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MEROMORPHIC EXTENSION OF ANALYTIC CONTINUED FRACTIONS ACROSS THE LINE OF NONCONVERGENCE

HANS-J. RUNCKEL

Dedicated to Professor Wolfgang J. Thron
on the occasion of his 70-th birthday

ABSTRACT. Continued fractions $K(-a_n(z)/\lambda(z))$ are considered, where $a_n(z) \not\equiv 0$, $n \in \mathbf{N}$, and $\lambda(z)$ are holomorphic functions on a region $M \subset \mathbf{C}$ such that $\lim_{n\to\infty} a_n(z) = 1/4$ holds uniformly in M. They converge on $M \backslash S$, $S := \{z \in M : \lambda(z) \in [-1,1]\}$, to a meromorphic function $F(z)$. Conditions on the speed of convergence of the sequence $a_n(z)$, $n \in \mathbf{N}$, are given which ensure that $F(z)$ can be extended meromorphically across S into a part of the Riemann-surface of $\lambda(z) - (\lambda^2(z) - 1)^{1/2}$. For special classes of continued fractions, explicit analytic extension results are given.

1. Introduction and main results. We first consider limit-periodic analytic continued fractions of the type

$$f(\lambda) = \frac{1}{\lambda} \underset{-}{} \frac{a_1}{\lambda} \underset{-}{} \frac{a_2}{\lambda} \underset{-}{} \frac{a_3}{\lambda} \underset{-}{} \cdots, \tag{1}$$

where $a_n \in \mathbf{C}$, $a_n \neq 0$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} a_n = 1/4$ holds. It is well known (see [**3, 4, 8**]), that the right side in (1) converges and represents a meromorphic function $f(\lambda)$ in $D^* := \mathbf{C}\backslash[-1,1]$, the complex plane with a cut along $[-1,1] \subset \mathbf{R}$. Let D^{**} be a second copy of D^* and assume that D^* and D^{**} are connected along the cut $[-1,1]$ by crosswise joining opposite boundaries of the cut. This generates the Riemann surface of $(\lambda^2-1)^{1/2}$ where $(\lambda^2-1)^{1/2} > 0$ for $\lambda > 1$, $\lambda \in D^*$.

In [**5**, Theorem 1] the author proved

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Theorem A. *Put $b_n := 1 - 4a_n$ for $n \in \mathbf{N}$, and assume that*

$$\sum_{j=1}^{\infty} |b_j| R_0^j < \infty \text{ holds for some } R_0 > 1. \tag{2}$$

Then $f(\lambda) = G(\lambda)/H(\lambda)$ holds, where G and H are (explicitly given in Lemmas 2–5) holomorphic in D^* and can be extended analytically across the cut $[-1, 1]$ from both sides into a region of D^{**} which is bounded there by the ellipse

$$\Big(2\mathrm{Re}\,\lambda/(R_0^{1/2} + R_0^{-1/2})\Big)^2 + \Big(2\mathrm{Im}\,\lambda/(R_0^{1/2} - R_0^{-1/2})\Big)^2 = 1.$$

Their focal points $1, -1$ are algebraic branch points of order 2 for $f(\lambda)$. Furthermore, G and H can be extended continuously onto this ellipse. If (2) is satisfied for all $R_0 > 1$, then G and H can be extended analytically onto D^{**} (e.g., if $b_n = 0$ holds for all $n \geq n_0$).

In the present paper we will use formulas and estimates which were derived in [**5**] and now are summarized in Section 2. These are taken as the starting point in order to prove several generalizations of Theorem A.

Together with (1) we also consider the following continued fraction

$$F(z) = \frac{1}{\lambda(z)} \, - \, \frac{a_1(z)}{\lambda(z)} \, - \, \frac{a_2(z)}{\lambda(z)} \, - \, \frac{a_3(z)}{\lambda(z)} \, - \, \cdots, \tag{1'}$$

where $a_n(z) \not\equiv 0$, $n \in \mathbf{N}$, and $\lambda(z)$ are holomorphic functions of z on a region $M \subset \mathbf{C}$, such that $\lim_{n\to\infty} a_n(z) = 1/4$ holds uniformly on each compact subset of M (abbreviated: "uniformly *in* M"). Put $S := \{z \in M : \lambda(z) \in [-1, 1]\}$ and $M^* := M \backslash S$, which, in general, is an at most countable union of disjoint regions. $M^* = \varnothing$ iff $\lambda(z)$ is a constant in $[-1, 1]$. We always assume $M^* \neq \varnothing$. By substituting $\lambda = \lambda(z)$ and $a_n = a_n(z)$ in exactly the same identities and estimates (see Section 2) which were used in the proof of Theorem A, we now obtain the much more general

Theorem A′. *Put $b_n(z) := 1 - 4a_n(z)$ for $n \in \mathbf{N}$, $z \in M$ and assume that, for a fixed $R_0 > 1$,*

$$\sum_{j=1}^{\infty} |b_j(z)| R_0^j < \infty \quad \text{holds uniformly in } M. \tag{2'}$$

Put $\omega(z) := \lambda(z) - (\lambda^2(z) - 1)^{1/2}$ *for* $z \in M$, *where the square root is to be chosen such that* $|\omega(z)| < 1$ *holds for all* $z \in M^*$.

Then the following hold.

(a) *The right side of* $(1')$ *converges and represents a meromorphic function* $F(z)$ *on* M^*. *More precisely,* $F(z) = \hat{G}(z)/\hat{H}(z)$ *holds, where* $\hat{G}(z)$ *and* $\hat{H}(z)$ *are holomorphic on* M^* *and are obtained from* $G(\lambda)$ *and* $H(\lambda)$ *in Theorem* A *by substituting* $\lambda = \lambda(z)$ *and* $b_n = b_n(z)$ *in their series representations as stated in Lemmas* 2–5 *and where we assume* $\hat{H}(z) \not\equiv 0$ (*this already follows from* $(2')$ *with* $R_0 = 1$).

(b) *If* $L \subset S$ *is an arc which is bijectively mapped by* λ *onto an open subinterval of* $[-1, 1]$ *such that* $\lambda'(z) \neq 0$ *holds for all* $z \in L$, *then* $\hat{G}(z)$ *and* $\hat{H}(z)$ *can be extended analytically across* L *from both sides into a region in a second copy of* M *as far as* $\lambda(z)$ *remains inside the ellipse of Theorem* A.

