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WHERE DO ALL THE VALUES GO? PLAYING WITH TWO-ELEMENT CONTINUED FRACTIONS

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To Professor W.J. Thron on the occasion of his 70th birthday.

1. Introduction. In the present paper we deal with continued fractions

$$\mathop{\mathbf{K}}_{n=1}^{\infty}\frac{a_n}{1}=\frac{a_1}{1}\,{}_{+}\,\frac{a_2}{1}\,{}_{+}\cdots{}_{+}\,\frac{a_n}{1}\,{}_{+}\cdots{}_{+}, \tag{1.1}$$

where $a_n \in \mathbf{C}$, $a_n \neq 0$ for all n (except for one example in Section 4).

In using continued fractions the situation is often as follows: All continued fractions in question are in a certain family, given by the condition that all a_n belong to a given set E, which is a convergence region or a conditional convergence region (for definitions, see [**3** pp. 78 and 80]). For a given such set E let $\mathcal{F}_E$ denote that particular family:

$$\mathcal{F}_E := \left\{\mathop{\mathbf{K}}_{n=1}^{\infty}\frac{a_n}{1}; a_n \in E \text{ for all } n\right\}. \tag{1.2}$$

Information on $\mathcal{F}_E$ generally is of value in computation of values of continued fractions from $\mathcal{F}_E$. We shall here concentrate on one particular type of information, i.e., the set L_E of possible values,

$$f=\mathop{\mathbf{K}}_{n=1}^{\infty}\frac{a_n}{1}, \tag{1.3}$$

of continued fractions in $\mathcal{F}_E$:

$$L_E=\left\{f; \mathop{\mathbf{K}}_{n=1}^{\infty}\frac{a_n}{1} \in \mathcal{F}_E\right\} \tag{1.4}$$

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The set L_E is of crucial importance in the process of establishing truncation error estimates, see, for instance, [**5**] and the references therein. In many cases the set L_E is hard to get at, in which case we often are well served by using sets $L \supset L_E$ (not too large, compared to L_E).

We illustrate the set L_E in some simple cases by examples:

Example 1.

$$E = \{a\}, \qquad a \notin \left(-\infty, -\frac{1}{4}\right).$$

In this case $\mathcal{F}_E$ has exactly *one* element $K(a/1)$, and L_E has also exactly *one* element

$$f = \frac{1}{2}(\sqrt{1+4a} - 1).$$

Example 2.

$$E = \left\{W; |W| \leq \frac{1}{4}\right\}.$$

This is the "Worpitzky case" [**8**]. Here the set L_E is the punctured disk

$$L_E = \left\{W; 0 < |W| \leq \frac{1}{2}\right\}.$$

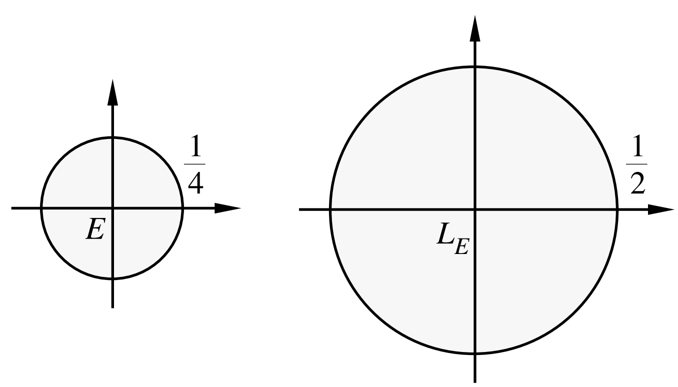

FIGURE 1.

