COEXISTENCE IN CHEMOSTAT-LIKE MODELS

PAUL WALTMAN

Dedicated to the memory of Geoffrey Butler

1. Introduction. Competition modeling is one of the more challenging aspects of mathematical biology. Competition is clearly important in nature yet there are so many ways for populations to compete that the modeling problem is difficult to do in any generality. On the other hand, the mathematical idea seems quite simple—when any population increases, the growth rate of the others should diminish, a concept that is quite easily expressed by partial derivatives of the specific growth rates. If an ecosystem is modeled by a system of differential equations, for example, by

$$y_i' = y_i f_i(y),$$

where $i = 1, 2, \ldots, n, f_i$ is a nonnegative, continuously differentiable function defined on $\mathbf{R}^n$, and $y = (y_1, y_2, \ldots, y_n)$, then competition is expressed by the condition

$$\frac{\partial f_i}{\partial y_j} \leq 0$$

when $i \neq j$. Dynamical systems with such properties have been studied extensively, see Hirsch [20, 21] and Smith [34]. When $n = 2$, such dynamical systems preserve an order (leave a cone invariant) under the flow in forward time, a property which can yield valuable information about potential asymptotic behavior.

Such models easily reflect the direct impact of one population upon the other; for example, one produces metabolic products that inhibit the growth of the other. The simplest form of competition is where two or more populations compete for the same resource, for example, the same food supply or the same growth limiting nutrient. One

can view the "competitors" as "predators" on the "nutrient," and this produces an entirely different type of behavior for the resulting dynamical system. Such systems have a tendency towards oscillation. In some of the simplest cases, however, there is a "conservation law" that enables one to reduce the system to the above competitive type. The lectures here survey a class of problems of this type with a view of determining when it is possible for all of the components of the model ecosystem to "survive." The basic model is the simple chemostat; from it, there are a great many variations which lead to interesting mathematical questions.

   2. The simple chemostat. The chemostat is a piece of laboratory apparatus used to culture microorganisms. It is of ecological interest because it is a laboratory model of a very simple lake. It also is one place where the mathematics is tractable, the parameters are measurable, and the experiments are reasonable. The importance of the chemostat in ecology is well documented in the survey articles of Jannash and Mateles [**27**], Taylor and Williams [**38**], Waltman, Hsu and Hubbell [**41**] and Frederickson and Stephanopoulos [**13**]. Important experiments on competition in the chemostat can be found in Hansen and Hubbell [**17**].

   The apparatus consists of three connected vessels. The first contains all of the nutrients needed for growth of a microorganism, all in excess except for one called the limiting nutrient. The concentration of the limiting nutrient, hereafter simply called the nutrient, is kept constant, and the nutrient is pumped at a constant rate into the second vessel, the culture vessel. The volume of this vessel is kept constant by pumping the contents out at the input rate. The culture vessel is charged with a variety of microorganisms, so it contains a mixture of nutrient and organisms. Its output is collected in the third vessel which represents the "production" of the chemostat. The culture vessel is well stirred and all other significant parameters affecting growth, for example, temperature, are kept constant. Since the output is continuous, the chemostat is often referred to as "continuous culture" to contrast it with the more common "batch culture" of microorganisms. A schematic is shown in Figure 2.1.

   We seek to write differential equations for the above model. A more complete derivation can be found elsewhere, for example, in Herbert, Elsworth, and Telling [**19**]. The rate of change of the nutrient can be
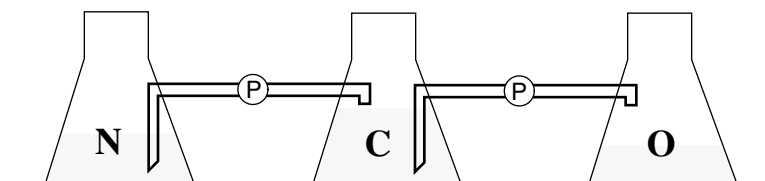
FIGURE 2.1. A schematic of a simple chemostat.

expressed as

$$\text{Rate of Change} = \text{input} \ - \text{washout} \ - \text{consumption},$$

while that of the organism can be expressed as,

$$\text{Rate of change} = \text{growth} \ - \text{washout}.$$

Let $S(t)$ and $x(t)$ denote the concentration of nutrient and organism at time t. The decrease in the rate of change because of washout is proportional to the concentration. If there were no organisms—hence no consumption—the equation for the nutrient would be

$$S'(t) = (S^{(0)} - S(t))D,$$

where $S^{(0)}$ is the input concentration and D is a constant of proportionality (reflecting the pump speed or "washout rate"). The formulation of the consumption term, based on experimental evidence, goes back at least to Monod [31] and takes the form

$$\frac{mSx}{a+S},$$

where m is the maximal growth rate and a is the Michaelis-Menten (or half saturation) constant. The form (and the terminology) of the consumption term is that of enzyme kinetics where S would be substrate. Both a and m can be measured experimentally. The differential equation for S takes the form

$$(2.1) \qquad S' = (S^{(0)} - S)D - \frac{mS}{a+S}\frac{x}{\gamma},$$

while that of the corresponding equation for the microorganism is

$$(2.2) \qquad x' = x\left(\frac{mS}{a+S} - D\right),$$

where γ is a "yield" constant reflecting the conversion of nutrient to organism. (We will scale it out in the simple chemostat but it is important for multiple nutrient problems.) The corresponding initial conditions are $S(0) \geq 0$ and $x(0) > 0$. The number of parameters in the system is excessive (mathematically), so some scaling is in order. First of all, note that $S^{(0)}$ and D are under the control of the experimenter (the input concentration and the washout rate). $S^{(0)}$ has units of concentration and D has units of reciprocal time. By measuring S, a, and x/γ in units of $S^{(0)}$ and time in units of D^{-1} one obtains the nondimensional differential equations (note that m and a have changed their meanings)

$$(2.3) \qquad \begin{aligned} S' &= 1 - S - \frac{mSx}{a+S} \\ x' &= x\left(\frac{mS}{a+S} - 1\right) \\ S(0) &\geq 0, \quad x(0) > 0. \end{aligned}$$

We can regard m and a as the "natural" parameters of the organism in this particular environment.

For system (2.3) the positive cone is positively invariant. Moreover, if one adds the two equations, and lets $\Psi = S + x$, one has that

$$\Psi' = 1 - \Psi$$

with $\Psi(0) > 0$. It follows at once that $\lim_{t\to\infty} \Psi(t) = 1$. This not only gives the required dissipativeness but also that, on the omega limit set, $x(t)$ satisfies

$$(2.4) \qquad x'(t) = x\left[\frac{m(1-x)}{1+a-x} - 1\right], \quad 0 \leq x \leq 1.$$

Define

$$\lambda = \frac{a}{m-1}.$$

(λ is called the "break even" concentration.) (2.4) has two critical points, $x = 0$ and $x = 1 - \lambda$, and the equation can be rewritten as

$$x' = x \left(\frac{m - 1}{1 + a - x} \right) [1 - \lambda - x].$$

Clearly, if $m < 1$ or $m > 1$ and $\lambda > 1$, then $\lim_{t \to \infty} x(t) = 0$. On the other hand, if $\lambda < 1$ and $m > 1$, then $\lim_{t \to \infty} x(t) = 1 - \lambda$ and $\lim_{t \to \infty} S(t) = \lambda$. If $m < 1$, the organism is washing out faster than its maximal growth rate, while, if $\lambda > 1$, there is insufficient nutrient available for the organism to survive. In either case extinction is not a surprising outcome. The case $m = 1$ is handled by using (2.4).