(c) *If, for some* $z_0 \in S$, $\lambda(z_0) = 1$ *or* $= -1$ *and* $\lambda'(z_0) \neq 0$ *holds, then* z_0 *is an algebraic branch point of order* 2 *for the extended function* $F(z)$.

(d) *Thus,* $\hat{G}$ *and* $\hat{H}$ *can be extended analytically into a part of the Riemann-surface of* $\omega(z)$ *over* M (*obtained by extending* $\omega(z)$ *from* M^* *across* S *into a second copy of* M). *If* $(2')$ *holds for all* $R_0 > 1$, *then* $\hat{G}$ *and* $\hat{H}$ *can be extended analytically from* M^* *into the whole Riemann-surface of* $\omega(z)$ *over* M.

(e) *In the special case* $M = \mathbf{C}$ *and* $\lambda(z) \equiv z$, *precisely the same result holds for* $F(z)$ *as is stated for* $f(\lambda)$ *in Theorem* A, *provided* $\hat{H}(z) \not\equiv 0$.

Remark 1. Condition $(2')$ can even be replaced by the weaker condition: $\sum_{j=1}^{\infty} |b_j(z)(\omega(z))^{2j}| < \infty$ uniformly on suitable subsets of the Riemann-surface of $\omega(z)$ over M.

The essential steps of the proofs of Theorems A and A$'$ are given in Section 2. Another generalization of Theorem A is

Theorem B. *Put* $b_n := 1 - 4a_n$ *for* $n \in \mathbf{N}$ *and assume that there exist* $R > 1$ *and* $p \in \mathbf{N}$ *together with suitable numbers* $c_0 = 1$,

$c_1, c_2, \dots, c_p \in \mathbf{C}$, such that

$$\sum_{j=1}^{\infty} |b_j|^2 R^{2j} < \infty \quad \textit{and} \quad \sum_{j=1}^{\infty} R^{2j} \left| \sum_{n=0}^{p} c_n b_{j+n} \right| < \infty \tag{3}$$

hold.

Then $f(\lambda) = G(\lambda)/H(\lambda)$ holds, where G and H (from Theorem A) *can be extended analytically from D^* across the cut $[-1,1]$ from both sides into a region of D^{**} which is bounded there by the ellipse*

$$(2\mathrm{Re}\,\lambda/(R+R^{-1}))^2 + (2\mathrm{Im}\,\lambda/(R-R^{-1}))^2 = 1.$$

Their focal points $1, -1$ are algebraic branch points of order 2 for $f(\lambda)$.

Remark 2. Of course, (3) holds for $R = R_0^{1/2}$ if R_0 satisfies (2). But for special sequences b_j, $j \in \mathbf{N}$, it may be possible to choose $c_1, \dots, c_p$ such that (3) holds for an $R > R_0^{1/2}$. If the b_j, $j \in \mathbf{N}$, satisfy the difference equation $\sum_{n=0}^{p} c_n b_{j+n} = 0$, $j \in \mathbf{N}$, then (3) holds with $R = R_0$ from (2).

The proof of Theorem B is given in Section 3. Its main steps, which are rather technical, are combined in Theorem 1. Afterwards, it is shown that from the same identities and estimates which are used in the proof of Theorem B we also obtain the much more general

Theorem B$'$. *Put $b_n(z) := 1 - 4a_n(z)$ for $n \in \mathbf{N}$, $z \in M$, and assume that there exist fixed $R > 1$ and $p \in \mathbf{N}$ together with suitable functions $c_1(z), \dots, c_p(z)$ which are holomorphic on M such that, with $c_0(z) \equiv 1$,*

$$\sum_{j=1}^{\infty} |b_j(z)|^2 R^{2j} < \infty \quad \textit{and} \quad \sum_{j=1}^{\infty} R^{2j} \left| \sum_{n=0}^{p} c_n(z) b_{j+n}(z) \right| < \infty \tag{3$'$}$$

hold uniformly in M. Furthermore, assume $\sum_{n=0}^{p} c_n(z)\omega(z)^{-2n} \not\equiv 0$, where $\omega(z)$ is defined in Theorem A$'$.

Then the same conclusions hold for $F(z)$ in $(1')$ as stated in Theorem A$'$, *except that now, in part (b), analytic extension across L is possible* *as far as* $\lambda(z)$ *remains inside the possibly larger ellipse of Theorem* B. *In the special case where* $c_1, \dots, c_p$ *are constants and* $M = \mathbf{C}$, $\lambda(z) \equiv z$, *precisely the same result holds for* $F(z)$ *as stated for* $f(\lambda)$ *in Theorem* B, *provided* $\hat{H}(z) \not\equiv 0$.

Remark 3. If, in $(1')$, $\lim_{n\to\infty} a_n(z) = a^2(z)$ holds uniformly in M, where $a(z)$ is holomorphic and $\neq 0$ on M, then an equivalence transformation leads to a continued fraction of type $(1')$ where a_n is replaced by $a_n(z)/4a^2(z)$ and λ by $\lambda(z)/2a(z)$. To this equivalent continued fraction, Theorems A$'$ and B$'$ can be applied directly.

Further results concerning meromorphic extensions analogous to those stated in Theorems A and B obviously are obtainable for continued fractions which are related to (1) by means of variable substitutions and equivalence transformations (e.g., regular C-fractions as considered in [**5**] and [**6**]).

A method of meromorphic extension which is different from the one used in this paper is the general method of "modified" continued fractions discussed in the work of Gill [**1**], Jacobsen [**2**], Thron and Waadeland [**6, 7**]. Many general results are obtained there. But these are not as explicit as the results obtained in the present paper.