Example 3. (a) Let E be the closed interval $E = [p, q]$ of the positive real axis. Then it is easy to prove that $L_E = [X, Y]$, where

$$X = \frac{p}{1+}\,\frac{q}{1+}\,\frac{p}{1+}\,\frac{q}{1+\cdots+} = \frac{1}{2}\left[\sqrt{(1+p+q)^2 - 4pq} - 1 - q + p\right],$$
$$Y = \frac{q}{1+}\,\frac{p}{1+}\,\frac{q}{1+}\,\frac{p}{1+\cdots+} = \frac{1}{2}\left[\sqrt{(1+p+q)^2 - 4pq} - 1 + q - p\right].$$

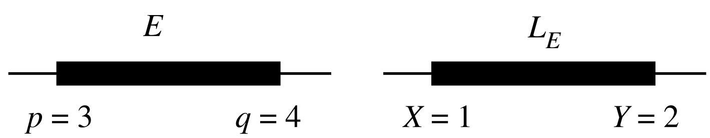

FIGURE 2.

(b) If E merely consists of the two endpoints p, q, $E = \{p, q\}$, it is easy to prove that if

$$pq \geq p + q,$$

the set $L_{\{p,q\}}$ is dense in $[X, Y]$, i.e., that

$$\text{cl } L_{\{p,q\}} = [X, Y] = L_{[p,q]} \qquad \text{(see [}\mathbf{6}\text{])}.$$

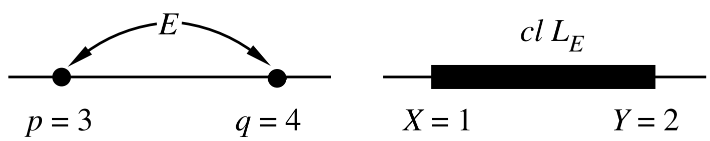

FIGURE 3.

If $pq < p + q$, however, the set cl $L_{\{p,q\}}$ is a Cantor type set, not dense in $[X, Y]$, in which case

$$\text{cl } L_{\{p,q\}} \neq L_{[p,q]} \qquad [\mathbf{6}].$$

Example 4. If $E_{\{p,q\}}$ consists of two points in the *complex* plane, different things may happen. In the illustration below, two possible structures of $L_{\{p,q\}}$ are shown:

FIGURE 4a.

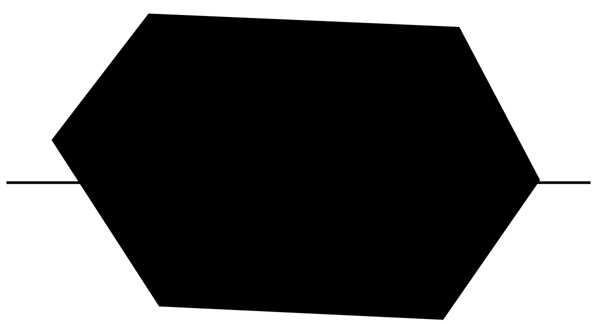

FIGURE 4b.

In the present paper we shall be concerned with sets $L_{\{p,q\}}$, where $p, q \in \mathbf{C}$, $p \neq q$. We shall partly discuss structure, partly probabilistic questions.

2. Remarks on motivation. Why should one be interested in $\mathcal{F}_{\{p,q\}}$ or $L_{\{p,q\}}$?

An obvious answer to this question is that it is natural to proceed from the one-element to the two-element case and look for results, hopefully pointing towards more general knowledge. Maybe it is tempting to guess that it would be the easiest case, next to the one-element case. This, however, is not true, as will be seen in the complexity of the glimpses to be presented in this paper as compared, e.g., to the Worpitzky case in Example 2. It should also be mentioned that we do not even know which 2-element sets are convergence regions (although we know a lot of sufficient conditions).

The main reason *here* to discuss $L_{\{p,q\}}$ is that in many cases a rather sparse subfamily of $\mathcal{F}_E$ can give substantial information on L_E, and even $L_{\{p,q\}}$ for some choice of p, q may be of interest. We recall that in Example 3b we sometimes had

$$\text{cl } L_{\{p,q\}} = L_{[p,q]}.$$

We include another example where a sparse subset of $\mathcal{F}_E$ can describe L_E completely. The example is not of the "$\mathcal{F}_{\{p,q\}}$-type," but since the underlying idea of the present paper is to emphasize the role of certain sparse subsets of $\mathcal{F}_E$ rather than merely $\mathcal{F}_{\{p,q\}}$ for its own sake, we feel that it should be included.

