi\, dx + \int_G \phi^2 A_{0,0}(x)\, dx \\ + \sum_{|\alpha|+|\beta|=2}^{2m-1} \int_G A_{\alpha\beta} D^\alpha\phi D^\beta\phi\, dx \\ + 2\sum_{|\alpha|=1}^{m} \int_G \phi A_{\alpha,0} D^\alpha\phi\, dx.\end{aligned}$$

We also need the following three results, which we could not find in the literature, and whose proofs may be obtained by imitating the proofs of [**2**; Lemma 7.7, Lemma 7.9 and Theorem 7.6].

LEMMA 3.3. *Let c_3 be as in* (2.3.3), *and let x^0 be a fixed* (*but otherwise arbitrary*) *point in Ω. Then, for every real-valued ϕ in $D_0^\infty(\Omega)$,*

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x^0) \int_\Omega D^\alpha\phi D^\beta\phi\, dx \leq c_3 E_1 |\phi|^2_{m,\Omega}. \tag{3.3.1}$$

LEMMA 3.4. *Suppose that the principal coefficients $A_{\alpha\beta}$, $|\alpha| = |\beta| = m$, are uniformly continuous on Ω. Then, for any $\delta > 0$, there exists $\rho_1(\delta) > 0$ such that, for every real-valued ϕ in $C_0^\infty(\Omega)$ for which*

$$\operatorname{diam}\operatorname{supp}\phi < \rho_1(\delta), \tag{3.4.1}$$

we have

$$\sum_{|\alpha|=|\beta|=m} \int_\Omega A_{\alpha\beta}(x) D^\alpha \phi D^\beta \phi \, dx \leq (\delta + c_3 E_1)|\phi|_m^2. \tag{3.4.2}$$

THEOREM 3.5. *Let G satisfy Hypothesis* 1.1. *Suppose that the principal coefficients $A_{\alpha\beta}$, $|\alpha| = |\beta| = m$, are uniformly continuous on Ω and that the intermediate coefficients $A_{\alpha\beta}$, $1 \leq (|\alpha| + |\beta|) \leq 2m - 1$, are bounded and continuous on Ω. Then there exist positive constants c_1 and c_2 which can be computed explicitly by means of Lemmas* 3.3, 3.4 *and* [**2**, *Lemma* 7.1] *and which depend only on m, n, E_1, $\sup\{|A_{\alpha\beta}(x)| : x \in \Omega;\ 2 \leq |\alpha| + |\beta| \leq 2m - 1\}$, $\sup\{|A_{\alpha,0}(x)| : x \in \Omega;\ 1 \leq |\alpha| \leq m\}$ and the modulus of continuity for the principal coefficients such that, for every real-valued ϕ in $C_0^\infty(G)$, we have*

(3.5.1)

$$\sum_{|\alpha|=|\beta|=m} \int_G A_{\alpha\beta}(x) D^\alpha \phi D^\beta \phi \, dx + \sum_{|\alpha|,|\beta|=1}^{m-1} \int_G A_{\alpha\beta}(x) D^\alpha \phi D^\beta \phi \, dx + 2\sum_{|\alpha|=1}^{m} \int_G \phi A_{\alpha,0} D^\alpha \phi \, dx \leq c_1 |\phi|_m^2 + c_2 |\phi|_0^2.$$

We will now compare the general, even-order, uniformly strongly elliptic operator L with a special case of the differential operator M_0.