2. Some auxiliary formulas and results. As usual, $1/\lambda \,_{-}\, a_1/\lambda \,_{-} \cdots \,_{-}\, a_{n-1}/\lambda = A_n/B_n$ holds for each $n \in \mathbf{N}$ (with $a_0 := 1$), where A_n, B_n are polynomials in λ of degree $n-1$, n respectively, which satisfy, for $n \in \mathbf{N}$,

$$(4)\qquad \begin{aligned} A_{n+1} &= \lambda A_n - a_n A_{n-1}, \qquad A_0 = 0, \quad A_1 = 1\\ B_{n+1} &= \lambda B_n - a_n B_{n-1}, \qquad B_0 = 1, \quad B_1 = \lambda. \end{aligned}$$

The substitution $2\lambda =: \omega + \omega^{-1}$, $\omega \in \mathbf{C}$, $\omega \neq 0$ maps $0 < |\omega| < 1$ onto D^*, $|\omega| > 1$ onto D^{**} and $|\omega| = 1$ onto both boundaries of the cut $[-1, 1] \subset \mathbf{R}$.

Using this substitution and $b_n := 1 - 4a_n$, $n \in \mathbf{N}$, we obtain, from (4),

$$\begin{aligned}(1-\omega^2)(2\omega)^n A_{n+1} =&(1+\omega^2)((1-\omega^2)(2\omega)^{n-1}A_n)\\ &+\omega^2(b_n-1)((1-\omega^2)(2\omega)^{n-2}A_{n-1}).\end{aligned}$$

Since $(1-\omega^2)(2\omega)^{n-1}A_n$ and, similarly, $(1-\omega^2)(2\omega)^n B_n$ are polynomials in ω^2, we put $w := \omega^2$ and define $C_n(w) := (1-w)(2\omega)^{n-1}A_n$, $D_n(w) := (1-w)(2\omega)^n B_n$. Then (4) implies, for $n \in \mathbf{N}$,

(5)

$$C_{n+1} = (1+w)C_n - wC_{n-1} + b_n wC_{n-1}, \quad C_0 = 0, \quad C_1 = 1-w,$$
$$D_{n+1} = (1+w)D_n - wD_{n-1} + b_n wD_{n-1}, \quad D_0 = 1-w, \quad D_1 = 1-w^2.$$

In [**5**], the following *explicit* representations for C_n and D_n were derived from (5).

Lemma 1. *For fixed $r = 0, 1, 2, \dots$, let $C_{n,r}$ and $D_{n,r}$ be defined as the sum of all terms $b_{j_1}b_{j_2}\cdots b_{j_r}w^m$ (r-fold products of b_j's with arbitrary m) which occur in C_n and D_n, respectively. Then $C_n = \sum_{0\le r\le (n-1)/2} C_{n,r}$, $D_n = \sum_{0\le r\le n/2} D_{n,r}$ holds, where $C_{n,0} = 1-w^n$, $D_{n,0} = 1-w^{n+1}$, and, for $r \ge 1$,*

$$(1-w)^r C_{n,r} = w^r \sum_{j_1=2}^{n-2r+1} b_{j_1}(1-w^{j_1-1}) \sum_{j_2=j_1+2}^{n-2r+3} b_{j_2}(1-w^{j_2-j_1-1}) \cdots \sum_{j_r=j_{r-1}+2}^{n-1} b_{j_r}(1-w^{j_r-j_{r-1}-1})(1-w^{n-j_r})$$

$$(n \ge 2r+1, j_0 := 0 \ \textit{if}\ r=1) \quad \textit{and}$$

$$(1-w)^r D_{n,r} = w^r \sum_{j_1=1}^{n-2r+1} b_{j_1}(1-w^{j_1}) \sum_{j_2=j_1+2}^{n-2r+3} b_{j_2}(1-w^{j_2-j_1-1}) \cdots \sum_{j_r=j_{r-1}+2}^{n-1} b_{j_r}(1-w^{j_r-j_{r-1}-1})(1-w^{n-j_r})$$

$$(n \ge 2r, j_0 := -1 \ \textit{if}\ r=1), \ \textit{respectively.}$$

Next, we want to determine the limits of $C_n, D_n, C_{n,r}, D_{n,r}$ for $n \to \infty$. Assuming (2) to hold, this is possible only for $|w| < 1$ because of the factors $(1-w^{n-j_r})$, which occur in the sum representations of $C_{n,r}, D_{n,r}$ and which, in general, are responsible for the divergence of (1) for $\lambda \in (-1,1)$. After having obtained *explicit* infinite series expressions for the above limits for $|w| < 1$, it will be shown that, under condition (2), these series even are absolutely uniformly convergent for $|w| \leq R_0$.

Lemma 2. *For $k, r \in \mathbf{Z}$, $k \geq -1$, $r \geq 0$, put*

$$S_{k,0} := 1 \quad \textit{and} \quad S_{k,r} = S_{k,r}(w) := \sum_{j=k+2}^{\infty} b_j (1 - w^{j-k-1}) S_{j,r-1}$$

for $r \geq 1$. Assume that (2) holds and put

$$\rho_{k+2} = \rho_{k+2}(R_0) := 2 \sum_{j=k+2}^{\infty} |b_j| R_0^{j-k-1} \quad \textit{for } k \geq -1.$$

Then, for each $r \in \mathbf{N}$, $k \geq -1$ and all $|w| \leq R_0$,

$$|S_{k,r}(w)| \leq \rho_{k+2}\rho_{k+4} \cdots \rho_{k+2r}$$

holds. Especially, all series $S_{k,r}$ are absolutely uniformly convergent for $|w| \leq R_0$. Hence, each $S_{k,r}$ is holomorphic for $|w| < R_0$ and continuous for $|w| \leq R_0$.

The proof of this Lemma is obvious, and the proof of the next Lemma also is given in [**5**].

Lemma 3. *Assume that $\sum_{j=1}^{\infty} |b_j| < \infty$ holds. Then, for each $r \geq 0$, $C_r^* := \lim_{n\to\infty} C_{n,r}$, $D_r^* := \lim_{n\to\infty} D_{n,r}$, and also $C := \lim_{n\to\infty} C_n$, $D := \lim_{n\to\infty} D_n$ exist uniformly on every compact subset of $|w| < 1$. Furthermore, $C_0^* = 1$, $D_0^* = 1$ and, for $r \geq 1$, $|w| < 1$, C_r^* and D_r^* are explicitly given by*

$$\begin{aligned} (1-w)^r C_r^*(w) &= w^r S_{0,r}(w) \\ (1-w)^r D_r^*(w) &= w^r S_{-1,r}(w), \end{aligned} \tag{6}$$

the series $S_{k,r}(w)$ being absolutely uniformly convergent for $|w| \leq 1$, $k \geq -1$, $r \geq 1$. Finally, $C(w) = \sum_{r=0}^{\infty} C_r^*(w)$ and $D(w) = \sum_{r=0}^{\infty} D_r^*(w)$ hold for $|w| < 1$, where the series are absolutely uniformly convergent on every compact subset of $|w| \leq 1$, $w \neq 1$.