Example 5. Let $\alpha \in (-\pi/2, \pi/2)$ and $p \in (0, 1/2]$ be fixed, and let E be the parabolic region

$$E = E(p, \alpha) = \left\{ z = re^{i\theta}; r \leq \frac{2p(1-p)\cos^2\alpha}{1 - \cos(\theta - 2\alpha)} \right\}.$$

E is a conditional convergence region for continued fractions (1.1). A complete description of L_E is given in the paper [**4**] and is illustrated below (Figure 5a).

FIGURE 5a.

The description itself shall be left out, since we do not need it explicitly. In [**4**] it was also established that any point of the boundary of L_E is the value of a certain 2-periodic continued fraction with elements from the boundary of E. Let $\mathcal{P}_{\partial E}$ be the family of all 2-periodic continued fractions (1.1) with elements from the boundary of E. A computer graphics illustration of the corresponding set of values (19044 points) is given in Figure 5b. In Figure 5c only the values with the "proper match" for the boundary of L_E are computed. Further discussions on the use of sparse subsets of some $\mathcal{F}_E$ to obtain information on L_E are given in [**1**].

Section 3 will contain a discussion of some structural aspects of $L_{\{p,q\}}$ when $|p-q|$ is small. Section 4 will contain a probabilistic discussion on $L_{\{p,q\}}$ for some special real values of p and q.

FIGURE 5b.

FIGURE 5c.

3. $\mathcal{F}_{\{\mathbf{p},\mathbf{q}\}}$ **with small** $|\mathbf{p}-\mathbf{q}|$**.** The tool in this discussion will be the following Proposition.

Proposition 3.1. *With* $a \notin (-\infty, -1/4], \Gamma = (\sqrt{1+4a}-1)/2$, $\mathrm{Re}\sqrt{1+4a} > 0$ *and* $\varepsilon_n \in \mathbf{C}$,

$$|\varepsilon_n| \le r < R(a) = \begin{cases} |a+\frac{1}{4}| & \textit{for } |a| < \frac{1}{4}, \\ \sqrt{\frac{|a|+\mathrm{Re}\, a}{2}} & \textit{for } |a| \ge \frac{1}{4}, \end{cases} \tag{3.1}$$

the following holds:

$$\mathop{\mathbf{K}}_{n=1}^{\infty} \frac{a+\varepsilon_n}{1} = \Gamma + \frac{1}{1+\Gamma} \sum_{n=1}^{\infty} \left(\frac{-\Gamma}{1+\Gamma}\right)^{n-1} \cdot \varepsilon_n + \text{ error term}, \tag{3.2}$$

where

$$|\text{Error term}| \le \frac{M(a)}{1-\frac{r}{R(a)}} \left(\frac{r}{R(a)}\right)^2, \tag{3.3}$$

$$M(a) = 2(|a|+R(a))\sqrt{\frac{2|a|}{|a|+\mathrm{Re}\, a}}. \tag{3.3$'$}$$

(See [**7**]. For (3.1) see [**3**, Theorem 4.45].)

The way this tool is used in the present situation is as follows:

For $p \notin (-\infty, -1/4]$ and $r = |p-q| < R(p)$, take $a = p$, $r = |q-p|$ and ε_n to be 0 or $q-p$. We then have, for $a_n \in \{p, q\}$

$$\mathop{\mathbf{K}}_{n=1}^{\infty} \frac{a_n}{1} = \Gamma + \frac{q-p}{1+\Gamma} \sum_{n=0}^{\infty} d_n \left(\frac{-\Gamma}{1+\Gamma}\right)^n + \text{ error term}, \tag{3.4}$$

where $d_n = 0$ for $a_{n+1} = p$, and $d_n = 1$ for $a_{n+1} = q$.

This shows that the set

$$S(\Gamma) = \left\{\sum_{n=0}^{\infty} d_n z^n; d_n \in \{0,1\}\right\}, \qquad z = \frac{-\Gamma}{1+\Gamma} \tag{3.5}$$

is of interest. Here we always have $|\Gamma| < |1+\Gamma|$, hence z is in the open unit disk U. The set $S(\Gamma)$ may also be described in the following way: Let $\mathcal{G}$ be the family of power series $g(z) = \sum_{n=0}^{\infty} d_n z^n$, where the only permitted values of d_n are 0 and 1, and let z be a fixed value in the open unit disk U. Then $S(\Gamma)$ is the range of the evaluation functional $z(g)$, $g \in \mathcal{G}$.