THEOREM 3.6. *Let M_1 be the differential operator defined by*

$$M_1 v = (-1)^m c_1 \Delta^m v + [A_{0,0}(x) + c_2]v. \tag{3.6.1}$$

If the equation

$$M_1 v = 0 \tag{3.6.2}$$

is nodally oscillatory in Ω, then (1.5) *is nodally oscillatory in Ω.*

PROOF. If (3.6.2) is nodally oscillatory in Ω, then, for every positive r, the region $\{x \in \Omega : |x| > r\}$ contains a nodal domain G' for the differential operator M_1. Thus, (3.6.2) has a nontrivial solution v in $H_m^0(G') \cap C^{2m}(G')$. Furthermore, using (3.5.1), (3.6.1), integration by parts, (2.3.2), and (2.3.4), we see that, for every real-valued ϕ in $C_0^\infty(G')$,

$$\begin{aligned}\int_{G'} \phi L\phi\, dx &- \int_{G'} \phi M_1 \phi\, dx \\ &\leq c_1|\phi|^2_{m,G'} + c_2|\phi|^2_{0,G'} - \left[c_1|\phi|^2_{m,G',w} + c_2|\phi|^2_{0,G'}\right] \\ &= c_1\left[|\phi|^2_{m,G'} - |\phi|^2_{m,G',w}\right] \leq 0.\end{aligned} \tag{3.6.3}$$

Using (3.6.3), (3.6.2) and a limiting argument, we obtain

$$\int_{G'} vLv\, dx \leq 0. \tag{3.6.4}$$

From (3.6.4) it follows that the smallest eigenvalue of the eigenvalue problem

$$Ly = \mu y, \quad y \in H_m^0(G') \cap C^{2m}(G') \tag{3.6.5}$$

is nonpositive. Consequently, standard variational arguments imply that G' has a nonempty open subset G'' such that zero is the smallest eigenvalue of the eigenvalue problem

$$Lu = \mu u, \quad u \in H_m^0(G'') \cap C^{2m}(G''). \tag{3.6.6}$$

Thus, we have shown that, for any $r > 0$, the equation (1.5) has a nodal domain $G'' \subset G' \subset \{x \in \Omega : |x| > r\}$. The proof of Theorem 3.6 is now complete. □

REMARK 3.7. Using Definition 1.3, we can compare M_0 with the differential operator M defined by

$$Mz = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha\left[e_{\alpha\beta}(x)D^\beta z\right] + e_0(x)z, \tag{3.7.1}$$

where the coefficient functions e_0 and $e_{\alpha\beta}$, $|\alpha| = |\beta| = m$, satisfy regularity, symmetry and ellipticity conditions analogous to those satisfied by the functions a_0 and $a_{\alpha\beta}$, $|\alpha| = |\beta| = m$. In fact, we can establish the following comparison theorem (see [**13**, Lemma 2.4] for details).

LEMMA 3.8. *Suppose that, for every* ϕ *in* $C_0^\infty(\Omega)$,

$$\int_\Omega \phi M_0 \phi \, dx \leq \int_\Omega \phi M \phi \, dx. \tag{3.8.1}$$

If the equation

$$Mz = 0 \tag{3.8.2}$$

has a nontrivial solution which is oscillatory in Ω *in the sense of Definition* 1.3, *then* (1.7) *has a nontrivial solution which is oscillatory in* Ω *in the sense of Definition* 1.3.

REMARK 3.9. Since the principal part of the differential operator M_1 has a simple radial form, it is not hard to generate differential operators of the form M_1 for which criteria for weak oscillation can be readily obtained. Using these criteria, together with Theorems 2.4, 3.6 and Lemma 3.8, we can obtain criteria for nodal oscillation of the general even-order equation (1.5). As an illustration of this method, we generalize, in Theorem 3.10, an oscillation criterion that was obtained in [**10**, Example 4.4].