Lemma 4. *Assume that* (2) *holds. Then, for each* $r \geq 1$, $C_r^*(w)$ *and* $D_r^*(w)$ *are holomorphic for* $|w| < R_0$ *and continuous for* $|w| \leq R_0$. *Furthermore,*

$$\tag{7} \begin{gathered} |C_r^*(w)| \leq (|w|/(R_0-1))^r \rho_2\rho_4\cdots\rho_{2r} \quad \textit{and} \\ |D_r^*(w)| \leq (|w|/(R_0-1))^r \rho_1\rho_3\cdots\rho_{2r-1} \quad \textit{hold for} \\ r \geq 1 \quad \textit{and} \quad |w| \leq R_0. \end{gathered}$$

Hence, $\sum_{r=0}^{\infty} C_r^*(w)$ *and* $\sum_{r=0}^{\infty} D_r^*(w)$ *are absolutely uniformly convergent for* $|w| \leq R_0$, *and their sums* $C(w)$ *and* $D(w)$ *are holomorphic for* $|w| < R_0$ *and continuous for* $|w| \leq R_0$.

Proof. According to (6) and Lemma 2, C_r^* and D_r^* are holomorphic for $|w| < R_0$ (with removable isolated singularity at $w = 1$) and continuous for $|w| \leq R_0$. Since $|1 - w| \geq R_0 - 1$ holds for $|w| = R_0 > 1$, (6) and Lemma 2 show that (7) holds for $|w| = R_0$. Applying the maximum principle to $C_r^*(w)w^{-r}$ and $D_r^*(w)w^{-r}$ shows that (7) also holds for $|w| \leq R_0$. The rest of Lemma 4 follows from $\lim_{k\to\infty} \rho_k = 0$. □

Lemma 5. *Assume that* (2) *holds. Then* $f(\lambda)$ *in* (1) *satisfies* $f(\lambda) = G(\lambda)/H(\lambda)$ *for* $\lambda \in D^*$, *where*

$$\tag{8} \begin{aligned} &G(\lambda) := 2\omega C(\omega^2), \qquad H(\lambda) := D(\omega^2), \\ &2\lambda = \omega + \omega^{-1} \quad \textit{or} \quad \omega = \lambda - (\lambda^2-1)^{1/2} \\ &\textit{with} \quad (\lambda^2-1)^{1/2} > 0 \quad \textit{for } \lambda > 1, \lambda \in D^*. \end{aligned}$$

The functions $C(\omega^2), D(\omega^2)$ *are holomorphic for* $|\omega| < R_0^{1/2}$ *and continuous for* $|\omega| \leq R_0^{1/2}$.

Hence, $G(\lambda)$ *and* $H(\lambda)$ *can be extended analytically from* D^* *across the cut* $[-1, 1]$ *into a region of* D^{**} *bounded by* $|\omega| = R_0^{1/2}$ *or, in terms of* λ, *by the ellipse stated in Theorem* A.

Proof. From the definition of C_n, D_n follows

$$A_n(\lambda)/B_n(\lambda) = 2\omega C_n(\omega^2)/D_n(\omega^2) \quad \text{for } n \in \mathbf{N}, 0 < |\omega| < 1.$$

Lemma 3 shows that $f(\lambda) = \lim_{n\to\infty} A_n(\lambda)/B_n(\lambda) = 2\omega C(\omega^2)/D(\omega^2)$ holds for $0 < |\omega| < 1$ or, equivalently, $\lambda \in D^*$. The rest follows from Lemma 4. □

Lemmas 1–5, altogether, again prove Theorem A.

In order to prove Theorem A$'$ we substitute $b_n = b_n(z)$ and $w = \omega^2(z)$. Then all identities and estimates in Lemmas 1–5 remain valid if (2$'$) is satisfied. All infinite series are absolutely uniformly convergent for z in compact subsets of M on which the two-valued function $\omega(z)$ satisfies $|\omega(z)| \le R_0$. Starting from (8) we define $\hat{G}(z) := 2\omega(z)C(z,\omega^2(z))$ and $\hat{H}(z) := 2\omega(z)D(z,\omega^2(z))$, where $C(z,w)$ and $D(z,w)$ are obtained from $C(w)$ and $D(w)$ by substituting $b_n = b_n(z)$, $n \in \mathbf{N}$. Then $F(z) = \hat{G}(z)/\hat{H}(z)$ has all the properties stated in Theorem A$'$.

3. Proof of Theorem B. The first part of condition (3) implies condition (2) for each R_0 that satisfies $1 < R_0 < R$. Hence, Lemma 2 shows that, for every $k \ge -1$ and $r \ge 0$, $S_{k,r}$ is absolutely uniformly convergent on every compact subset of $|w| < R$ and, therefore, is holomorphic for $|w| < R$.

Now, the technical main steps of the proof of Theorem B are combined in

Theorem 1. *Assume that* (3) *is satisfied and put* $P(w) := \sum_{n=0}^{p} c_n w^n$. *Let* R_0 *be a fixed number with* $1 < R_0 < R$. *Then the following statements are true.*