To study competition in the chemostat, introduce two different micro-organisms into the system, labeled x_1 and x_2, with corresponding natural parameters a_i and $m_i, i = 1, 2$. We assume that the corresponding λ's, λ_1 and λ_2, are different. The overall system becomes

(2.5)
$$S' = 1 - S - \frac{m_1 S x_1}{a_1 + S} - \frac{m_2 S x_2}{a_2 + S}$$
$$x_1' = x_1 \left(\frac{m_1 S}{a_1 + S} - 1 \right)$$
$$x_2' = x_2 \left(\frac{m_2 S}{a_2 + S} - 1 \right)$$
$$S(0) \geq 0, \quad x_1(0) > 0, \quad x_2(0) > 0.$$

In the same manner as above, one shows that

$$\lim_{t \to \infty} S(t) + x_1(t) + x_2(t) = 1,$$

where the convergence is exponential.

Again this yields the dissipative condition and the fact that, on the omega limit set, trajectories satisfy

(2.6)
$$x_1' = x_1 \left(\frac{m_1 - 1}{1 + a - x_1 - x_2} \right) [1 - \lambda_1 - x_1 - x_2]$$
$$x_2' = x_2 \left(\frac{m_2 - 1}{1 + a - x_1 - x_2} \right) [1 - \lambda_2 - x_1 - x_2]$$
$$x_1(0) \geq 0, \quad x_2(0) \geq 0, \quad x_1(0) + x_2(0) \leq 1.$$

This system is competitive, dissipative, and, since it is two-dimensional, preserves the cone

$$K = \{(x_1, x_2) \mid x_1 \geq 0, x_2 \leq 0\}.$$

The system (2.6) has three rest points,

$$\begin{aligned}
E_0 &= (0, 0), \\
E_1 &= (1 - \lambda_1, 0), \\
E_2 &= (0, 1 - \lambda_2).
\end{aligned}$$

THEOREM 2.1. *Suppose that $0 < \lambda_1 < \lambda_2 < 1$. Then any solution of the system* (2.5) *satisfies*

$$\begin{aligned}
\lim_{t \to \infty} S(t) &= \lambda_1 \\
\lim_{t \to \infty} x_1(t) &= 1 - \lambda_1 \\
\lim_{t \to \infty} x_2(t) &= 0.
\end{aligned}$$

PROOF. Let $(x_1(t), x_2(t))$ be a solution with initial conditions in the positive cone. Its omega limit set is not empty and the trajectory is asymptotic to it. Thus is it necessary only to analyze the system (2.6). If there were an interior attractor, it must contain an equilibrium point [21]. However, there are no equilibrium points in the interior of the positive quadrant since $\lambda_1 < \lambda_2$, so the omega limit set is on the boundary. (Two-dimensional competitive systems have no periodic orbits.) A simple computation shows that E_0 is a repeller, E_1 is asymptotically stable (locally) and E_2 repels the interior of the cone. Thus all trajectories tend to E_1. □

The experiments of Hansen and Hubbell [17] confirm the mathematical result. By working with various microorganisms, Hansen and Hubbell showed that it is the lambda value which determines the outcome of the competition. It is worth noting that this was an example of the mathematics preceding the biological experiment. I would like to add some personal remarks. The biologist and the mathematicians

collaborated from the beginning on the problem; hence, there was no reluctance on the part of the biologist to carry out the time consuming experiments. It is my experience that this sort of collaboration is rare—biologists and mathematicians have different agendas in this field—yet it is necessary for mathematical biology to achieve "scientific respectability" outside the realm of interested specialists.

This theorem is an example of the principle of competitive exclusion—only one competitor can survive on a single resource. Many of the well-known models of competitive systems seem to satisfy this conclusion, for example, the two-dimensional Lotka-Volterra competition model. However, in nature, many populations seem to coexist together, ostensibly on the same resource. It becomes then an interesting problem to modify the model, taking into account some new aspect which will produce coexistence. We will survey some of the modifications for they lead to mathematically interesting difficulties. We can list the following possibilities:

1. Introduce more competitors.

2. Modify the functional response.

3. Make either the nutrient concentration or the washout rate time dependent (to introduce seasonal variations).

4. Introduce an additional trophic level.

5. Introduce delays in the conversion of nutrient to organism.

6. Introduce diffusion (remove the well-stirred hypothesis).

These possibilities are discussed in the next sections. Another possibility, not discussed here, is to introduce multiple nutrients. This is a subject of considerable scope and the interested reader might want to consult [**7, 25, 29**, or **39**].

3. Extensions. In this section we survey some of the literature for the first two of the possibilities mentioned above. The modifications will not produce (robust) coexistence but each adds an interesting dimension to chemostat models.

If, in the chemostat, several populations of microorganisms were introduced, then the equations take the form (ignoring the yield constants)

$$S' = (S^{(0)} - S)D - \sum_{i=1}^{n} \frac{m_i S x_i}{a_i + S}$$

$$x_i' = x_i \left(\frac{m_i S}{a_i + S} - D \right), \quad i = 1, 2, \ldots, n,$$

with $S(0) \geq 0, x_i(0) > 0$. Scaling D and $S^{(0)}$ as above yields the normalized equations

(3.1)
$$S' = 1 - S - \sum_{i=1}^{n} \frac{m_i S x_i}{a_i + S}$$

$$x_i' = x_i \left(\frac{m_i S}{a_i + S} - 1 \right), \quad i = 1, 2, \ldots, n.$$

The corresponding λ's are defined by

$$\lambda_i = \frac{a_i}{m_i - 1},$$

where it is assumed that $m_i > 1$ if λ_i is to be defined. The basic results are contained in two statements. If $m_i \leq 1$, or if $\lambda_i \geq 1$, then $\lim_{t \to \infty} x_i(t) = 0$. In this case the entire system merely tends to a lower order dynamical system, that is, one with fewer competitors. The statement provides necessary conditions for survivability and one need only consider competitors which satisfy these conditions. Thus, one can assume that all of the λ_i's are defined. The second statement is that, if

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$$

then

$$\lim_{t \to \infty} x_1(t) = 1 - \lambda_1$$
$$\lim_{t \to \infty} x_i(t) = 0, \quad i = 2, \ldots, n.$$

A proof was given in [26] but a much neater (and more general) proof, using a Liapunov function, appears in Hsu [23]. The proof in [23] also allows the parameter (in the unscaled version) D to be different for each

type of organism, that is, change D to D_i in the unscaled equations. Thus this result is (biologically) more general in that it can take into account differing natural "death" rates. (The basic chemostat model only allows removal through "washout.") When there are more than two competitors the monotonicity arguments given in the preceding section do not work—at least directly—since a cone is not left invariant by the flow. However, there is the preservation of "unorderedness" by the flow (trajectories which are unordered at some time are unordered at all future times.) It would be of interest to find a simple proof based on this property which does not require a Liapunov function.

