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RIGHT AND LEFT DISCONJUGACY IN DIFFERENCE EQUATIONS

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1. Introduction. We are concerned with the n-th order difference equation

$$(1) \qquad Py(t) \equiv \sum_{i=0}^{n} \alpha_i(t) y(t+i) = 0, \quad t \in [a,b],$$

where $a < b$ are integers and $[a,b] \equiv \{a, a+1, \dots, b\}$, $\alpha_n = 1$, and α_0 satisfies

$$(2) \qquad (-1)^n \alpha_0(t) > 0, \quad t \in [a,b].$$

Solutions of the difference equation (1) are defined on $[a, b+n]$.

In part, we will be concerned with a partial factorization of P if (1) is right $(j, n-j)$-disconjugate. In addition, we give several results relating right and left disconjugacy and disconjugacy.

As defined by Hartman, (1) is said to be disconjugate on an interval J if no nontrivial solution has n generalized zeros on J. In the classic paper [**2**], Hartman has shown that (1) is disconjugate on J if and only if P has a certain factorization. Further, necessary and sufficient conditions for disconjugacy in terms of the coefficients $\alpha_i(t)$ are given, and sign conditions on the Green's functions for certain boundary value problems for a disconjugate difference equation are given.

More recently, Peterson [**5**] defined the more general notions of right and left disconjugacy. Necessary conditions for right $(j, n-j)$-disconjugacy in terms of the coefficients $\alpha_i(t)$ are given in [**4**]. Peterson [**7**] also gave necessary and sufficient conditions for $(j, n-j)$-disconjugacy in terms of certain Wronskians. Finally, Peterson [**6**] gave sign conditions on the Green's functions for boundary value problems where (1) satisfies certain $(j, n-j)$-disconjugacy conditions.

2. Preliminaries. Define the difference operator Δ by $\Delta y(t) = y(t+1) - y(t)$, and define the operators Δ^i by $\Delta^i y(t) = \Delta(\Delta^{i-1} y(t))$

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for $i = 1, 2, \dots,$ where $\Delta^0 y(t) = y(t)$. As defined by Hartman [**2**], we say that a function $y(t)$ defined on $[a, b+n]$ has a *generalized zero* at t_0 in case either $y(t_0) = 0$, or, if $t_0 > a$, there exists an integer $1 \le j \le t_0 - a$ such that

$$(-1)^j y(t_0 - j) y(t_0) > 0,$$
$$y(t) = 0, \quad t_0 - j < t < t_0.$$

The difference equation (1) is *disconjugate* on an interval $J \subseteq [a, b+n]$ if no nontrivial solution has n generalized zeros on J. It is easy to see that condition (2) is a necessary condition for disconjugacy.

The following definition is due to Peterson (see [**5**]).

DEFINITION 1. Let J be a subinterval of $[a, b+n]$, and let $1 \le j \le n-1$. We say that (1) is *right* $(j, n-j)$*-disconjugate* on J provided there is no nontrivial solution $y(t)$ of (1) and integers $\alpha, \beta \in J$ with $\alpha + j \le \beta \le \beta + n - j - 1 \in J$ such that

$$y(\alpha + i) = 0, \quad 0 \le i \le j - 1,$$
$$y(\beta + i) = 0, \quad 0 \le i \le n - j - 2,$$

and y has a generalized zero at $\beta + n - j - 1$. Similarly, we say that (1) is *left* $(j, n-j)$*-disconjugate* on J provided there is no nontrivial solution $y(t)$ of (1) and integers $\alpha, \beta \in J$ with $\alpha + j \le \beta \le \beta + n - j - 1 \in J$ such that

$$y(\alpha + i) = 0, \quad 0 \le i \le j - 2,$$
$$y(\beta + i) = 0, \quad 0 \le i \le n - j - 1,$$

and y has a generalized zero at $\alpha + j - 1$.

If (1) is disconjugate on J, then it is right $(j, n-j)$-disconjugate for $1 \le j \le n-1$. It is easy to see that right $(j, n-j)$-disconjugacy for some fixed $1 \le j \le n-1$ does not imply right $(n-j, j)$-disconjugacy.

EXAMPLE 1. Consider the difference equation

$$u(t+3) - u(t+2) - u(t+1) - u(t) = 0.$$

This difference equation is right $(2, 1)$-disconjugate on $[0, 3]$, but there is a solution $u(t)$ with $u(0) = u(2) = 0$ and $u(1) = u(3) = 1$, so that

$u(t)$ has a generalized zero at $t = 3$. Hence, the difference equation is not right $(1, 2)$-disconjugate on $[0, 3]$.