(a) *For* $k \ge -1$ *and* $r \ge 0$, $(P(1/w))^r S_{k,r}(w)$ *is uniformly convergent for* $R_0 \le |w| \le R^2$, *and, hence, holomorphic for* $R_0 < |w| < R^2$ *and continuous for* $R_0 \le |w| \le R^2$. *Furthermore, there exist constants* $K_1, K_2 > 0$, *such that*

$$|(P(1/w))^r S_{k,r}| \le \varepsilon_{k,r} \quad \textit{holds for } R_0 \le |w| \le R^2,$$

where we have defined for $R_0 \le |w| \le R^2$, $\varepsilon_{k,0} := 1$,

$$\varepsilon_{k,1}(w) := K_1 \left(\sum_{j=k+2}^{\infty} |b_j|^2 |w|^{j-k} \right)^{1/2} + \sum_{j=k+2}^{\infty} |w|^{j-k-1} \left| \sum_{n=0}^{p} c_n b_{j+n} \right|,$$

and, for $r \geq 2$,

$$\begin{aligned}
&\varepsilon_{k,r}(w) := \\
&\left(K_1 \left(\sum_{j=k+2}^{\infty} |b_j|^2 |w|^{j-k} \right)^{1/2} + \sum_{j=k+2}^{\infty} |w|^{j-k-1} \left| \sum_{n=0}^{p} c_n b_{j+n} \right| \right) \varepsilon_{k+2,r-1} \\
&+ K_2 \left(\sum_{j=k+2}^{\infty} |b_j|^2 |w|^{j-k} \right)^{1/2} \left(\sum_{l=k+4}^{\infty} |b_l|^2 |w|^{l-k-2} \right)^{1/2} \varepsilon_{k+4,r-2}.
\end{aligned}$$

(b) *With* $K := K_1 + K_2/K_1$,

$$\begin{aligned}
&\varepsilon_{k,r}(w) \leq \\
&\left(K \left(\sum_{j=k+2}^{\infty} |b_j|^2 |w|^{j-k} \right)^{1/2} + \sum_{j=k+2}^{\infty} |w|^{j-k-1} \left| \sum_{n=0}^{p} c_n b_{j+n} \right| \right) \varepsilon_{k+2,r-1}(w)
\end{aligned}$$

holds for $R_0 \leq |w| \leq R^2$. *Hence, if we define, for* $k \geq -1$,

$$\begin{aligned}
\rho^*_{k+2} &= \rho^*_{k+2}(R) \\
&:= K \left(\sum_{j=k+2}^{\infty} |b_j|^2 R^{2(j-k)} \right)^{1/2} + \sum_{j=k+2}^{\infty} R^{2(j-k-1)} \left| \sum_{n=0}^{p} c_n b_{j+n} \right|,
\end{aligned}$$

then

$$|(P(1/w))^r S_{k,r}(w)| \leq \rho^*_{k+2} \rho^*_{k+4} \cdots \rho^*_{k+2r}$$

holds for $R_0 \leq |w| \leq R^2$, $k \geq -1$, $r \geq 1$.

(c) *Let* R_0 *and* R_1 *be chosen close to* R *such that* $1 < R_0 < R < R_1^2 < R^2$ *and* $P(1/w) \neq 0$ *holds for* $|w| = R_0$ *and* $|w| = R_1^2$. *If* $m > 0$ *is a lower bound of* $|P(1/w)|$ *for* $|w| = R_0$ *and* $|w| = R_1^2$, *then*

$$|S_{k,r}(w)| \leq m^{-r} \rho^*_{k+2}(R) \rho^*_{k+4}(R) \cdots \rho^*_{k+2r}(R)$$

holds for $R_0 \leq |w| \leq R_1^2$, $k \geq -1$, $r \geq 1$. *In particular,* $k = 0, -1$ *and* (6) *yield*

$$|w^{-r}(1-w)^r C^*_r(w)| \leq m^{-r} \rho^*_2 \rho^*_4 \cdots \rho^*_{2r}$$

and

$$|w^{-r}(1-w)^r D^*_r(w)| \leq m^{-r} \rho^*_1 \rho^*_3 \cdots \rho^*_{2r-1}$$

for $R_0 \le |w| \le R_1^2$ *and* $r \ge 1$. *Hence,* $\sum_{r=0}^{\infty} C_r^*(w)$, $\sum_{r=0}^{\infty} D_r^*(w)$ *are absolutely uniformly convergent on each compact subset of* $R_0 \le |w| < R^2$ *and their sums* $C(w)$, $D(w)$ *altogether (see Lemma* 4) *are holomorphic for* $|w| < R^2$ *and continuous for* $|w| \le R^2$ *with possible exceptions only at zeros of* $P(1/w)$ *on* $|w| = R^2$.

(d) *As in Lemma* 5, $f(\lambda) = G(\lambda)/H(\lambda)$ *holds with* $G(\lambda) = 2\omega C(\omega^2)$, $H(\lambda) = D(\omega^2)$, $2\lambda = \omega + \omega^{-1}$ *or* $\omega = \lambda - (\lambda^2-1)^{1/2}$ *with* $(\lambda^2-1)^{1/2} > 0$ *for* $\lambda > 1$, $\lambda \in D^*$. *The functions* $C(\omega^2)$, $D(\omega^2)$ *are holomorphic for* $|\omega| < R$ *and continuous for* $|\omega| \le R$ *with possible exceptions only at zeros of* $P(1/\omega^2)$ *on* $|\omega| = R$. *Hence,* $G(\lambda)$, $H(\lambda)$ *can be extended analytically from* D^* *across the cut* $[-1, 1]$ *into a region of* D^{**} *bounded by* $|\omega| = R$, *or, in terms of* λ, *by the ellipse stated in Theorem* B.

Proof of Theorem 1. (a). We first assume $0 < |w| < R$ and $r \ge 2$. Then $S_{k,r}$ converges absolutely for these w (Lemma 3). Therefore,

$$\begin{aligned}
P(1/w)S_{k,r} &= P(1/w)\sum_{j=k+2}^{\infty} b_j S_{j,r-1} - \sum_{n=0}^{p} c_n w^{-n} \sum_{j=k+2}^{\infty} b_j w^{j-k-1} S_{j,r-1} \\
&= P(1/w)\sum_{j=k+2}^{\infty} b_j S_{j,r-1} \\
&\quad - \sum_{n=0}^{p} c_n w^{-n} \sum_{j=k+2+n}^{\infty} b_j w^{j-k-1} S_{j,r-1} \\
&\quad - \sum_{n=1}^{p} c_n w^{-n} \sum_{j=k+2}^{k+n+1} w^{j-k-1} b_j S_{j,r-1} \\
&= P(1/w)\sum_{j=k+2}^{\infty} b_j S_{j,r-1} \\
&\quad - \sum_{n=0}^{p} c_n w^{-n} \sum_{j=k+2}^{\infty} b_{j+n} w^{j+n-k-1} S_{j+n,r-1} \\
&\quad - \sum_{n=1}^{p} c_n w^{-n} \sum_{j=k+2}^{k+n+1} w^{j-k-1} b_j S_{j,r-1}
\end{aligned}$$

holds and, finally,

$$P(1/w)S_{k,r} = P(1/w)\sum_{j=k+2}^{\infty} b_j S_{j,r-1} - \sum_{j=k+2}^{\infty} w^{j-k-1} S_{j,r-1}\left(\sum_{n=0}^{p} c_n b_{j+n}\right)$$
$$- \sum_{n=1}^{p} c_n w^{-n} \sum_{j=k+2}^{k+n+1} w^{j-k-1} b_j S_{j,r-1}$$
$$+ \sum_{n=1}^{p} c_n \sum_{j=k+2}^{\infty} w^{j-k-1} b_{j+n}(S_{j,r-1} - S_{j+n,r-1}). \tag{9}$$