As noted above, the functional response in the chemostat equations is based on experimental evidence and follows the Michaelis-Menten dynamics of enzyme kinetics. It is one of a class of functional responses known as Holling Type II [**22**]. There is also experimental evidence for other types of functional responses, so it is of interest to consider models of more generality. If one replaces this particular form with a general term, the equations become (ignoring yield constants)

$$S' = (S^{(0)} - S)D - \sum_{i=1}^{n} x_i(t)p_i(S(t))$$
$$x_i' = x_i(p_i(S(t)) - D), \quad i = 1, 2, \ldots, n.$$

Introducing the same scaling as before yields the new system

$$(3.2) \qquad \begin{aligned} S' &= 1 - S - \sum_{i=1}^{n} x_i(t)p_i(S(t)) \\ x_i' &= x_i(p_i(S(t)) - 1), \quad i = 1, 2, \ldots, n. \end{aligned}$$

There is evidence, [**42**], that a particular nutrient can be inhibiting at higher concentrations. Butler and Wolkowicz [**6**] consider the system (3.2) under the following assumptions:

(i) $p_i : \mathbf{R}^+ \to \mathbf{R}^+$;

(ii) p_i is continuously differentiable;

(iii) $p_i(0) = 0$;

(iv) There exist unique, positive, extended real numbers λ_i and μ_i such that $p_i(S) < 1$ if $S \notin [\lambda_i, \mu_i]$, and $p_i(S) > 1$ if $S \in (\lambda_i, \mu_i)$;

(v) If λ_i (or μ_i) is finite, then $p_i'(\lambda_i) \neq 0$ ($p_i'(\mu_i) \neq 0$);

(vi) All λ_i, μ_i (other than those which are infinite) are distinct from each other and from 1.

In (iv) one makes the appropriate convention in case one or both of the numbers are infinite. The number 1 which appears in (iv) reflects the fact that the washout rate (the original D in the chemostat) has been scaled to one while the number 1 in condition (vi) reflects the fact that the nutrient concentration (the original $S^{(0)}$ in the chemostat) has been scaled to one. (v) and (vi) make the proofs easier but [**6**] notes that they can be removed. (iv) can be relaxed to a finite number of real numbers at the expense of a more complicated proof.

It is easy to show that all solutions of the system (3.2) with positive initial conditions are positive and that the system is dissipative. As before, if $\lambda_i \geq 1$, then $\lim_{t\to\infty} x_i(t) = 0$. Thus, one may classify the λ's by

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_\nu < 1 \leq \lambda_j$$

where $\nu + 1 \leq j \leq n$. If $\nu = 0$, then no competitor survives. Define

$$Q = \bigcup_{i=1}^{\nu} (\lambda_i, \mu_i).$$

If $\nu = 0$, then Q is empty. The principal result in [**6**] may now be stated.

THEOREM 3.1. *Let Λ denote the set of left endpoints of the components of Q which are less than one together with the number 1 if $1 \notin Q$. With the exception of a set of initial conditions of Lebesgue measure zero, all solutions of (3.2) satisfy $\lim_{t\to\infty} S(t) = \gamma, \gamma \in \Lambda$, with the following asymptotic behavior:*

If $\gamma = \lambda_i$, and $j \neq i$, then

$$\lim_{t\to\infty} x_i(t) = 1 - \lambda_i$$
$$\lim_{t\to\infty} x_j(t) = 0.$$

If $\gamma = 1$,

$$\lim_{t\to\infty} x_i(t) = 0, \quad i = 1, 2, \dots, n.$$

Moreover, for each $\gamma \in \Lambda$, there is an open, nonempty set of initial conditions for which the solutions of (3.2) satisfy $\lim_{t \to \infty} S(t) = \gamma$.

The exceptional set in the theorem is the stable manifold of equilibrium points whose S component takes on the value of one of the μ_i's. In the monotonic case, and, hence, in the basic chemostat model, Q is connected so competitive exclusion holds (see also [1]). Of course, when the p_i's are monotone, the μ_i's are infinite and the exceptional set does not exist.

The authors [6] conclude with an interesting theoretical water treatment problem [42] to illustrate the use of the theorem. Suppose that one contaminant is phenol and that the input concentration is high, much higher than the acceptable level of concentration which is denoted by A, where A is assumed to be much less than the input concentration (which is scaled to one). There are two microorganisms which feed on phenol. The first, denoted x_1 is growth limiting at low concentrations but inhibited at high concentrations. In the parameters above, $\lambda_1 < A < \mu_1 < 1$ (after scaling). The other, denoted by x_n has $A < \lambda_n < 1 < \mu_n$. Organism x_n would limit the concentration of phenol but at an unacceptably high level. If, however, one could find organisms $x_2, x_3, \ldots, x_{n-1}$ so that the (λ, μ) intervals overlap in such a way as to provide a single connected component of Q, then one would have $\lim_{t \to \infty} S(t) = \lambda_1$, $\lim_{t \to \infty} x_1(t) = 1 - \lambda_1$, $\lim_{t \to \infty} x_i(t) = 0, i > 1$. The system would equilibrate to a perfectly tolerable situation.

4. Forced oscillations. A natural modification of the chemostat is to allow the system to be forced, that is, to remove the hypotheses that the nutrient input concentration is constant or that the flow rate is constant. Changing these corresponds to allowing time dependent environmental changes in the lake being modeled (seasonal or daynight changes are the most obvious). The most likely assumption, at least for a first start, is to use periodic functions. Note that both of these quantities are under the control of the experimenter, and, thus, theoretical results on the problem suggest new experiments.

The case of an oscillating input nutrient concentration has been studied by Hsu [24], Smith [33], and Hale and Somolinas [16]. The case of a varying washout rate has been investigated by Butler, Hsu, and Walt-

man [4] and Gatto, Annaboratone, and Borghesi [15]. Stephanopoulos, Fredrickson and Aris [37] vary both (as a step function) and present some experimental results. The results in [4] will be described here. The general philosophy is clear, however, and useful in the analysis of other chemostat models. If the nutrient can be made to oscillate, then, during part of its "cycle," it may be in a region where one competitor has the advantage, while in a different part of its cycle, the other competitor might have the advantage. Could it not be then that both could survive? The answer is yes, if the parameters are "right." Right will be defined by a bifurcation theorem. The principal tool will be the use of the Rabinowitz bifurcation theorem, in a manner originally suggested by the work of Cushing, for example [11].

The variable washout rate will be denoted by a function $D(t)$ which is assumed to be positive, continuous, and periodic with period ω. It is convenient to scale time by the mean value of $D(t)$,

$$\frac{1}{\omega} \int_0^\omega D(t)\, dt,$$

and to again scale out the nutrient input concentration $S^{(0)}$. For a variable washout rate, the equations take the form

$$S' = (1 - S)D(t) - \frac{m_1 S x_1}{a_1 + S} - \frac{m_2 S x_2}{a_2 + S}$$

(4.1)
$$x_1' = x_1 \left(\frac{m_1 S}{a_1 + S} - D(t) \right)$$

$$x_2' = x_2 \left(\frac{m_2 S}{a_2 + S} - D(t) \right)$$

$$S(0) \geq 0, \quad x_1(0) > 0, \quad x_2(0) > 0.$$

Although the system is not autonomous, solutions will still be viewed as parametric curves in the nonnegative cone $\mathbf{R}_+^3$. Since the system is periodic one could also set up a discrete dynamical system and use the monotonicity. Another possibility is to set up a mapping in an appropriate function space as will be done below when differential-difference equations are used. The approach in [4] is to consider the nonautonomous system and deal with the resulting stability questions. The positive invariance of the positive cone and the faces $S - x_1$

and $S - x_2$ follows much as before, and the uniform boundedness follows from essentially the same invariance principle used in the simple chemostat model, i.e., one again has that

$$(4.2) \qquad \lim_{t \to \infty} [S(t) + x_1(t) + x_2(t)] = 1,$$

where the convergence is exponential.