To set the stage for Theorem 3.10, we recall some ideas from [**10**]. Define the polynomial function $P_{m,n}$ by

$$P_{m,n}(r) = \prod_{j=1}^{m} (r - 2j + 2)(r - 2j + n). \tag{3.9.1}$$

We refer the reader to [**10**, Proposition 3.1] for zero-distribution properties of $P_{m,n}$. Let

$$N = \{r \in \mathbf{R}^1 : P_{m,n}(r) = 0\}; \tag{3.9.2}$$

$$V = \{r \in R^1 \backslash N : (r, P_{m,n}(r)) \text{ is a local maximum}\}, \tag{3.9.3}$$

$$W = \{r \in R^1 \backslash N : (r, P_{m,n}(r)) \text{ is a local minimum}\}. \tag{3.9.4}$$

Note that V and W are finite sets. If V is not empty, let

$$K_4 = \min\{P_{m,n}(r) : r \in V\}. \tag{3.9.5}$$

If W is not empty, let

$$K_5 = \min\{|P_{m,n}(r)| : r \in W\}. \tag{3.9.6}$$

If both V and W are nonempty, let

$$K_6 = \begin{cases} K_4 & \text{if } K_4 < K_5 \\ -K_5 & \text{if } K_4 \geq K_5 \end{cases}. \tag{3.9.7}$$

(Note that V and W are simultaneously empty if and only if $(m, n) = (1, 2)$.)

THEOREM 3.10. *Let $m \geq 2$, let δ be any positive number, and let*

$$K_7 = \begin{cases} (-1)^{m+1}(K_4 + \delta) & \text{if } n = 2 \\ (-1)^m(K_5 + \delta) & \text{if } n = 4 \\ (-1)^{m+1}(K_6 + \delta) & \text{if } n \neq 2 \text{ and } n \neq 4. \end{cases} \tag{3.10.1}$$

For any $r > 0$, let

$$S_r = \{x \in R^n : |x| = r\}. \tag{3.10.2}$$

Define the functions $h_2 : (0, \infty) \to \mathbf{R}^1$ and $h_3 : \Omega \to \mathbf{R}^1$ as follows:

$$h_2(r) = \max\{[A_{0,0}(x) + c_2] : x \in S_r\}, \tag{3.10.3}$$

$$h_3(x) = h_2(|x|). \tag{3.10.4}$$

If there exists $r_1 > 0$ such that

$$c_1|x|^{2m}h_3(x) \leq K_7 \text{ whenever } x \in \Omega \text{ and } |x| > r_1, \tag{3.10.5}$$

then (1.5) *is nodally oscillatory in Ω.*

PROOF. The definition of K_7 implies that the polynomial equation

$$P_{m,n}(r) + (-1)^m K_7 = 0 \tag{3.10.6}$$

has at least one complex root with nonzero imaginary part; hence, the differential equation

$$(-1)^m \Delta^m z + K_7 |x|^{-2m} z = 0 \tag{3.10.7}$$

has at least one nontrivial solution which is oscillatory, in the sense of Definition 1.3, in the unbounded open set

$$\Omega^* := \{x \in \Omega : |x| > r_1\}. \tag{3.10.8}$$

Let M_2 and M_3 be differential operators defined as follows:

$$M_2 u = c_1(-1)^m \Delta^m u + K_7 |x|^{-2m} u, \tag{3.10.9}$$

$$M_3 u = (-1)^m c_1 \Delta^m u + h_3(x) u. \tag{3.10.10}$$

Then the hypothesis (3.10.5) and the definitions of the functions h_2 and h_3 imply that, for any nonempty open set G contained in Ω^* and any ϕ in $C_0^\infty(G)$,

$$\int_G \phi M_1 \phi \, dx \le \int_G \phi M_3 \phi \, dx \le \int_G \phi M_2 \phi \, dx. \tag{3.10.11}$$

Applying Lemma 3.8, we deduce from (3.10.11) that (3.6.2) has at least one nontrivial solution which is oscillatory in Ω^*, in the sense of Definition 1.3. It follows from Theorem 2.4 that (3.6.2) is nodally oscillatory in Ω^*. Hence, (3.6.2) is nodally oscillatory in Ω. It follows from Theorem 3.6 that (1.5) is nodally oscillatory in Ω. □

REMARK 3.11. It can easily be shown, using [**10**, Theorem 4.1], that the constant K_7 is optimal.

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