From the definition of $S_{k,r-1}$, follows

$$\begin{aligned} S_{j,r-1} - S_{j+n,r-1} &= \sum_{l=j+2}^{\infty} b_l(1-w^{l-j-1})S_{l,r-2} \\ &\quad - \sum_{l=j+n+2}^{\infty} b_l(1-w^{l-j-n-1})S_{l,r-2} \\ &= \sum_{l=j+2}^{j+n+1} b_l S_{l,r-2} - \sum_{l=j+2}^{j+n+1} b_l w^{l-j-1} S_{l,r-2} \\ &\quad + (w^{-n}-1)\sum_{l=j+n+2}^{\infty} b_l w^{l-j-1} S_{l,r-2}. \end{aligned}$$

Substituting this into (9) yields

$$\begin{aligned} (9) = &\sum_{n=1}^{p} c_n \sum_{j=k+2}^{\infty} w^{j-k-1} b_{j+n} \sum_{l=j+2}^{j+n+1} b_l S_{l,r-2} \\ &- \sum_{n=1}^{p} c_n \sum_{j=k+2}^{\infty} w^{j-k-1} b_{j+n} \sum_{l=j+2}^{j+n+1} b_l w^{l-j-1} S_{l,r-2} \\ &+ \sum_{n=1}^{p} c_n (w^{-n}-1) \sum_{j=k+2}^{\infty} w^{j-k-1} b_{j+n} \sum_{l=j+n+2}^{\infty} b_l w^{l-j-1} S_{l,r-2}. \end{aligned} \tag{10}$$

In the last two sums j and $-j$ in the exponent of w cancel and (10) can be written as

$$\begin{aligned}
(10) &= \sum_{n=1}^{p} c_n(w^{-n}-1) \sum_{j=k+2}^{\infty} b_{j+n} \sum_{l=j+n+2}^{\infty} b_l w^{l-k-2} S_{l,r-2} \\
&= \sum_{n=1}^{p} c_n(w^{-n}-1)w^{n+1} \sum_{j=k+2}^{\infty} b_{j+n} \\
&\qquad \cdot \sum_{l=(k+2+n)+2}^{\infty} b_l w^{l-(k+2+n)-1} S_{l,r-2} \\
&\quad - \sum_{n=1}^{p} c_n(w^{-n}-1)w^{n+1} \\
&\qquad \cdot \sum_{j=k+3}^{\infty} b_{j+n} \sum_{l=(k+2+n)+2}^{j+n+1} b_l w^{l-(k+2+n)-1} S_{l,r-2}.
\end{aligned}$$

In the first sum on the right side of the last equation,

$$\sum_{l=(k+2+n)+2}^{\infty} b_l w^{l-(k+2+n)-1} S_{l,r-2} = -S_{k+2+n,r-1} + \sum_{l=(k+2+n)+2}^{\infty} b_l S_{l,r-2}$$

is substituted. Then

$$\begin{aligned}
(10) = &- \sum_{n=1}^{p} c_n(w^{-n}-1)w^{n+1}\left(\sum_{j=k+2}^{\infty} b_{j+n}\right) S_{k+2+n,r-1} \\
&+ \sum_{n=1}^{p} c_n(w^{-n}-1)w^{n+1} \sum_{j=k+2}^{\infty} b_{j+n} \sum_{l=(k+2+n)+2}^{\infty} b_l S_{l,r-2} \\
&- \sum_{n=1}^{p} c_n(w^{-n}-1) \sum_{j=k+3}^{\infty} b_{j+n} \sum_{l=(k+2+n)+2}^{j+n+1} b_l w^{l-k-2} S_{l,r-2}.
\end{aligned}$$

After having carried out these substitutions, we multiply the final equation obtained for $P(1/w)S_{k,r}$ by $(P(1/w))^{r-1}$ and use the abbreviation $E_{k,r} = E_{k,r}(w) := (P(1/w))^r S_{k,r}(w)$. Then, we obtain, for

$r \geq 2$ and $0 < |w| < R$,

(11)

$$
\begin{aligned}
E_{k,r} = {} & P(1/w) \sum_{j=k+2}^{\infty} b_j E_{j,r-1} - \sum_{j=k+2}^{\infty} w^{j-k-1} E_{j,r-1} \left(\sum_{n=0}^{p} c_n b_{j+n} \right) \\
& - \sum_{n=1}^{p} c_n \sum_{j=k+2}^{k+n+1} w^{j-k-n-1} b_j E_{j,r-1} \\
& + P(1/w) \sum_{n=1}^{p} c_n \sum_{j=k+2}^{\infty} w^{j-k-1} b_{j+n} \sum_{l=j+2}^{j+n+1} b_l E_{l,r-2} \\
& - P(1/w) \sum_{n=1}^{p} c_n \sum_{j=k+2}^{\infty} b_{j+n} \sum_{l=j+2}^{j+n+1} b_l w^{l-k-2} E_{l,r-2} \\
& - \sum_{n=1}^{p} c_n (w^{-n} - 1) w^{n+1} \left(\sum_{j=k+2}^{\infty} b_{j+n} \right) E_{k+2+n,r-1} \\
& + P(1/w) \sum_{n=1}^{p} c_n (w^{-n} - 1) w^{n+1} \sum_{j=k+2}^{\infty} b_{j+n} \sum_{l=k+n+4}^{\infty} b_l E_{l,r-2} \\
& - P(1/w) \sum_{n=1}^{p} c_n (w^{-n} - 1) \sum_{j=k+3}^{\infty} b_{j+n} \sum_{l=k+n+4}^{j+n+1} b_l w^{l-k-2} E_{l,r-2}.
\end{aligned}
$$

From now on, we assume that condition (3) is satisfied and that $R_0 \leq |w| \leq R^2$ holds, where R_0 is an arbitrary fixed number with $1 < R_0 < R$.