It also follows without much difficulty that $\liminf_{t \to \infty} S(t) \geq \eta > 0$. The competitive exclusion conditions are also essentially as in the basic chemostat model. If $\lambda_i > 1$, that competitor will wash out of the vessel, and if $0 < \lambda_1 < \lambda_2 < 1$ and $m_1 \geq m_2$, then competitor number 2 must wash out of the system. This sets the stage that coexistence is possible only if both λ_i's are less than 1 (we take $\lambda_1 < \lambda_2$ as a matter of labeling) and $m_1 < m_2$. (This last condition makes $a_1 < a_2$.) Roughly speaking, a high m_i makes a competitor do well at high concentrations by giving it a high reproductive rate, and a low a_i makes a better competitor at low concentrations by allowing it to reach half of its maximal reproductive rate at a lower concentration. (If this last point seems counterintuitive at first, recall that $\lambda_i < 1$ is being assumed, so that the competitor is able to survive in the system without competition. If a_i were too low, then its λ would be "too high" ($\lambda \geq 1$) for survival.) As a consequence, coexistence can be anticipated only if the competitors have different competitive strengths. This difference will manifest itself in the basic hypotheses of the principal theorems given below.

The first step is to analyze the system with only one competitor, that is, with an initial condition zero for one of the x_i's. Since the system is nonautonomous, this is more difficult than before, and stability and convergence rates play a much more delicate role. Suppressing the subscripts, such a system would take the form

$$(4.3) \qquad \begin{aligned} S' &= (1 - S)D(t) - \frac{mxS}{a + S} \\ x' &= x \left[\frac{mS}{a + S} - D(t) \right]. \end{aligned}$$

Rewriting the above "conservation" limit as

$$S(t) + x(t) = 1 + R(t),$$

where $R(t) = O(e^{-\alpha t}), t \to \infty$, some $\alpha > 0$. Then the x component of (4.3) is a solution of

$$(4.4) \qquad y' = y \left[\frac{m(1-y)}{a+1-y} - D(t) \right] + r(t),$$

where $r(t) = O(e^{-\alpha t})$ as $t \to \infty$ and $\alpha > 0$. In turn, (4.4) is a perturbation of

$$(4.5) \qquad z' = z \left[\frac{m(1-z)}{a+1-z} - D(t) \right].$$

That (4.5) has a unique asymptotically stable periodic solution was shown in [**3**] using Massera's theorem. Call the solution ϕ_i, where the i indicates the choice of $m = m_i$ and $a = a_i$. The functions S_i are defined by $S_i(t) = 1 - \phi_i$. The following lemma sets the "foundation" for the application of the bifurcation theorem.

LEMMA 4.1. *Let $0 < \lambda_1 < \lambda_2 < 1$ and let $m_1 \geq m_2$. There are positive ω-periodic functions $S_1(t), \phi_1(t)$ such that every solution of (4.1) with positive initial conditions satisfies $\lim_{t \to \infty} |S(t) - S_1(t)| = 0$, $\lim_{t \to \infty} |x_1(t) - \phi_1(t)| = 0$ and $\lim_{t \to \infty} x_2(t) = 0$, where S_1 and ϕ_1 are given above. The rate of convergence is exponential.*

The proof of this lemma involves some careful estimates in order to obtain uniform asymptotic stability. With this lemma, however, the philosophy of approach is now clear. One wants to fix the parameters for the first competitor and vary the remaining parameters in the system to make the periodic solution in the $S - x_1$ face change stability in the direction orthogonal to that face—that is, to make it bifurcate into the positive cone. First one notes that there are three relevant periodic solutions of the system (4.1). One is constant, the fixed point $(1, 0, 0)$ corresponding to the extinction of both competitors which we label as E_0. The periodic solutions, $(S_1(t), \phi_1(t), 0)$ and $(S_2(t), 0, \phi_2(t))$ are labeled E_1 and E_2, respectively.

THEOREM 4.2. *Let m_1 and a_1 be given so that $\lambda_1 < 1$. There exists $\alpha = \alpha(m_1, a_1)$ such that, for any $a_2 > \alpha, m_2$ (the bifurcation*

parameter) *can be chosen so that* $\lambda_1 < \lambda_2$ *and* (4.1) *has an* ω-*periodic solution* $(S(t), x_1(t), x_2(t))$, *with all components positive, near* E_1 *and bifurcating from it.*

THEOREM 4.3. *Let* m_1, a_1, a_2 *be given as above. There exists a continuous one-parameter family of positive* ω-*periodic solutions of* (4.1) *connecting the solutions* E_1 *and* E_2.

In principle, one could do the asymptotic expansion and obtain information about stability and the direction of bifurcation. The lack of knowledge of the solution ϕ_1 did not seem to make this a profitable approach. However, numerical simulation did show the bifurcating solution to be stable.

These results, along with those of Hsu, Smith, and Hale and Somolinas, do indicate that, when one observes coexistence, one might look for a time varying environmental component as a possible explanation of the lack of competitive exclusion. Of course, periodic functions are the most elementary forcing functions—any input that would cause the nutrient level to fluctuate between the regions where each competitor has an advantage might produce a coexistence result. To the author's knowledge, such investigations have not been undertaken. Those interested in almost periodic solutions, for example, might find an interesting problem here.

5. Three trophic levels in the chemostat. The introduction of periodic coefficients in the previous section showed that coexistence was possible but only in an oscillatory fashion. The nutrient had to cycle between the regions where each of the competitors was dominant. A natural question is whether the necessary oscillation can be produced without resorting to outside "forcing." This cannot happen directly in the simple chemostat model as was shown above, but it is possible if one introduces an additional trophic level. The chemostat will be configured with a nutrient, a microorganism growing on the nutrient and two competitors feeding on the microorganism. The discussion follows [**5**]. See also Keener [**28**].

The mathematical model is quite similar to that discussed above. The

first level is the nutrient. The second level—called the prey—grows
on the nutrient and is to be fed upon by two organisms at the third
level—called the predators. The principal result will be that, if the
parameters are right, the competing predators can coexist on the same
prey. We let S denote the nutrient, x, the prey, and y and z, the
predators. If we assume that all of the reactions are of Michaelis-
Menten type, the model, with the usual scaling, takes the form

$$S' = 1 - S - \frac{m_1 S x}{a_1 + S}$$

$$x' = x\left(\frac{m_1 S}{a_1 + S} - 1 - \frac{m_2 y}{a_2 + x} - \frac{m_3 z}{a_3 + x}\right)$$

(5.1)

$$y' = y\left(\frac{m_2 x}{a_2 + x} - 1\right)$$

$$z' = z\left(\frac{m_3 x}{a_3 + x} - 1\right)$$

$$S(0) \geq 0, \quad x(0) > 0, \quad y(0) > 0, \quad z(0) > 0.$$

Even with only one predator, the system (5.1) is of interest as a simple
food chain. Since we need to analyze this system below, we note that
it takes the form

$$S' = 1 - S - \frac{m_1 S x}{a_1 + S}$$

(5.2)

$$x' = x\left(\frac{m_1 S}{a_1 + S} - 1 - \frac{m_2 y}{a_2 + x}\right)$$

$$y' = y\left(\frac{m_2 x}{a_2 + x} - 1\right)$$

$$S(0) \geq 0, \quad x(0) > 0, \quad y(0) > 0.$$

(Food chains with more general functional response terms can be found
in [**8**].) In the same manner as before, one shows that solutions of (5.1)
satisfy

$$\lim_{t \to \infty} [S(t) + x(t) + y(t) + z(t)] = 1,$$

and those of (5.2) satisfy

$$\lim_{t \to \infty} [S(t) + x(t) + y(t)] = 1,$$

where the convergence is exponential. Again, this yields the dissipative condition and allows the elimination of one variable. It is convenient to eliminate S so that, on the omega limit set, trajectories satisfy