We shall use the following notation:

(3.6a) $$L(p,q) := L_{\{p,q\}}$$

(3.6b)
$$L_*(p,q) := \Gamma + \frac{q-p}{1+\Gamma} \cdot \text{cl } S(\Gamma) \qquad \text{(Multiplication by a factor and translation.)}$$

(3.6c)
$$L^*(p,q) := \text{Union of all "errorterm disks" around the points of } L_*(p,q).$$

Obviously,
$$L_{\{p,q\}} \subseteq L^*(p,q).$$

The purpose of the present section is to throw some light on the structure of $S(\Gamma)$, and thereby on $L_*(p,q)$, which again, in turn, for small $r/R(a)$, gives valuable information on $L^*(p,q)$. The results will be obtained by using the following simple lemma, which deals with the case when z is positive, $z = x$, $0 \le x < 1$, in which case $\Gamma = -x/(1+x)$.

Lemma 3.2.

(3.7) $$\text{cl } S\left(\frac{-x}{1+x}\right) = \begin{cases} \left[0, \frac{1}{1-x}\right] & \textit{for } \frac{1}{2} \le x < 1, \\ \textit{A Cantor set} & \textit{for } 0 < x < 1/2. \end{cases}$$

The Cantor set is described by using

$$M_0 = \left[0, \frac{1}{1-x}\right]$$
$$M_1 = \left[0, \frac{x}{1-x}\right] \cup \left[1, \frac{1}{1-x}\right],$$

where M_{n+1} is obtained from M_n by removing from each of the 2^n subintervals an open interval of length $x^n(1-2x)/(1-x)$, centered at the mid-point of the subinterval. Then

$$\text{Cantor set } = \bigcap_{n=0}^{\infty} M_n,$$

where M_n contains 2^n intervals of total length $(2x)^n/(1-x)$.

The proof of Lemma 3.2 is straightforward, is based upon arguments of standard nature, and shall be omitted here.

In what follows we shall restrict the discussion of the structure of $S(\Gamma)$ to the cases when

$$\frac{1}{\pi} \cdot \arg\left(\frac{-\Gamma}{1+\Gamma}\right)$$

is rational. Then Lemma 3.2 is the key to the following result.

Proposition 3.3. *Let s and t be natural numbers, and $(s,t) = 1$. Furthermore, let*

$$z = -\frac{\Gamma}{1+\Gamma} = \rho \cdot \exp(2\pi si/t), \qquad 0 < \rho < 1. \tag{3.8}$$

Then

$$\text{cl } S(\Gamma) = \sum_{k=0}^{t-1} \rho^k \cdot \exp(2k\pi si/t) \cdot \text{cl } S\left(\frac{-\rho^t}{1+\rho^t}\right). \tag{3.9}$$

Outline of proof.

$$\begin{aligned}\sum_{n=0}^{\infty} d_n z^n &= \sum_{n=0}^{\infty} d_n \rho^n \cdot \exp(2\pi nsi/t) \\ &= \sum_{k=0}^{t-1} \rho^k \cdot \exp(2k\pi si/t) \cdot \left(\sum_{m=0}^{\infty} d_{k+mt}\rho^{mt}\right).\end{aligned}$$

The closure of the set determined by the inner sum is

$$\text{cl } S\left(\frac{-\rho^t}{1+\rho^t}\right).$$

We shall illustrate this result in some "good cases" and some "bad cases."

Good case.

$$2^{-\frac{1}{t}} \leq \rho < 1. \tag{3.10}$$

The reason for calling this a *good* case is as follows: If, in the definition of $S(\Gamma)$ in (3.5), we permit the whole interval $[0,1]$ and not only the set $\{0,1\}$ as d_n-values, we get, for $z = x$, $1/2 \leq x < 1$, exactly the set cl $S(-x/(1+x))$. This, in turn, means that if E is the segment from p to q, including the endpoints, we actually get

$$L_E \subseteq L^*(p,q), \tag{3.11}$$

which is in the spirit of the intention of the paper. (See Section 2.)