We want to show that all sums on the *right side* of equation (11) are absolutely uniformly convergent for $R_0 \leq |w| \leq R^2$. Let K_0 be an upper bound of $|P(1/w)|$ for $R_0 \leq |w| \leq R^2$. Assume also that $|E_{j,r-1}(w)| \leq \varepsilon_{j,r-1}(w) = \varepsilon_{j,r-1}$ and $|E_{j,r-2}(w)| \leq \varepsilon_{j,r-2}(w) = \varepsilon_{j,r-2}$ holds for $R_0 \leq |w| \leq R^2$, where $\varepsilon_{j,r-1}, \varepsilon_{j,r-2}$ are assumed to be already defined such that, for $s = r-1$ and $r-2$,

$$
\begin{aligned}
& 0 \leq \varepsilon_{j,s} \leq \varepsilon_{k,s} < \infty \quad \text{holds for } j \geq k \geq -1 \quad \text{and that} \\
& \varepsilon_{j,s}(w) \leq \varepsilon_{j,s}(R^2) \text{ holds for } R_0 \leq |w| \leq R^2 \text{ and } j \geq -1.
\end{aligned}
\tag{12}
$$

Using (12) and the Cauchy-Schwarz inequality, we can estimate the right side of (11) by

(13)

$$\begin{aligned}
|E_{k,r}| \le{}& K_0\varepsilon_{k+2,r-1}\sum_{j=k+2}^{\infty}|b_j| + \varepsilon_{k+2,r-1}\sum_{j=k+2}^{\infty}|w|^{j-k-1}\left|\sum_{n=0}^{p}c_nb_{j+n}\right| \\
&+ \varepsilon_{k+2,r-1}\sum_{n=1}^{p}|c_n|\sum_{j=k+2}^{\infty}|b_j| \\
&+ K_0\varepsilon_{k+4,r-2}\sum_{n=1}^{p}n|c_n|\left(\sum_{j=k+3}^{\infty}|b_j|^2|w|^{j-k}\right)^{1/2} \\
&\qquad\cdot\left(\sum_{l=k+4}^{\infty}|b_l|^2|w|^{l-k-2}\right)^{1/2} \\
&+ K_0\varepsilon_{k+4,r-2}\sum_{n=1}^{p}n|c_n|\left(\sum_{j=k+3}^{\infty}|b_j|^2|w|^{j-k}\right)^{1/2} \\
&\qquad\cdot\left(\sum_{l=k+4}^{\infty}|b_l|^2|w|^{l-k-2}\right)^{1/2} \\
&+ \varepsilon_{k+3,r-1}|w|\sum_{n=1}^{p}|c_n||1-w^n|\sum_{j=k+2}^{\infty}|b_j| \\
&+ K_0\varepsilon_{k+5,r-2}|w|\sum_{n=1}^{p}|c_n||1-w^n|\sum_{j=k+3}^{\infty}|b_j|\sum_{l=k+5}^{\infty}|b_l| \\
&+ K_0\varepsilon_{k+5,r-2}(|w|^{1/2}-1)^{-1}\sum_{n=1}^{p}|c_n||w^{-n}-1| \\
&\qquad\cdot\left(\sum_{j=k+4}^{\infty}|b_j|^2|w|^{j-k}\right)^{1/2} \\
&\qquad\cdot\left(\sum_{l=k+5}^{\infty}|b_l|^2|w|^{l-k-2}\right)^{1/2}.
\end{aligned}$$

This shows that all series $E_{k,r}(w)$ are uniformly convergent for $R_0 \le |w| \le R^2$. Hence, every $E_{k,r}(w)$ is holomorphic for $R_0 < |w| < R^2$ and continuous for $R_0 \le |w| \le R^2$.

By applying (12) and elementary estimates to the right side of (13), we finally obtain $|E_{k,r}(w)| \le \varepsilon_{k,r}(w)$ for $R_0 \le |w| \le R^2$, $k \ge -1$, $r \ge 2$, where $\varepsilon_{k,r}$ is defined as stated in Theorem 1(a) and also satisfies (12) with $s = r$. The constants $K_1, K_2 > 0$ occurring in the definition of $\varepsilon_{k,r}$ are upper bounds of two polynomials in $|w|$ of degree $\le p+2$ with coefficients K_0 and $|c_n|$, $1 \le n \le p$, $R_0 \le |w| \le R^2$.

Finally, we consider the case $r = 1$. Then

$$P(1/w)S_{k,1} = P(1/w)\sum_{j=k+2}^{\infty} b_j - \sum_{j=k+2}^{\infty} w^{j-k-1}\sum_{n=0}^{p} c_n b_{j+n} - \sum_{n=1}^{p} c_n \sum_{j=k+2}^{k+1+n} b_j w^{j-k-1-n}$$

holds for $0 < |w| \le R^2$. Similarly, as before, we obtain, for $1 < R_0 \le |w| \le R^2$,

$$|E_{k,1}(w)| \le K_0 \sum_{j=k+2}^{\infty} |b_j| + \sum_{j=k+2}^{\infty} |w|^{j-k-1}\left|\sum_{n=0}^{p} c_n b_{j+n}\right| + \sum_{n=1}^{p} |c_n| \sum_{j=k+2}^{\infty} |b_j|.$$

This immediately yields $|E_{k,1}(w)| \le \varepsilon_{k,1}(w)$ for $R_0 \le |w| \le R^2$, $k \ge -1$, where $\varepsilon_{k,1}$ is defined as stated in Theorem 1(a) with the same constant K_1 as above (in the case $r \ge 2$). This proves part (a).