EXAMPLE 2. It was shown in [**1**] that the difference equation

$$Ly(t) + p(t)y(t) = 0, \quad t \in J,$$

where L is disconjugate, is right $(j, n-j)$-disconjugate for those integer values of j, $1 \le j \le n-1$, such that $(-1)^{n-j}p(t) > 0$ (we still assume (2) holds for this equation).

For functions $y_1, \dots, y_j$ defined on $[c, d]$, define the *Wronskian*

$$W(y_1, \dots, y_j)(t) = \begin{vmatrix} y_1(t) & \cdots & y_j(t) \\ \Delta y_1(t) & \cdots & \Delta y_j(t) \\ \vdots & \ddots & \vdots \\ \Delta^{j-1} y_1(t) & \cdots & \Delta^{j-1} y_j(t) \end{vmatrix}$$

for $t \in [c, d-j+1]$.

3. Results on disconjugacy. To begin with, a careful examination of the proof of [**7**, Theorem 2] shows that this theorem has the following generalization.

THEOREM 1. *For each s, let $u_j(t, s)$ be a solution of* (1) *satisfying the partial set of initial conditions $u_j(s+i, s) = \delta_{ij}$, $0 \le i \le j$, for $1 \le j \le n-1$.*

(a) *The difference equation* (1) *is right $(j, n-j)$-disconjugate on $[c, d]$ if and only if $W[u_j(t, s), \dots, u_{n-1}(t, s)] > 0$ for $c \le s \le t-j \le d-n+1$.*

(b) *The difference equation* (1) *is left $(j, n-j)$-disconjugate on $[c, d]$ if and only if $(-1)^{j(n-j)} W[u_{n-j}(t, s), \dots, u_{n-1}(t, s)] > 0$ for $c \le t \le s-j \le d-n+1$.*

As a corollary to this theorem, we can prove the following partial factorization result.

COROLLARY 1. *If* (1) *is right $(j, n-j)$-disconjugate on $[c, d]$, then there exist solutions $u_j(t), \dots, u_{n-1}(t)$ of* (1) *and a difference equation*

$P_1 y(t) = 0$ *of order* j *such that*

$$Pu(t) = P_1 P_2 u(t), \quad t \in [c, d-n],$$

where $P_2 u(t) = W(u(t), u_j(t), \dots, u_{n-1}(t))$.

PROOF. For $\epsilon > 0$ define $u_l^\epsilon(t)$ to be the solution of (1) satisfying

$$u_l^\epsilon(c+i) = \frac{\epsilon^{l-i}}{(l-i)!}, \quad 0 \le i \le l,$$
$$u_l^\epsilon(c+i) = 0, \quad l+1 \le i \le n-1,$$

for $j \le l \le n-1$. Note that $u_l^\epsilon(t)$ converges uniformly to the solution $u_l(t,c)$ of (1) on $[c,d]$ satisfying

$$u_l(c+i,c) = \delta_{li}, \quad 0 \le i \le n-1.$$

It follows from Theorem 1 that there is an $\epsilon > 0$ such that

$$W(u_j^\epsilon(t), \dots, u_{n-1}^\epsilon(t)) > 0, \quad t \in [c+j, d-n+j+1].$$

It can be shown that

$$W(u_j^\epsilon(t), \dots, u_{n-1}^\epsilon(t)) > 0, \quad t \in [c, c+j-1].$$

Hence, if $u_j(t) = u_j^\epsilon(t), \dots, u_{n-1}(t) = u_{n-1}^\epsilon(t)$, then

$$W(u_j(t), \dots, u_{n-1}(t)) > 0, \quad t \in [c, d-n+j+1].$$

The corollary then follows from [**2**, Proposition 4.2]. □

REMARK 1. The form of P_1 in the above theorem is given in [**2**, Proposition 4.2] except that the coefficients β_{n-k} and β_0 should be given by

$$\beta_{n-k} = \frac{(-1)^k}{w_k(m+n-k)}, \quad \beta_0 = \frac{\alpha_0(m)}{w_k(m+1)}.$$

The following relationship between right disconjugacy and disconjugacy can also be considered as a corollary to Theorem 1.