In this case cl $S(\Gamma)$ is a polygon. We shall look at two particular simple examples.

Example 6. $(s,t) = (1,3)$ and $\rho = .8 > 2^{-1/3}$ in Proposition 3.3.

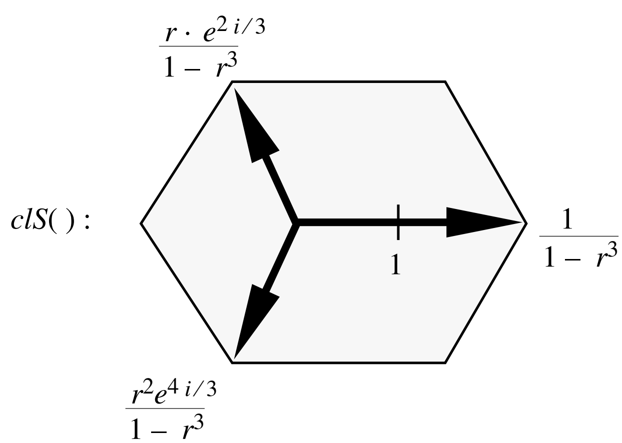

FIGURE 6. $\rho = r = .8$.

Example 7. $(s,t) = (1,4)$ and $\rho = .9 > 2^{-1/4}$ in Proposition 3.3.

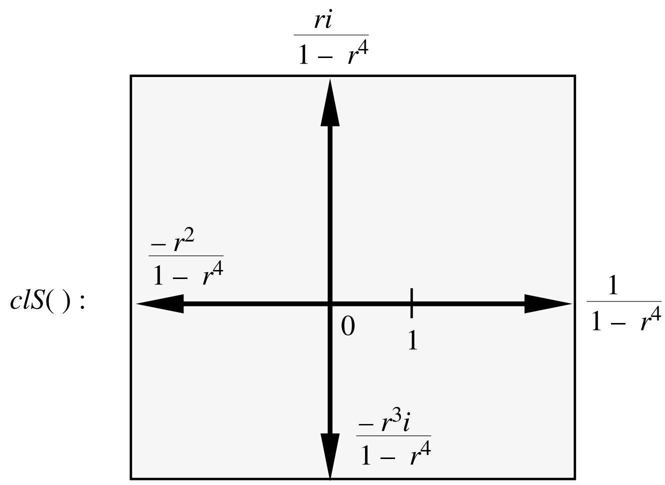

FIGURE 7. $\rho = r = .9$.

We recall that the road from cl $S(\Gamma)$ to $L^*(p,q)$ goes as follows:

$$\text{cl } S(\Gamma) \sim\sim\sim\sim\sim> L_*(p,q) = \Gamma + \frac{q-p}{1+\Gamma}\text{cl } S(\Gamma) \sim\sim\sim\sim\sim> L^*(p,q)$$

Rotation, change of scale and translation — Covering with "error disks"

Bad case.

$$\rho < 2^{-\frac{1}{t}}.$$

The reason for calling this a *bad* case is that, here, cl $S(-x/(1+x))$ is not the interval $[0, 1/(1-x)]$, but only a small subset of it, actually a subset of measure 0, and consequently we will *not* have $L_E \subseteq L^*(p,q)$ as in (3.11). Since in this case the terms in 3.9 are Cantor sets, we could say that cl $S(\Gamma)$ in this case is a "vector sum of Cantor sets." To give an idea of what this may look like we show two examples.

Example 8. $(s,t) = (1,3)$, $\rho = .7 < 2^{-1/3}$ in Proposition 3.3.

FIGURE 8.

Example 9. $(s,t) = (1,4)$ in Proposition 3.3.

(a) $\rho = .6 < 2^{-\frac{1}{4}}$

FIGURE 9a. $\rho = .6 < 2^{-\frac{1}{4}}$.

