ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 1, Winter 1991

APPROXIMATION BY CONSTRAINED PARAMETRIC POLYNOMIALS

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1. Introduction. The mathematical representations of curves and surfaces employed by computer-aided design (CAD) systems are quite varied, having evolved in complexity over the past 25 years. Initially, the mathematics was kept rather simple, making use of piecewise quadratic and cubic curves, and linearly and cubicly blended surfaces. Use of more sophisticated entities such as higher degree polynomials, splines, B-splines and nonuniform rational B-splines had to await the development of intuitive, nontechnical user interfaces. While the older systems have incorporated some of the newer mathematical entities, most are still used with much of their original mathematics intact.

Of particular importance to the users of CAD systems is the communication of geometric design data between themselves and other users on dissimilar CAD systems. Such an exchange of data takes place routinely between manufacturers and suppliers, and often between departments of larger manufacturers. Because of the variety of entities used, approximations are frequently required. These approximations must be *accurate* (to tolerances on the order of .01 mm) and the process must be *fast* (to accommodate the large volume of data being transferred).

Since low degree polynomials are among the most common entities used by CAD systems, the problem of data exchange is often reduced to the problem of inexpensively producing low degree parametric polynomial approximations of curves and surfaces. There are two constraints which must be respected, in addition to user-specified bounds on the uniform error. First, the approximates should have nearly arc-length parametrizations. This is critical to most CAD systems since parametrizations are used in visualizing wireframe models and in manufacturing processes. Second, since complicated curves and surfaces will, in general, be approximated in a piecewise fashion, composite approximations must have a prescribed amount of geometric or visual

Received by the editors on September 15, 1988, and in revised form on March 3, 1989.

continuity. In practice, the approximations must possess at least C° continuity, while some systems require slope and curvature continuity.

In this article we derive lower degree polynomial approximates to higher degree polynomials using a generalization of Chebyshev economization; approximates to nonpolynomial entities are derived using a least square formulation. Both of the schemes described incorporate constraints which guarantee user-controlled geometric continuity between the piecewise approximates. In each case the parametrization of the approximate is tied to that of the original, the assumption being that the parametrization of the original entity is satisfactory.

The outline of this article is as follows. We introduce notation in Section 2 for vector-valued function spaces and norms on those spaces. In Section 3 we introduce the collection of constrained Chebyshev polynomials, and in Section 4 we discuss least square analogs of these polynomials, the Jacobi and ultraspherical polynomials. Polynomial approximations to polynomial and nonpolynomial curves are proposed in Sections 5 and 6, respectively. Finally, in Section 7 we discuss how these methods can be extended to parametric surfaces.

2. General notation. For a nonnegative integer m, let π_m denote the collection of real polynomials of degree at most m, with $\pi_{-1} \equiv \{0\}$. Parametric polynomials of degree at most m in Euclidean n-space are designated by π_m^n. The *power basis* for π_m is the usual set $\{1, s, \dots, s^m\}$. The *Bernstein basis* for π_m consists of the set $\{B_{i,m}(s)\}_{i=0}^m$ where

$$B_{i,m}(s) = \binom{m}{i} s^i (1-s)^{m-i}, \qquad i = 0, 1, \dots, m, \quad m \geq 0.$$

Bezier (of Renault) and de Casteljau (of Citroen) were among the first to make use of the Bernstein basis in CAD, principally because of the geometric significance which could be placed upon the coefficients in a polynomial's Bernstein representation [**3**, p. 132]. Of importance in this note is a formula which relates the power basis coefficients of a polynomial to its Bernstein coefficients. In particular, if $p(s) = \sum_{i=0}^m v_i B_{i,m}(s)$, then its derivatives are given by

$$p^{(k)}(s) = (m!/(m-k)!) \sum_{i=0}^{m-k} \left[\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} v_{i+j} \right] B_{i,m-k}(s),$$

for $0 \leq k \leq m$. Evaluating this expression at $s = 0$, we find that

$$p^{(k)}(0) = (m!/(m-k)!) \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} v_j,$$

and, thus,

$$[p(0), \dots, p^{(k-1)}(0)] = [v_0, \dots, v_{k-1}]UD, \qquad 1 \leq k \leq m, \tag{2.1}$$

where $U = [u_{ij}]$ is upper triangular and $D = \operatorname{diag}[d_i]$ is diagonal,

$$u_{ij} = \begin{cases} (-1)^{j-i}\binom{j-1}{i-1}, & 1 \leq i \leq j \leq k, \\ 0, & 1 \leq j < i \leq k; \end{cases}$$
$$d_i = m!/(m+1-i)!, \qquad 1 \leq i \leq k. \tag{2.2}$$

It is a simple matter to show that $\mathrm{U}^{-1} = [(-1)^{j-i}u_{ij}]$ so that the coefficients v_i may be computed from *prescribed* derivative information at $s = 0$. Because of the symmetry of the Bernstein polynomials on the interval $[0, 1]$, a similar formula relates the coefficients v_i to derivatives at $s = 1$.