(5.3)
$$x' = x \left(\frac{m_1(1 - x - y - z)}{1 + a_1 - x - y - z} - \frac{m_2 y}{a_2 + x} - \frac{m_3 z}{a_3 + x} \right)$$

$$y' = y \left(\frac{m_2 x}{a_2 + x} - 1 \right)$$

$$z' = z \left(\frac{m_3 x}{a_3 + x} - 1 \right)$$

$$x(0) \geq 0, \quad y(0) \geq 0, \quad z(0) \geq 0, \quad x(0) + y(0) + z(0) \leq 1.$$

Similarly, the omega limit set of solutions of (5.2) satisfy

(5.4)
$$x' = x \left(\frac{m_1(1 - x - y)}{1 + a_1 - x - y} - \frac{m_2 y}{a_2 + x} \right)$$

$$y' = y \left(\frac{m_2 x}{a_2 + x} - 1 \right).$$

The approach to the problem is to study the food chain (5.2) first and to find conditions under which there will be an oscillatory solution to this system. Solutions of (5.2) correspond to solutions of (5.1) with the initial condition $z(0) = 0$. Thus, one can consider one of the parameters in the z equation as a bifurcation parameter and try to bifurcate from the periodic solution in the food chain. Equivalently, one seeks a periodic orbit of the planar system (5.4) and attempts to bifurcate to a periodic orbit (5.3) which lies in the interior of the positive cone of $\mathbf{R}^3$. The details are complicated and we sketch only the basic ideas.

Before beginning, note that there are three values of the λ-parameter to consider, i.e.,
$$\lambda_i = \frac{a_i}{m_i - 1}, \quad i = 1, 2, 3.$$

The following assumption will be made throughout:

(H) $m_i > 1$ and $\lambda_i < 1,$ $i = 1, 2, 3$ and $\lambda_2 < \lambda_3,$

(H) assures that all of the populations are viable in the current environment (m, a and D are related appropriately for survival to be possible) and prejudices population 2 over population 3 and assures that they are "different."

The system (5.4) has two boundary equilibria, $(0,0)$ and $(1 - \lambda_1, 0)$. The hypothesis (H) makes the origin unstable. The remaining boundary equilibrium will be asymptotically stable if $\lambda_1 + \lambda_2 > 1$ and a saddle point if $\lambda_1 + \lambda_2 < 1$. It is not difficult to show that there is a unique interior equilibrium point if $\lambda_1 + \lambda_2 < 1$ and no interior equilibrium if $\lambda_1 + \lambda_2 > 1$. (Recall that we are assuming that $1 > \lambda_i > 0$, for each i.) The biological interpretation is that $\lambda_1 + \lambda_2 > 1$ means the extinction of the highest level predator (or predators).

Suppose the interior equilibrium does exist; label it (x_0, y_0).

THEOREM 5.1. *If*

$$(5.5) \qquad \frac{y_0}{(a_2 + \lambda_2)^2} < \frac{m_1 a_1}{(1 + a_1 - \lambda_2 - y_0)^2}$$

then (x_0, y_0) is globally asymptotically stable with respect to the interior of the first quadrant.

THEOREM 5.2. *If the inequality in (5.5) is reversed, there exists a periodic orbit for (5.4).*

The periodic orbit given by Theorem 5.2 corresponds to a periodic orbit in the plane $z = 0$ for the full system (5.1). It is from this orbit that the bifurcation is to occur. One suspects that this orbit is unique and is orbitally asymptotically stable. If there are several periodic orbits, then the inner one must be stable from the inside (since the above condition is a local instability condition for (x_0, y_0)) and the outer one stable from the outside (by the dissipativeness of the system). This is strong enough to guarantee a stable periodic orbit. However, we shall have to assume slightly more:

(S)   There exists a limit cycle for (5.4) which has a Floquet multiplier strictly inside the unit circle.

The following theorem summarizes the case of competitive exclusion for the full system.

THEOREM 5.3. *Let* $m_3 \leq m_2$ *and let* (H) *hold. If* $(S(t), x(t), y(t), z(t))$ *is a solution of* (5.1) *with positive initial conditions, then*

$$\lim_{t \to \infty} z(t) = 0.$$

Thus $m_2 < m_3$ is necessary for coexistence. The coexistence case is established by a bifurcation argument, and the principal result takes the form

THEOREM 5.4. *Let* a_i, m_i, *be fixed so that* $m_i > 1, i = 1, 2$, *and* $\lambda_i < 1$. *Let* (S) *hold. Fix* $m_3 > m_2$. *Then there exists a number* a_3^* *such that for* $a_3 < a_3^*$, *and* $|a_3 - a_3^*|$ *sufficiently small, one has* $\lambda_2 < \lambda_3$ *and* (5.1) *has a periodic orbit in the positive cone in* $\mathbf{R}^4$ *arbitrarily near the plane* $S + x + y = 1, z = 0$.

6. Delays in the chemostat. Another way that oscillations can be introduced into model ecosystems is to incorporate into the model the delays naturally inherent in the biological system. In the chemostat, there are two possible sources of delays, delays due to the possibility that the organism stores the nutrient (and, consequently, the "free" nutrient concentration does not reflect the nutrient available for growth), and delays due to the lag between consumption and cell division. Caperon [**9**] introduced delays into a model of the chemostat to reflect the internal storage of nutrient (see Droop [**12**]). The model of Caperon suffers from the fact that it leads to negative concentrations—a certain sign of a modeling error. Constructing an "internal stores model" of the chemostat with delays remains an open modeling problem and one surely of interest. Bush and Cook [**2**] investigate the growth of one organism in a chemostat with a delay term to reflect the delay between consumption and growth. (As a consequence, there is a delay in the growth equation but no delay in the consumption term in the nutrient equation.) This was extended to the competitive situation by Freedman, So and Waltman [**14**]. This is

the model discussed below where the format of the equations is very much like that of the equations discussed before.

The model is written for an arbitrary monotone consumption term, but it is the simple chemostat model if one uses Michaelis-Menten kinetics for this function. It is supposed that each competitor has a delay τ_i which affects its growth rate, that is, its specific growth rate is a function of the nutrient level at time $t - \tau_i$. The model takes the form of a system of differential-difference equations:

$$
\begin{aligned}
S'(t) &= 1 - S(t) - x_1(t)p_1(S(t)) - x_2(t)p_2(S(t)) \\
x_1'(t) &= x_1(t)[p_1(S(t - \tau_1)) - 1] \\
x_2'(t) &= x_2(t)[p_2(S(t - \tau_2)) - 1],
\end{aligned}
$$

(6.1)

where $\tau_1, \tau_2 \geq 0, S(t) = \phi(t) \geq 0$ on $[-\tau, 0], \tau = \max(\tau_1, \tau_2)$, and $x_i(0) = x_{i0} \geq 0, i = 1, 2$. The last two equations can be written in integral form as

$$
x_i(t) = x_i(0) \exp\left(\int_0^t p_i(S(\theta - \tau_i) - 1)\, d\theta \right).
$$

Hence the "method of steps" is applicable and the proper initial value problem is as indicated by the initial conditions given above. Using the above integral representation and a simple inequality argument for S', it is not difficult to show that solutions of the system (6.1) are non-negative for all positive time. The "conservation" argument used before to obtain boundedness (and to reduce the complexity of the problem) is no longer valid. The boundedness and the continuability of solutions of the system (6.1) can be established but it is not quite as easy as with the previous chemostat problems.