(b) $\rho = .7 < 2^{-\frac{1}{4}}$

FIGURE 9b. $\rho = .7 < 2^{-\frac{1}{4}}$.

Based upon the underlying intention of this paper, the case $\rho < 2^{-\frac{1}{t}}$ is regarded as a bad case. But, from another point of view, keeping in mind the increasing interest in fractals, it is, in fact, the most interesting case and deserves attention. We hope to come back to this particular case (and extensions thereof) in a later paper.

4. The distribution of values. In the paper [**2**] the distribution of values of a continued fraction (1.1) was discussed for given convergence regions E and given distribution of a_n on E. The discussion was limited to the case of *uniform distribution*, i.e., each a_n was supposed to be uniformly distributed on E. As for E, in most cases certain intervals on the real axis were considered, but also the complex Worpitzky disk was discussed. In the present paper we look at *three* different 2-point sets E, all of them located on the real axis and all being supports for some probability measure. They lead to three very different kinds of distribution of values, thus indicating the variety of possible distributions the 2-element sets can lead to.

Example 10. Normally the possibility $a_n = 0$ in (1.1) is ruled out in the definition of a continued fraction. In the present example we shall accept it. Let, for a given $a > 0$,

$$E = \{0, a\},$$

and let $\Pr(a_n = 0) = \alpha$, $\Pr(a_n = a) = 1 - \alpha$, where Pr stands for "probability" and $0 < \alpha < 1$. The possible values of the continued fractions in $\mathcal{F}_E$, i.e., the set L_E, consists of the numbers

$$0,\ a,\ \frac{a}{1+a},\ \frac{a}{1+\dfrac{a}{1+a}},\ \cdots,\ \frac{a}{1+}\cdots\frac{a}{+1},\ \cdots,$$

and of the number

$$\xi = \frac{a}{1+}\,\frac{a}{1+}\cdots\frac{a}{+1}\,+\cdots+ = \frac{1}{2}(\sqrt{1+4a}-1).$$

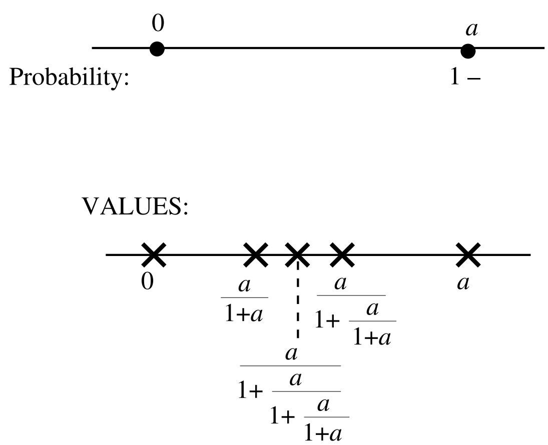

FIGURE 10.

Simple use of the rules of probability leads to the following probabilities for the values f of the continued fraction:

$$\Pr(f = 0) = \alpha$$

$$\Pr(f = a) = (1-\alpha)\cdot\alpha$$

$$\Pr\left(f = \frac{a}{1+}\,\frac{a}{1+}\cdots\frac{a}{+1}\right) = (1-\alpha)^n\cdot\alpha$$

Of course, $\Pr(f = \xi) = 0$. Furthermore, $\Pr(f \in I) = 0$ for all intervals I containing values in L_E.

The cumulative distribution function has a discontinuity at every possible value of the continued fraction, except at ξ, where it is continuous. It is constant on any interval not containing a value of a continued fraction.

The influence of α is illustrated, for instance, by

$$\begin{aligned}\Pr(f \le \xi) &= \alpha + \alpha(1-\alpha)^2 + \alpha(1-\alpha)^4 + \cdots \\ &= \frac{\alpha}{1-(1-\alpha)^2} = \frac{1}{2-\alpha} \\ \Pr(f > \xi) &= \alpha(1-\alpha) + \alpha(1-\alpha)^3 + \alpha(1-\alpha)^5 + \cdots \\ &= \frac{\alpha(1-\alpha)}{1-(1-\alpha)^2} = \frac{1-\alpha}{2-\alpha}.\end{aligned}$$

Part of the graph of the cumulative distribution function is shown in Figure 11. In the illustration we have chosen $a = 2$ and $\alpha = 1-\alpha = 1/2$.