COROLLARY 2. *If* (1) *is right* $(j, n-j)$*-disconjugate on* $[c, d]$ *for* $j = 1, \dots, n-1$*, then* (1) *is disconjugate on* $[c, d]$.

We will conclude with two theorems which give relationships between right and left disconjugacy.

THEOREM 2. *Let* $j \in \{1, n-1\}$*. Then* (1) *is right* $(j, n-j)$*-disconjugate on* J *if and only if* (1) *is left* $(j, n-j)$*-disconjugate on* J.

PROOF. We prove this for $j = 1$. The case $j = n-1$ is similar. Assume first that (1) is right $(1, n-1)$-disconjugate on J. Suppose that (1) is not left $(1, n-1)$-disconjugate on J. Then there is a nontrivial solution $u(t)$ and integers c, d with $c < d < d+n-2 \in J$ such that

$$\begin{aligned} u(c) &\leq 0 \\ u(t) &> 0, \quad t \in [c+1, d-1] \\ u(d+i) &= 0, \quad 0 \leq i \leq n-2. \end{aligned}$$

Note that $u(c) < 0$ since (1) is right $(1, n-1)$-disconjugate. Among all such solutions $u(t)$ and integers c, d given above, assume that $u(t)$ is such that $d-c$ is minimal. We can assume $u(d-1) = 1$. Since (1) is right $(1, n-1)$-disconjugate, there is a solution $v(t)$ satisfying

$$\begin{aligned} v(c) &= 0 \\ v(d-1) &= 1 \\ v(d+i) &= 0, \quad 0 \leq i \leq n-3. \end{aligned}$$

Note that, by the right $(1, n-1)$-disconjugacy, $v(t)$ does not have a generalized zero at $d+n-2$ so that $(-1)^{n-1} v(d+n-2) < 0$. From the difference equation (1),

$$\alpha_1(d-2)u(d-1) + \alpha_0(d-2)u(d-2) = 0$$

so that

$$u(d-2) = \frac{-\alpha_1(d-2)}{\alpha_0(d-2)}.$$

Similarly,

$$v(d+n-2)+\alpha_1(d-2)v(d-1)+\alpha_0(d-2)v(d-2)=0$$

so that

$$v(d-2)=\frac{-\alpha_1(d-2)}{\alpha_0(d-2)}-\frac{v(d+n-2)}{\alpha_0(d-2)}.$$

Since $(-1)^{n-1}v(d+n-2)<0$ and (2) holds,

$$\operatorname{sgn}\{\alpha_0(d-2)v(d+n-2)\}=(-1)^n(-1)^n=1$$

so that $u(d-2)>v(d-2)$. Pick $\alpha>0$ such that $w(t)\equiv\alpha u(t)-v(t)\geq 0$ on (c,d) and there exists a $t_0\in(c,d)$ such that $w(t_0)=0$.

First consider the case $t_0\in(c,d-2]$. Then

$$\begin{aligned} w(t_0)&=0\\ w(d-1)&\geq 0\\ w(d+i)&=0, \quad 0\leq i\leq n-3,\\ w(d+n-2)&=-v(d+n-2) \end{aligned}$$

and $(-1)^{n-1}w(d-1)w(d+n-2)\geq 0$ so that $w(t)$ has a zero at $d-1$ or a generalized zero at $d+n-2$. This contradicts the right $(1,n-1)$-disconjugacy.

Now consider the case $t_0=d-1$. Then $\alpha=1$ and

$$\begin{aligned} w(c)&<0\\ w(d-2)&=u(d-2)-v(d-2)>0\\ w(d+i-1)&=0, \quad 0\leq i\leq n-2, \end{aligned}$$

which contradicts the minimality of $d-c$. Hence, (1) is left $(1,n-1)$-disconjugate on J.

Conversely, assume (1) is left $(1,n-1)$-disconjugate on J. Assume $[d,d+n-1]\subseteq J$ and $v(t)$ is a solution such that

$$\begin{aligned} v(d)&=1\\ v(d+i+1)&=0, \quad 0\leq i\leq n-3,\\ (-1)^{n-1}v(d+n-1)&>0. \end{aligned}$$

It suffices to show that $v(t) > 0$ on $(-\infty, d] \cap J$. By equation (1),

$$v(d+n-1) + \alpha_1(d-1)v(d) + \alpha_0(d-1)v(d-1) = 0$$

so that

$$v(d-1) = \frac{-\alpha_1(d-1)}{\alpha_0(d-1)} - \frac{v(d+n-1)}{\alpha_0(d-1)}.$$