The investigation of solutions of the problem takes the following form. First, one population growing on the nutrient is analyzed (after some scaling) and a Hopf bifurcation (with the delay as parameter) is shown to exist, establishing the existence of a periodic solution $(\hat{S}(t), \hat{x}_1(t), 0)$. For one population of microorganisms the two-dimensional system governing growth is

(6.2)
$$
\begin{aligned}
S'(t) &= 1 - S(t) - x_1(t)p_1(S(t)) \\
x_1'(t) &= x_1(t)[p_1(S(t - \tau_1)) - 1].
\end{aligned}
$$

After scaling one has

$$S'(t) = \tau[1 - S(t) - x_1(t)p_1(S(t))]$$
$$x_1'(t) = \tau x_1(t)[p_1(S(t-1)) - 1].$$

(6.2) has a unique equilibrium point $E^* = (S^*, x^*)$ since p_1 is strictly increasing. This point bifurcates into a periodic orbit for τ large.

THEOREM 6.1. *There exists $\tau_0 > 0$ such that a family of periodic solutions of* (6.2) *bifurcates from the equilibrium point E^* for τ near τ_0.*

Although stability is, in principle, computable, and a procedure is provided in [18] for a delay equation, the calculation is extremely complicated and stability has not been proved. Numerical solutions show the asymptotic stability quite clearly. Assuming that the solution is asymptotically stable, a secondary bifurcation can be shown to occur. The argument requires a form of a Poincaré map in the appropriate function space. The Rabinowitz bifurcation theorem (in the form given by Smoller [35, p. 173]), is applied to this mapping to yield the existence of a periodic solution with a positive x_2 component. The parameter involves an integral of a function of the oscillating solution—a sort of weighted "mean value." In the case of Michaelis-Menten dynamics, the value of this parameter can be controlled by the constant m_2. In the general case, there must be a natural parameter so that one can write $f_2 = \mu \hat{f}_2$ and keep $\hat{f}_2$ fixed while varying μ. Thus, coexistence of competing predators is possible in a chemostat if there is sufficient delay between nutrient uptake and reproduction. The principal theorem takes the form of a bifurcation statement.

THEOREM 6.2. *Suppose that* (6.2) *has a* (linearly) *asymptotically orbitally stable periodic solution $(\hat{S}(t), \hat{x}(t))$ with period $T > 0$. There exists a branch of periodic orbits of* (6.1), *with positive x_2 component, bifurcating from the above orbit in a neighborhood of $\mu_0 = \left[\int_0^T \hat{p}_2(\hat{S}(s)\, ds\right]^{-1}$.*

Stability considerations and qualitative properties of solutions of the system (6.1) remain open problems. A more careful biological analysis of the possible causes of delays should lead to a more realistic model of competition in the chemostat. One could certainly speculate that the damped oscillations observed in the experiments reported in [**17**] could be caused by delays.

Figure 6.1 shows the time course of a sample problem and Figure 6.2 shows the projection of the coexisting orbit onto each of the possible pairs of variables.
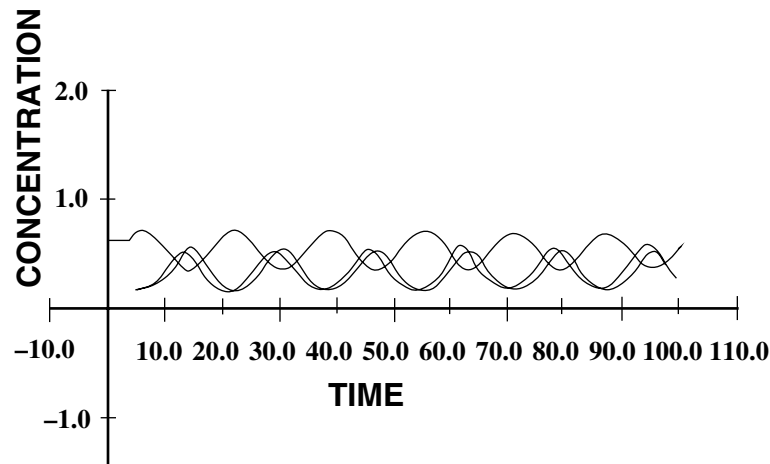

FIGURE 6.1.   The time course for the coexistence case.   (Michaelis-Menten dynamics, $m_1 = 3.1, m_2 = 3.09, a_1 = 1., a_2 = 1., \tau_1 = 3, \tau_2 = 4$.

7. The unstirred chemostat. In this section the "well-mixed" hypothesis in the chemostat will be removed and the nutrient and organisms will be allowed to diffuse through the chemostat. The work follows [**36**]. Only one space variable will be considered; obviously, three would be desirable. The model then becomes a system of partial
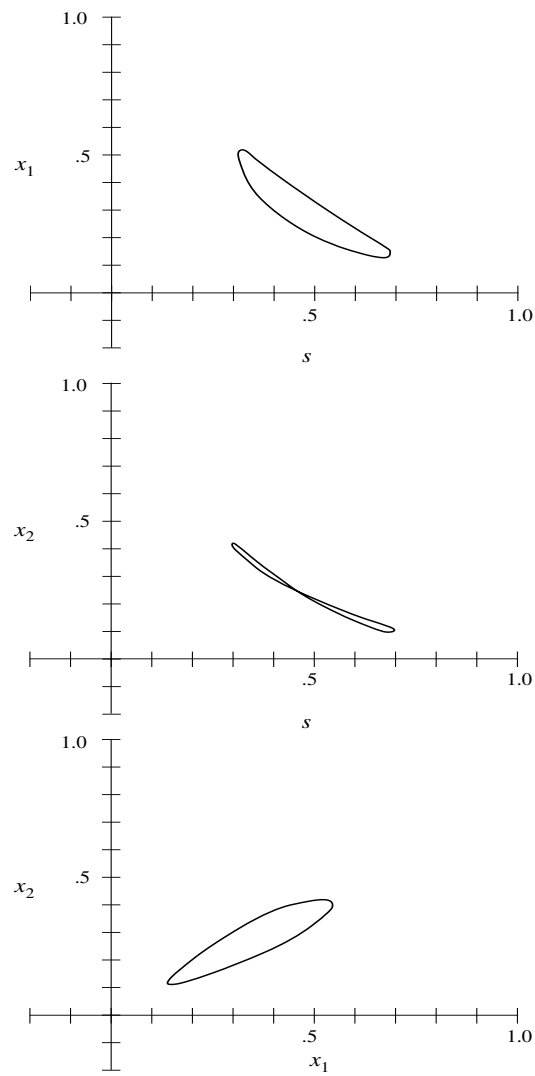
FIGURE 6.2. Three projections onto two variables, same parameters as Figure 6.1: (a) $S - x_1$ (b) $S - x_2$, (c) $x_1 - x_2$.

differential equations of the form

$$S_t = S_{xx} - \frac{m_1 S u}{a_1 + S} - \frac{m_2 S v}{a_2 + S}$$

(7.1)
$$u_t = u_{xx} + \frac{m_1 S u}{a_1 + S}$$

$$v_t = v_{xx} + \frac{m_2 S v}{a_2 + S},$$

with boundary conditions

$$S_x(t, 0) = -S^{(0)}$$
$$u_x(t, 0) = v_x(t, 0) = 0$$
(7.2)
$$S_x(t, 1) + \gamma S(t, 1) = 0$$
$$u_x(t, 1) + \gamma u(t, 1) = 0$$
$$v_x(t, 1) + \gamma v(t, 1) = 0$$

and initial conditions

$$S(0, x) = S_0(x) \geq 0$$
(7.3)
$$u(0, x) = u_0(x) \geq 0$$
$$v(0, x) = v_0(x) \geq 0.$$

(The common diffusion constant has been scaled to one, the source placed at $x = 0$ and the overflow placed at $x = 1$.) The change from the basic chemostat to (7.1)–(7.3) is clear except possibly for moving the "input" and "overflow" from the equations to the boundary conditions. This point is clarified below. One seeks steady states of this problem, in particular, coexistence steady states, since the basic biological question is whether the introduction of a nutrient gradient can result in coexistence. There is an affirmative answer but without a stability conclusion. Moreover, in examples, the coexistence region is quite small. The interesting mathematics is in the resulting nonlinear boundary value problem for two coupled second order, ordinary differential equations. Although the particular form of the nonlinearity is exploited, this type of boundary value problem would seem to be of mathematical interest with a more general nonlinearity (and without differentiability of the nonlinear term). The consideration of different

diffusion coefficients would be more realistic but this remains an open problem.