For a compact set B in $\mathbf{R}^m$, $C^k(B)$ denotes the collection of k-times continuously differentiable functions on that set. Vector-valued functions in n-space, each of whose components are k-times continuously differentiable on this set, are denoted by $C^k(B)^n$.

For any vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{R}^n$, we denote the usual Euclidean norm by $||\mathbf{a}||_E$. In terms of this norm we define the uniform and least square norms for $\mathbf{f} \in C(B)^n$ by

$$||\mathbf{f}||_{L_\infty(B)} = \max_{s \in B} ||\mathbf{f}(s)||_E \quad \text{and}$$
$$||\mathbf{f}||^2_{L_2(B)} = \int_B ||\mathbf{f}(s)||^2_E \, ds. \tag{2.3}$$

Note that when considering the norm of the difference of two parametric functions, the parameterizations are linked. That is, for parametric curves defined on an interval I,

$$||\mathbf{f} - \mathbf{p}||_{L_\infty(I)} = \max_{s \in I} ||\mathbf{f}(s) - \mathbf{p}(s)||_E.$$

The Euclidean distance between the two curves $\mathbf{f}$ and $\mathbf{p}$ is bounded by this value but is not, in general, equal to it. A better measure of the "true" distance can be obtained by considering the Hausdorff metric [**5, 6**], $H(\mathbf{f},\mathbf{p}) = \max\{d(\mathbf{f},\mathbf{p}), d(\mathbf{p},\mathbf{f})\}$, where $d(\mathbf{f},\mathbf{p}) = \max_{s\in I} \min_{t\in I} |\mathbf{f}(s) - \mathbf{p}(t)|$.

3. A weighted minimax problem. In [**7, 8**] the collection of *constrained* Chebyshev polynomials was introduced as a solution to a *weighted* minimax problem. These polynomials, denoted by $T_m^{(\alpha,\beta)}(x)$, are the unique monic polynomials $(x-1)^\alpha(x+1)^\beta(x^m - g_0(x))$ where g_0 is extremal for

Problem 3.1. For each triple of nonnegative integers (m,α,β), determine

$$\underset{g\in\pi_{m-1}}{\text{minimum}} \ \underset{-1\le x\le 1}{\text{maximum}} (1-x)^\alpha(1+x)^\beta|x^m - g(x)|.$$

These polynomials are a proper generalization since $2^{m-1}T_m^{(0,0)}(x)$ is precisely the classical Chebyshev polynomial of degree m.

Of interest here is the symmetric case, and for the remainder of this note we will take $\beta = \alpha$. The constrained Chebyshev polynomials in this case are denoted by $T_m^{(\alpha)}(x)$ and take the form

$$T_m^{(\alpha)}(x) = (x^2-1)^\alpha(x^m - \cdots) \in \pi_{m+2\alpha}. \tag{3.1}$$

There are instances when the constrained Chebyshev polynomials can be determined explicitly (see [**6, 7**]). In general, this is not the case, but they can be approximated by using a modified Remez exchange algorithm [**2**].

Most CAD systems parametrize curves on the interval [0,1]. We shall do the same and introduce more convenient notation for the constrained Chebyshev polynomials on this interval.

Definition 3.1. For each pair of nonnegative integers (m,α), with $m \ge 2\alpha$, the *symmetric constrained Chebyshev polynomials* are given by

$$C_m^{(\alpha)}(s) = 2^{-m} T_{m-2\alpha}^{(\alpha)}(2s-1)$$
$$= s^\alpha (s-1)^\alpha (s^{m-2\alpha} + \cdots), \tag{3.2}$$

$$E_m^{(\alpha)} = \max_{s\in[0,1]} |C_m^{(\alpha)}(s)|. \tag{3.3}$$

Thus, m indicates the degree of the monic polynomial $C_m^{(\alpha)}(s)$, while α indicates the order of the zero at both $s = 0, 1$. Some of the normalized constrained polynomials $C_m^{(\alpha)}(s)/E_m^{(\alpha)}$ are displayed in Figure 1; the magnification error $E_m^{(\alpha)}$ due to the constraint is summarized in Table 1.