Let $u(t)$ be the solution of (1) satisfying

$$\begin{aligned} u(d) &= 1 \\ u(d+i+1) &= 0, \quad 0 \le i \le n-2. \end{aligned}$$

Then, using equation (1) and solving for $u(d-1)$, we obtain

$$u(d-1) = \frac{-\alpha_1(d-1)}{\alpha_0(d-1)}.$$

Note that $\operatorname{sgn}\{v(d+n-1)\alpha_0(d-1)\} = (-1)^{n-1}(-1)^n = -1$ so that $v(d-1) > u(d-1)$. It follows that $w(t) \equiv v(t) - u(t)$ is a solution of (1) satisfying

$$\begin{aligned} w(d-1) &> 0 \\ w(d+i) &= 0, \quad 0 \le i \le n-2. \end{aligned}$$

By the left $(1, n-1)$-disconjugacy of (1), we have that both $w(t) > 0$ and $u(t) > 0$ for $t \in (-\infty, d-1] \cap J$. It follows that $v(t) \ge u(t) > 0$ for $t \in (-\infty, d] \cap J$. Hence, (1) is right $(1, n-1)$-disconjugate on J. □

THEOREM 3. *Assume* $1 \le j \le n-2$ *and that* (1) *is right* $(n-i, i)$-*disconjugate on* J *for* $i = 1, \dots, j+1$. *Then* (1) *is left* $(n-j, j)$-*disconjugate on* J.

PROOF. The proof is by induction on j. Theorem 2 shows that the result holds for $j = 1$. Assume $1 < j \le n-2$ and that the result holds if j is replaced by $l < j$. Suppose the result does not hold at j. Then there exists $c, d \in J$, with $c+n-j \le d < d+j-1 \in J$, and a nontrivial solution $u(t)$ of (1), with

$$\begin{aligned} u(c+i) &= 0, \quad 0 \le i \le n-j-2, \\ u(d+i) &= 0, \quad 0 \le i \le j-1, \end{aligned}$$

and u has a generalized zero at $c+n-j-1$. By the right $(n-j,j)$-disconjugacy, we must have that $u(c+n-j-1) \neq 0$ and $c-1 \in J$. Without loss of generality, assume $u(c+n-j-1) = 1$. Note, by the induction step, (1) is left $(n-i,i)$-disconjugate for $1 \leq i \leq j-1$. Let $v(t)$ be the solution of (1) with

$$\begin{aligned} v(c+i-1) &= 0, \quad 0 \leq i \leq n-j-1, \\ v(c+n-j-1) &= 1 \\ v(d+i+1) &= 0, \quad 0 \leq i \leq j-2. \end{aligned}$$

Then $v(t) > 0$ on $[c+n-j-1,d]$ by [**3**, Theorem 7]. Let $\alpha > 0$ be such that $w(t) \equiv \alpha v(t) - u(t) \geq 0$ on $[c,d]$ and there exists $t_0 \in [c+n-j-1, d-1]$ such that $w(t_0) = 0$. Note that $v(c-1) = 0$ so $w(c-1)$ and $u(c-1)$ are of opposite sign.

First consider the case $t_0 = c+n-j-1$. Then $w(t)$ has $n-j$ zeros at c and $w(c+n-j) > 0$. Note that

$$\begin{aligned} \operatorname{sgn}\{w(c-1)w(c+n-j)\} &= -\operatorname{sgn}\{u(c-1)\} \\ &= -\operatorname{sgn}\{u(c-1)u(c+n-j-1)\}. \end{aligned}$$

Since u has a generalized zero at $c+n-j-1$, it follows that w has a generalized zero at $c+n-j$. But then w has $n-j$ zeros at c, $j-1$ zeros at $d+1$, and a generalized zero at $c+n-j$, contradicting that (1) is left $(n-j+1, j-1)$-disconjugate on J.

Now consider the case $c+n-j \leq t_0 < d$. In this case, w has $n-j-1$ zeros at c, $j-1$ zeros at $d+1$, and a zero at t_0. Since $w(t) \geq 0$ on $[c+n-j-1,d]$, either $w(t_0-1) = 0$ or w has a generalized zero at t_0+1, contradicting that (1) is ρ_{n-j-1}-disconjugate on J (see [**3**, Theorem 7]).

Hence the result holds at j, and by induction the proof is complete. □

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