The boundary conditions require some discussion. Consider a small region of width h near the boundary. If $S(t,x)$ is the nutrient concentration, then the total nutrient in the region at time t is given by $T = \int_0^h S\,dx$. The net flux into the region is $S_x|_{x=h} + S^{(0)}$. Since there is no flow through the "left hand" wall and since $S^{(0)}$ is constant, as $h \to 0$, one has $S_x|_{x=0} = -S^{(0)}$. That $u_x(t,0) = v_x(t,0) = 0$ simply reflects the fact that there is no flow of microorganisms through the wall. Similar arguments apply at the right-hand end where nutrient and microorganisms are "pumped out" of the vessel.

Some scaling is in order. As before, concentrations can be measured in units of $S^{(0)}$, the input concentration, which has the effect of changing the first boundary condition in (7.2) to $S_x(t,0) = -1$ and changes a_i but not the form of the nonlinearity. Let

$$f_i(S) = \frac{S}{a_i + S}, \quad i = 1, 2.$$

A conservation principle is present but it is more complicated than before because of the partial derivatives. If $w(t,x) = S(t,x) + u(t,x) + v(t,x)$, then

$$w_t = w_{xx},$$

with boundary conditions

$$w_x(t,0) = -1$$
$$w_x(t,1) + \gamma w(t,1) = 0$$

and initial condition $w(0,x) = h(x) \equiv S_0(x) + u_0(x) + v_0(x)$. To solve this equation, subtract the steady state solution $z(x) = (1+\gamma)/\gamma - x$ from $w(t,x)$ to get the homogeneous problem in $\phi(t,x) = w(t,x) - z(x)$,

$$\phi_t = \phi_{xx}$$
$$\phi_x(t,0) = 0$$
$$\phi_x(t,1) + \gamma\phi(t,1) = 0$$
$$\phi(0,x) = \phi_0(x).$$

It is easy to see that $\lim_{t\to\infty} \phi(t,x) = 0$ or that, for $x \in [0,1]$,

(7.4) $$S(x) + u(x) + v(x) = z(x).$$

Thus, any steady state solution of (7.1)–(7.3) satisfies (7.4). We use this to eliminate S and study the boundary value problem for two ordinary, differential equations,

$$
\begin{aligned}
&u'' + m_1 u f_1(z(x) - u - v) = 0 \\
&v'' + m_2 v f_2(z(x) - u - v) = 0 \\
&u'(0) = v'(0) = 0 \\
&u'(1) + \gamma u(1) = 0 \\
&v'(1) + \gamma v(1) = 0.
\end{aligned}
$$

(7.5)

Solutions of interest have $u(x) > 0$ and $v(x) > 0$, since this corresponds to coexistence of the competing populations.

THEOREM 7.1. *Let $a_i > 0, i = 1, 2$, and $\gamma > 0$ be fixed. There exists a nonempty unbounded open set in $\mathbf{R}_+^2$ such that, for $(m_1, m_2) \in \mathbf{R}_+^2$, (7.5) has a solution $(u(x), v(x))$ with $u(x) > 0, v(x) > 0$, for $x \in [0, 1]$.*

It will be convenient to use the integral formulation of the boundary value problem. Let

$$
G(x, \tau) = \begin{cases}
\frac{1+\gamma}{\gamma} - x, & 0 \leq \tau \leq x \\
\frac{1+\gamma}{\gamma} - \tau, & x \leq \tau \leq 1
\end{cases}.
$$

The equivalent system of integral equations is

$$
\begin{aligned}
u(x) &= m_1 \int_0^1 G(x, \tau) u(\tau) f_1(z(\tau) - u(\tau) - v(\tau)) \, d\tau \\
v(x) &= m_2 \int_0^1 G(x, \tau) v(\tau) f_2(z(\tau) - u(\tau) - v(\tau)) \, d\tau.
\end{aligned}
$$

(7.6)

The proof follows from three lemmas. The approach is to utilize two applications of bifurcation from a simple eigenvalue, first for nonlinear Sturm-Liouville problems [10] and then for a mapping defined from the integral operators in (7.6) [35].

LEMMA 7.2. *Let λ_0 be the first eigenvalue of*

(7.7)
$$
y'' + \lambda f(z(x)) y = 0,
$$

subject to the boundary conditions

$$y'(0) = 0$$
$$(7.8) \qquad\qquad y'(1) + \gamma y(1) = 0.$$

For $m_1 > \lambda_0$, there exists a positive solution $u(x)$ of

$$u'' + m_1 u f_1(z(x) - u) = 0$$

satisfying (7.8).

Fix any $m_1 > \lambda_0$ and let $\hat{u}(x)$ denote the solution guaranteed by Lemma 7.2. The following technical lemma is important in the bifurcation arguments.

LEMMA 7.3. *Let $\hat{u}(x)$ be as above. Then m_1 is not an eigenvalue of*

$$u'' + \lambda[f_1(z(x) - \hat{u}(x)) - \hat{u}(x)f'(z(x) - \hat{u}(x))]u = 0$$

with boundary conditions (7.8).

The theorem is established by the following statement about the boundary value problem (7.5).

LEMMA 7.4. *Fix m_1 and $\hat{u}$ as above. Let $\hat{\lambda}$ be the least eigenvalue of*

$$v'' + \lambda f_2(z(x) - \hat{u}(x))v = 0$$
$$v'(0) = 0$$
$$v'(1) + \gamma v(1) = 0.$$

As m_2 increases past $\hat{\lambda}$ and sufficiently close to $\hat{\lambda}$, there exists a family of solutions $(u(x), v(x))$ of the boundary value problem (7.5) with $u(x) > 0$ and $v(x) > 0$.
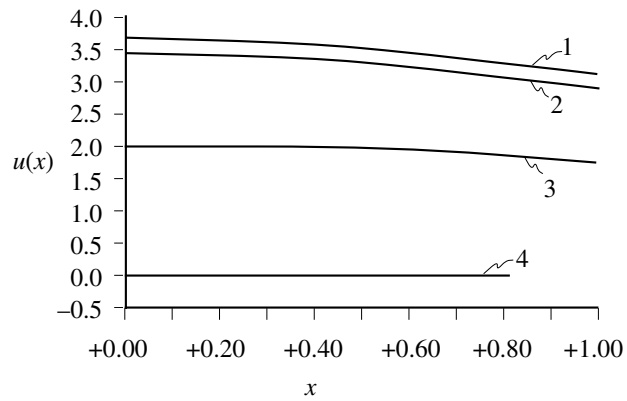
FIGURE 7.1. A coexistence case: $u(x)$. ($\gamma = .3, m_1 = 1.32475$, $a_1 = 1.5$, $a_2 = 2.0$, four values of m_2).
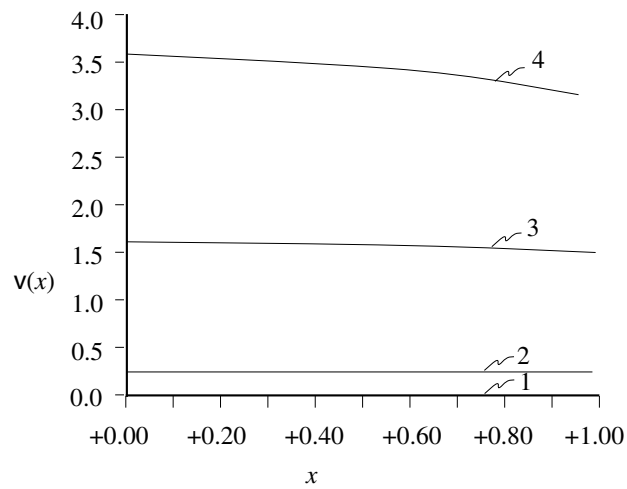

FIGURE 7.2. A coexistence case: $v(x)$. (Same parameters, corresponding labels.)