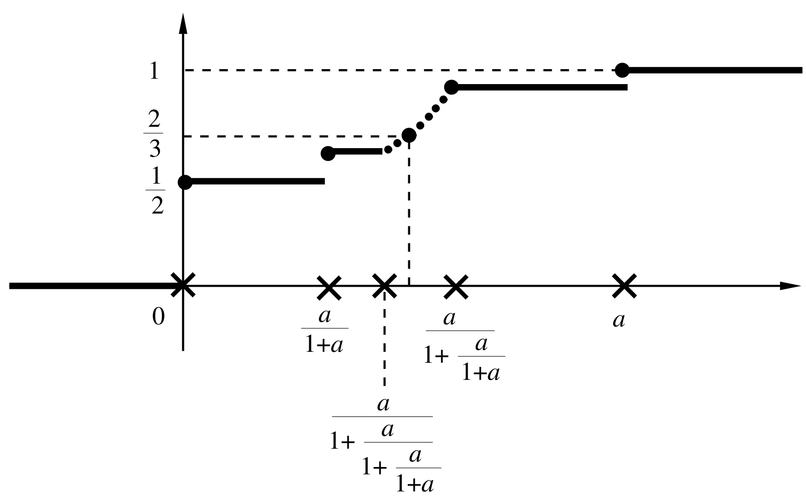

FIGURE 11.

Example 11. Let

$$E = \left\{ -\frac{1}{4}, \frac{1}{4} \right\},$$

and let

$$\Pr\left(a_n = -\frac{1}{4}\right) = \alpha \quad \text{and} \quad \Pr\left(a_n = \frac{1}{4}\right) = 1 - \alpha,$$

where $0 < \alpha < 1$. Since E consists of the two endpoints of the real diameter of the Worpitzky element disk, this may be called the "two-point Worpitzky case".

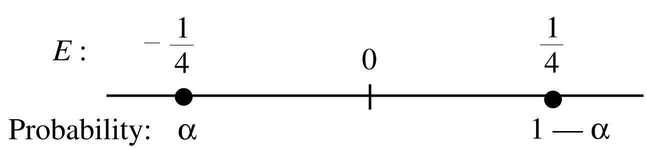

FIGURE 12.

In the next illustration we shall use the notation

$$\boxed{-\tfrac{1}{2}} := \frac{-\frac{1}{4}}{1} + \frac{-\frac{1}{4}}{1} + \cdots + \frac{-\frac{1}{4}}{1} + \cdots.$$

Furthermore, boldface print shall indicate "no c.f. values in the open interval." X shall denote values. The signs $\overset{-}{-}, \mp, \overset{+}{+}, \pm$ indicate how the continued fractions start in that particular closed interval, such that

$$\overset{-}{-} \text{ means } \frac{-\frac{1}{4}}{1} + \frac{-\frac{1}{4}}{1} + \cdots +,$$

$$\mp \text{ means } \frac{-\frac{1}{4}}{1} + \frac{\frac{1}{4}}{1} + \cdots +,$$

and so on.

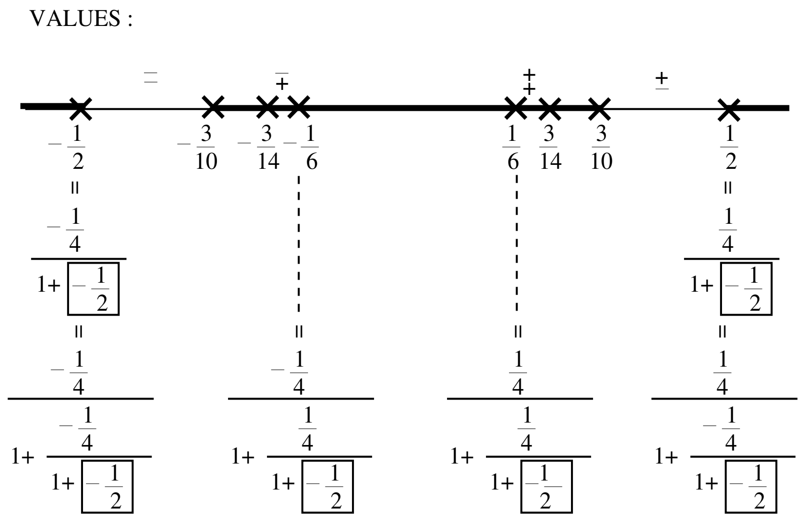

FIGURE 13.