(b). The definition of $\varepsilon_{k,r}$ yields, for $r \ge 2$,

$$\varepsilon_{k,r} \ge K_1\left(\sum_{j=k+2}^{\infty} |b_j|^2 |w|^{j-k}\right)^{1/2} \varepsilon_{k+2,r-1}.$$

Replacing k by $k+2$ and r by $r-1$ yields

$$\varepsilon_{k+2,r-1} \ge K_1\left(\sum_{j=k+4}^{\infty} |b_j|^2 |w|^{j-k-2}\right)^{1/2} \varepsilon_{k+4,r-2} \quad \text{for } r \ge 2.$$

We substitute this in the definition of $\varepsilon_{k,r}$ and obtain

$$\varepsilon_{k,r} \leq \left(K \left(\sum_{j=k+2}^{\infty} |b_j|^2 |w|^{j-k} \right)^{1/2} + \sum_{j=k+2}^{\infty} |w|^{j-k-1} \left| \sum_{n=0}^{p} c_n b_{j+n} \right| \right) \varepsilon_{k+2,r-1} \tag{14}$$

where $K := K_1 + K_2/K_1$. This also holds for $r = 1$.

With $\rho^*_{k+2}(R)$, as defined in Theorem 1(b), we obtain from (a) and (14), $|E_{k,r}(w)| \leq \rho^*_{k+2}\rho^*_{k+4}\cdots\rho^*_{k+2r}$ for all $k \geq -1$, $r \geq 1$, and $R_0 \leq |w| \leq R^2$.

(c). We now choose R_0 and R_1 close to R such that $1 < R_0 < R < R_1^2 < R^2$ and $P(1/w) \neq 0$ hold for $|w| = R_0$ and $|w| = R_1^2$. Let $m > 0$ be a lower bound of $|P(1/w)|$ for $|w| = R_0$ and $|w| = R_1^2$. Because of (a), $E_{k,r}(w)$ is holomorphic for $R_0 \leq |w| \leq R_1^2$. Therefore, also, $S_{k,r}(w)$ is holomorphic for $R_0 \leq |w| \leq R_1^2$ with removable isolated singularities at the zeros of $P(1/w)$. From (b) follows

$$|S_{k,r}(w)| \leq m^{-r}\rho^*_{k+2}\rho^*_{k+4}\cdots\rho^*_{k+2r} \quad \text{for } |w| = R_0 \quad \text{and} \quad |w| = R_1^2.$$

Because of the maximum principle this estimate holds for all w with $R_0 \leq |w| \leq R_1^2$. The remaining part of statement (c) is a consequence of $\lim_{k\to\infty} \rho^*_k = 0$. Finally, (d) follows from (c) and (8) in Lemma 5. This concludes the proof of Theorem 1. □

Theorem B now follows from Theorem 1(d).

In order to prove Theorem B', we substitute $b_n = b_n(z)$, $w = \omega^2(z)$ and $c_n = c_n(z)$ in all identities occurring in the proof of Theorem 1 and assume now that ($3'$) holds. Theorem A$'$ implies that (11) remains valid for all $z \in M$ for which the two-valued function $\omega(z)$ satisfies $|\omega^2(z)| \leq R_0$, all series on the right side being absolutely uniformly convergent on compact subsets. Therefore, it suffices to consider (11) on $\Omega := \{z \in M^* : 1 < R_0 \leq |\omega^2(z)| \leq R^2\}$. Then on each compact subset of Ω the estimates of Theorem 1(a), (b) remain valid where the constants K_1, K_2 and K now depend on these compact sets. Hence, also, all series on the right side of

(11) are absolutely uniformly convergent on each compact subset of Ω. The left side of (11) is $E_{k,r}(z,\omega^2(z)) = (Q(z))^r S_{k,r}(z,\omega^2(z))$, where $Q(z) := \sum_{n=0}^{p} c_n(z)\omega(z)^{-2n} \not\equiv 0$ and $E_{k,r}(z,w)$, $S_{k,r}(z,w)$ are obtained from $E_{k,r}(w) := (P(1/w))^r S_{k,r}(w)$ (see Lemma 2) by substituting $b_n = b_n(z)$. Since $E_{k,r}(z,\omega^2(z))$ is holomorphic on each open $\Omega_0 \subset \Omega$, $S_{k,r}(z,\omega^2(z))$ is also holomorphic there with removable isolated singularities at zeros of $Q(z)$. To each $z_0 \in \Omega_0$, there exist constants $\rho > 0$, $m > 0$ with $\{z \in \mathbf{C} : |z - z_0| \leq \rho\} \subset \Omega_0$ such that $|Q(z)| \geq m$ holds for $|z - z_0| = \rho$. Hence, according to Theorem 1(b), $|S_{k,r}(z,\omega^2(z)| \leq m^{-r}\rho^*_{k+2}\rho^*_{k+4}\cdots\rho^*_{k+2r}$ holds for $|z-z_0| = \rho$ and, because of the maximum principle, also for $|z-z_0| \leq \rho$. Consequently, conclusions analogous to Theorem 1(c), (d) remain valid in this generalization. The proof of (d) uses the same reasoning as the corresponding part in the proof of Theorem A$'$.

REFERENCES

1. John Gill, *Enhancing the convergence region of a sequence of bilinear transformations*, Math. Scand. **43** (1978), 74–80.

2. Lisa Jacobsen, *Functions defined by continued fractions meromorphic continuation*, Rocky Mountain J. Math. **15** (1985), 685–703.

3. William B. Jones and Wolfgang J. Thron, *Continued fractions. Analytic theory and applications*, Encyclopedia of Mathematics and its Applications, No. 11; Addison Wesley, Reading, MA, 1980.

4. Oskar Perron, *Die Lehre von den Kettenbrüchen*, Band 2, B.G. Teubner, Stuttgart, 1957.

5. Hans-J. Runckel, *Continuity on the boundary and analytic continuation of continued fractions*, Math. Z. **148** (1976), 189–205.

6. Wolfgang J. Thron and Haakon Waadeland, *Accelerating convergence of limit periodic continued fractions $K(a_n/1)$*, Numer. Math. **34** (1980), 155–170.

7. Wolfgang J. Thron and Haakon Waadeland, *Analytic continuation of functions defined by means of continued fractions*, Math. Scand. **47** (1980), 72–90.

8. H.S. Wall, *Analytic theory of continued fractions*. Van Nostrand, New York, 1948.

Abteilung Mathematik IV, Universität Ulm, Oberer Eselsberg, D-7900 Ulm, F.R.G.