REFERENCES

1. R.A. Armstrong and R. McGehee, *Competitive exclusion,* Amer. Natur. **115** (1980), 151–170.

2. A.W. Bush and A.E. Cook, *The effect of time delay and growth inhibition in the bacterial treatment of wastewater,* J. Theoret. Biol. **63** (1975), 385–395.

3. G.J. Butler and H.I. Freedman, *Periodic solutions of a predator prey system with periodic coefficients,* Math. Biosci. **55** (1981), 27–38.

4. G.J. Butler, S.B. Hsu, and P. Waltman, *A mathematical model of the chemostat with periodic washout rate,* SIAM J. Appl. Math. **45** (1985), 435–449.

5. ——, ——, and ——, *Coexistence of competing predators in a chemostat,* J. Math. Biol. **17** (1983), 133–151.

6. —— and G.S.K. Wolkowicz, *A mathematical model of the chemostat with a general class of functions describing nutrient uptake,* SIAM J. Appl. Math. **45** (1985), 138–151.

7. —— and ——, *Exploitative competition in a chemostat for two complimentary, and possibly inhibitory, resources,* Math. Biosci. **83** (1987), 1–48.

8. —— and ——, *Predator-mediated competition in the chemostat,* J. Math. Biol. **24** (1986), 167–191.

9. J. Caperon, *Time lag in population growth response of Dsochryris galana to a variable nitrate environment,* Ecology **50** (1969), 188–192.

10. M.G. Crandall and P.H. Rabinowitz, *Nonlinear Sturm-Liouville boundary value problems,* J. Math. Mech. **19** (1970), 1083–1102.

11. J.M. Cushing, *Periodic time-dependent predator prey systems,* SIAM J. Appl. Math. **23** (1977), 972–979.

12. M.R. Droop, *The nutrient status of algae cells in continuous culture,* J. Marine Biol. Assoc. U.K. **54** (1974), 825–855.

13. A.G. Frederickson and G. Stephanopoulos, *Microbial competition,* Science **243** (1981), 972–979.

14. H.I. Freedman, J.W.-H. So and P. Waltman, *Coexistence in a model of competition in the chemostat incorporating discrete delays,* SIAM J. Appl. Math. **49**) 1989, 859–870.

15. M. Gatto, M. Annaboratone, and M. Borghesi, *The competitive coexistence of two species in periodically varying environments,* Boll. Zool. **46** (1979), 191–200.

16. J.K. Hale and A.S. Somolinas, *Competition for fluctuating nutrient,* J. Math. Biol. **18** (1983), 255–280.

17. S.R. Hansen and S.P. Hubbell, *Single nutrient microbial competition: agreement between experimental and theoretical forecast outcomes,* Science **20** (1980), 1491–1493.

18. B.D. Hassard, N.D. Kazarinoff, and Y.-H. Wan, *Theory and Application of Hopf Bifurcation,* Cambridge University Press, London, 1981.

19. D. Herbert, R. Elsworth and R.C. Telling, *The continuous culture of bacteria: a theoretical and experimental study,* J. Gen. Microbiology **4** (1956) 601–622.

20.   M.W. Hirsch, *Systems of differential equations which are competitive or cooperative* I: *Limit sets,*, SIAM J. Math. Anal. **13** (1982), 167–179.

21.   ———, *The dynamical systems approach to differential equations,* Bull. Amer. Math. Soc. **11** (1984), 1–64.

22.   C.S. Holling, *Some characteristics of simple types of predation and parasitism,* Canad. Ent. **91** (1959), 385–395.

23.   S.B. Hsu, *Limiting behavior for competing species,* SIAM J. Appl. Math. **34** (1978), 760–763.

24.   ———, *A competition model for a seasonally fluctuating nutrient,* J. Math. Biol. **9** (1980), 115–132.

25.   ———, K.S. Cheng, and S.P Hubbell, *Exploitative competition of microorganisms for two complementary nutrients in continuous culture,* SIAM J. Appl. Math. **41** (1981), 422–444.

26.   ———, S. Hubbell, and P. Waltman, *A mathematical theory for single-nutrient competition in continuous cultures of microorganisms,* SIAM J. Appl. Math. **32** (1977), 366–383.

27.   H.W. Jannash and R.T. Mateles, *Experimental bacterial ecology studied in continuous culture,* Adv. Microbial. Phys. **11** (1974), 423–439.

28.   J.P. Keener, *Oscillatory coexistence in a food chain model with competing predators,* J. Math. Biol. **22** (1985), 123–135.

29.   J.A. Leon and D.B. Tumpson, *Competition between two species for two complementary or substitutable resources,* J. Theoret. Biol. **50** (1975), 185–201.

30.   R. McGehee and R.A. Armstrong, *Some mathematical problems concerning the ecological principle of competitive exclusion,* J. Differential Equations **23** (1977), 30–52.

31.   J. Monod, *Recherches sur la croissance des cultures bacteriennes,* Herman, Paris, 1942.

32.   J. Selgrade, *Asymptotic behavior of solutions to single loop positive feedback systems,* J. Differential Equations **38** (1980), 80–103.

33.   H.L. Smith, *Competitive coexistence in an oscillating chemostat,* SIAM J. Appl. Math. **40** (1981), 498–522.

34.   ———, *Systems of ordinary differential equations which generate an order preserving flow.* A survey of results, SIAM Rev. **30** (1988), 87–113.

35.   J. Smoller, *Shock Waves and Reaction Diffusion Equations,* Springer-Verlag, New York, 1983.

36.   J.W.-H. So and P. Waltman, *A nonlinear boundary value problem arising from competition in the chemostat,* Appl. Math. Comput. **32** (1989), 169, 183.

37.   G. Stephanopoulos, A.G. Fredrickson, and R. Aris, *The growth of competing microbial populations in a CSTR with periodically varying inputs,* AIChE J. **25** (1979), 863–872.

38.   P.A. Taylor and J.L. Williams, *Theoretical studies on the coexistence of competing species under continuous-flow conditions,* Canad. J. Microbiology **21** (1975), 90–98.

39. D. Tilman, *Resource Competition and Community Structure,* Princeton University Press, Princeton, N.J., 1982.

40. P. Waltman, *Competition Models in Population Biology,* Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, PA, 1983.

41. ———, S.B. Hsu, and S.P. Hubbell, *Theoretical and experimental investigations of microbial competition in continuous culture,* in T. Burton (ed.), *Modeling and Differential Equations in Biology,* Marcel Dekker, New York, 1980.

42. R.D. Yang and A.E. Humphrey, *Dynamics and steady state studies of phenol biodegradation in pure and mixed cultures,* Biotechnology and Bioeng. **17** (1975), 1211–1235.

EMORY UNIVERSITY, ATLANTA, GA 30322