TABLE I. $2^{2m-1}E_m^{(\alpha)}$

m/α	0	1	2	3
1	1.0000			
2	1.0000	2.0000		
3	1.0000	1.5396		
4	1.0000	1.3726	8.0000	
5	1.0000	1.2852	4.5795	
6	1.0000	1.2312	3.3886	32.0000
7	1.0000	1.1945	2.7860	15.2332
8	1.0000	1.1679	2.4235	9.8125
9	1.0000	1.1477	2.1821	7.2353
10	1.0000	1.1319	2.0101	5.7634
11	1.0000	1.1191	1.8814	4.8247
12	1.0000	1.1086	1.7816	4.1799
13	1.0000	1.0998	1.7020	3.7126
14	1.0000	1.0923	1.6369	3.3599
15	1.0000	1.0858	1.5828	3.0845
16	1.0000	1.0802	1.5371	2.8651
17	1.0000	1.0754	1.4981	2.6860
18	1.0000	1.0709	1.4642	2.5373
19	1.0000	1.0671	1.4348	2.4121
20	1.0000	1.0636	1.4089	2.3051

4. A least square extremal problem. The extremal problem analogous to the weighted minimax Problem 3.1 in the least square norm is

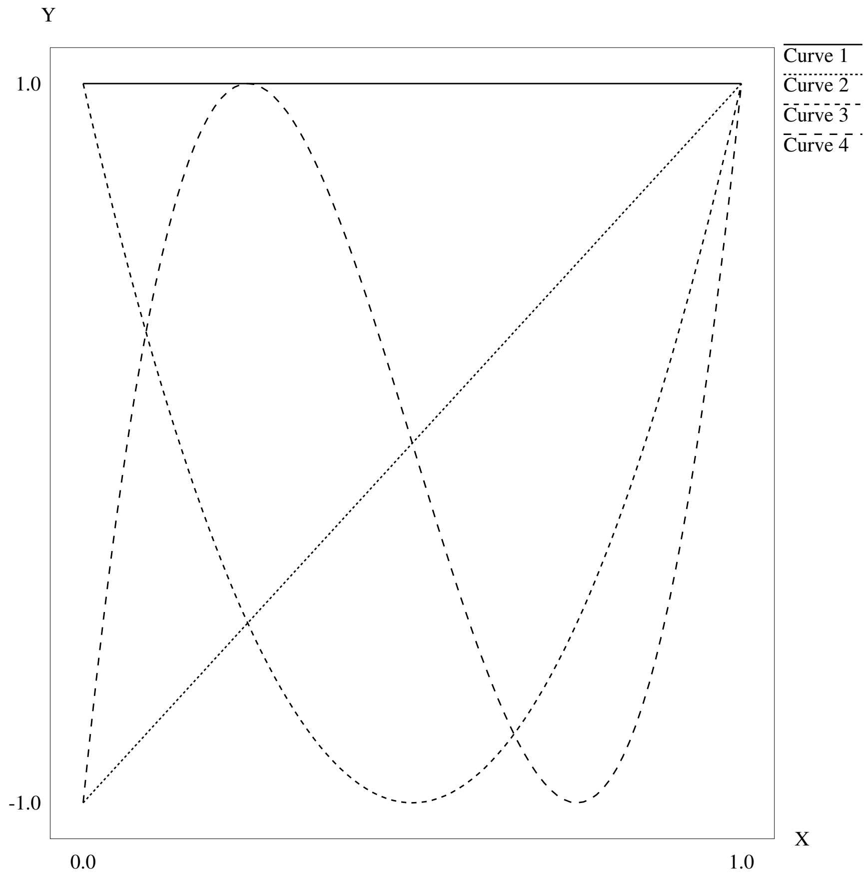

FIGURE 1a. $C_m^{(\alpha)}(s)/E_m^{(\alpha)}, m = 0, 1, 2, 3$

Problem 4.1. For each triple of nonnegative integers (m, α, β), determine

$$\underset{g \in \pi_{m-1}}{\text{minimum}} \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}[x^m - g(x)]^2 \, dx.$$

The solution to this problem gives rise to the classical *Jacobi* polynomials and, in the symmetric case, the *ultraspherical* polynomials. Using