In the illustration below we show (part of) the cumulative distribution function. In the illustration we have chosen $\alpha = 1 - \alpha = 1/2$.

In this case the cumulative distribution function F is continuous, nondecreasing and with $F'(x) = 0$ a.e. (with respect to linear Lebesgue measure). We omit the proof here, since it is very much related to the proof of the same properties for the ternary Cantor function (although there are differences making the proof far from trivial).

Example 12. Let p and q be positive numbers such that $p + q \leq pq$, and let

$$E = \{p, q\}.$$

We then know from Section 1 that

$$\text{cl } L_E = L_{[p,q]} = [X, Y],$$

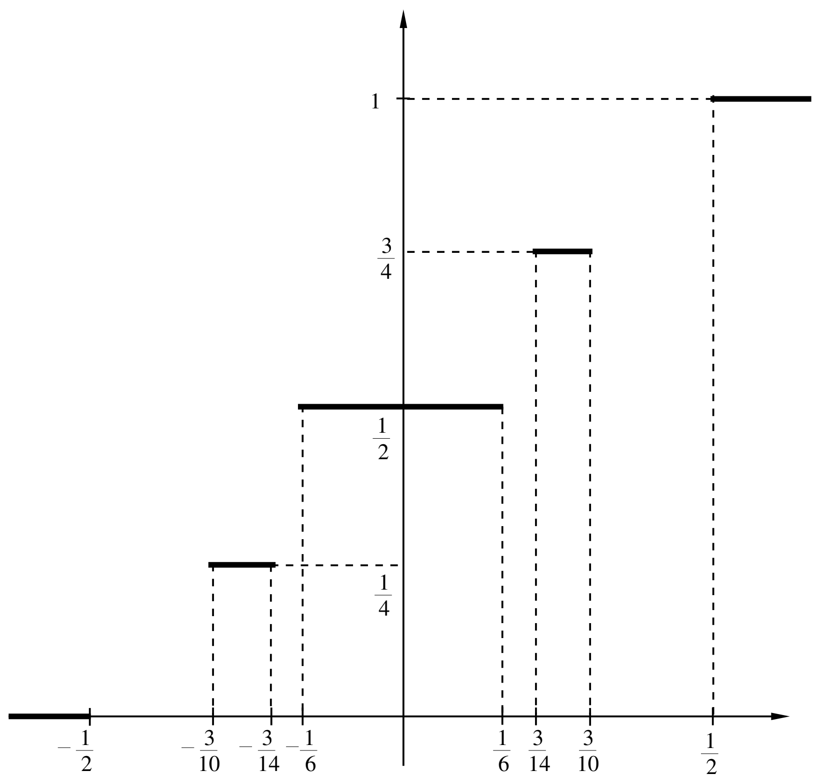

FIGURE 14.

where X and Y are as in Example 3. With $\Pr(a_n = p) = \alpha$ and $\Pr(a_n = q) = 1 - \alpha$, $0 < \alpha < 1$ it can be proved in a way, very much related to arguments in [**2**], that the distribution of the values of f in $[X, Y]$ is given by a measure which is absolutely continuous with respect to Lebesgue measure.

Final remarks. We hope to come back later to problems of the type discussed in the present section, hopefully in a wider context, which then will also contain the complex case. *Here* the main purpose was to show some of the variety in cumulative distribution functions occurring for $E = \{p, q\}$, where p and q are real: (1) a piecewise constant function; (2) a continuous, but not absolutely continuous function; and (3) an absolutely continuous function.

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