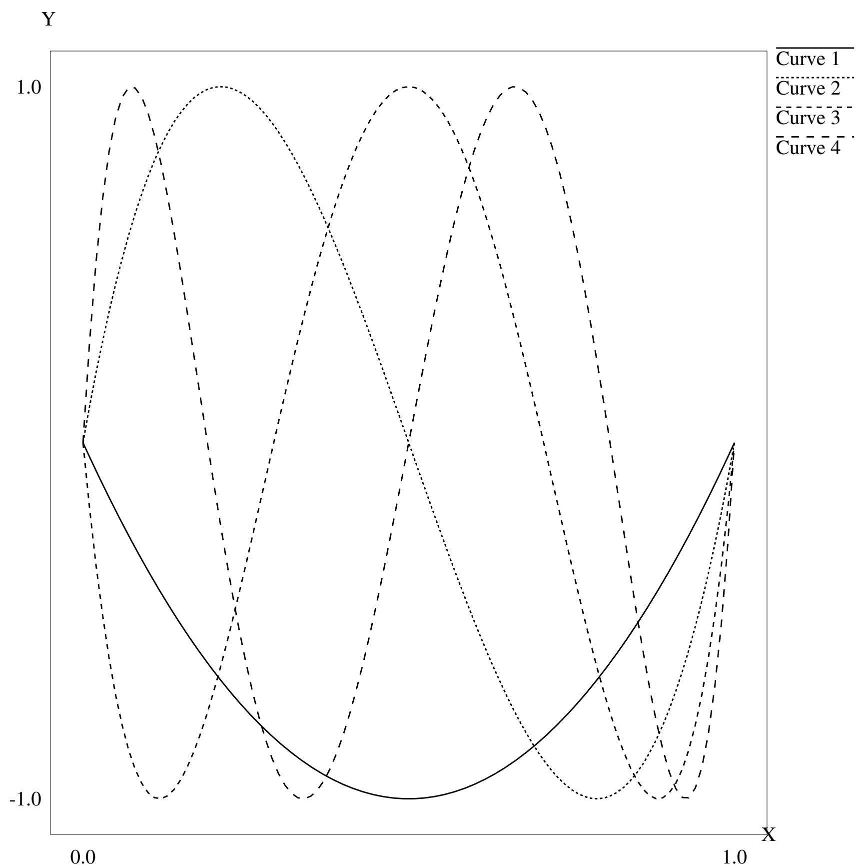

FIGURE 1b. $C_m^{(\alpha)}(s)/E_m^{(\alpha)}, m = 2, 3, 4, 5$

the notation of Szego [**9**, pp. 80–81], the latter polynomials are denoted $P_m^{(\alpha)}(x)$, with weight function $(1 - x^2)^{\alpha - .5}$. They are normalized by

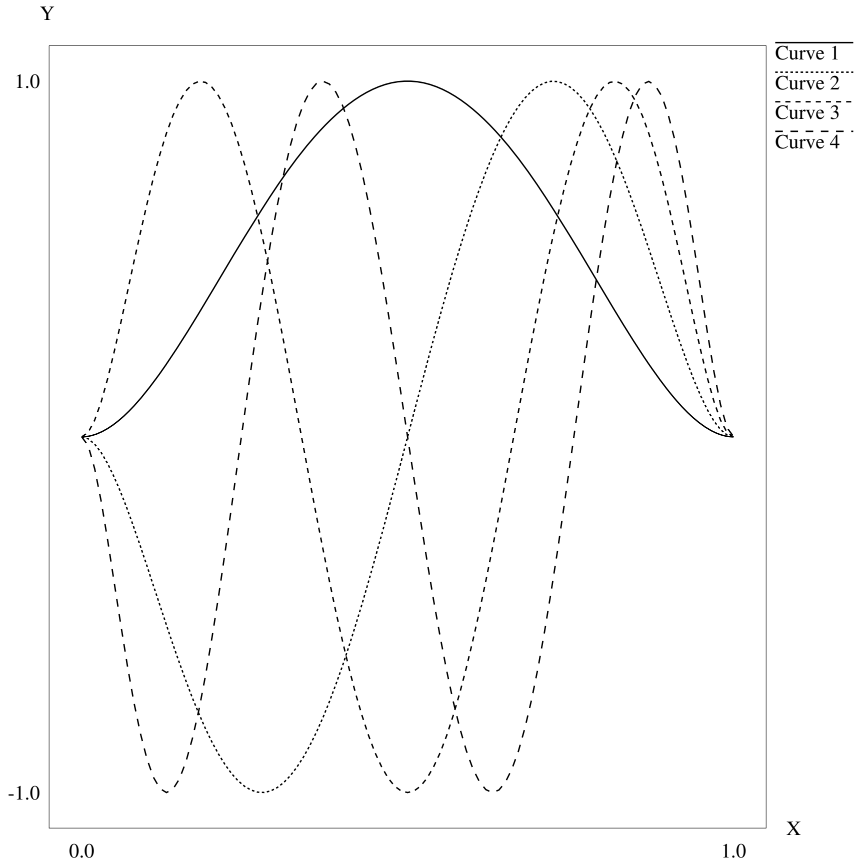

FIGURE 1c. $C_m^{(\alpha)}(s)/E_m^{(\alpha)}, m = 4, 5, 6, 7$

$P_m^{(\alpha)}(x) = \tau_m^{(\alpha)} x^m + \cdots \in \pi_m$ so that

$$\begin{aligned}\min_{g\in\pi_{m-1}} &\int_{-1}^{1}(1-x^2)^{\alpha-.5}[x^m - g(x)]^2\,dx \\ &= [\tau_m^{(\alpha)}]^{-2}\int_{-1}^{1}(1-x^2)^{\alpha-.5}\{P_m^{(\alpha)}(x)\}^2\,dx \\ &= \sigma_m^{(\alpha)}/[\tau_m^{(\alpha)}]^2,\end{aligned} \tag{4.1}$$

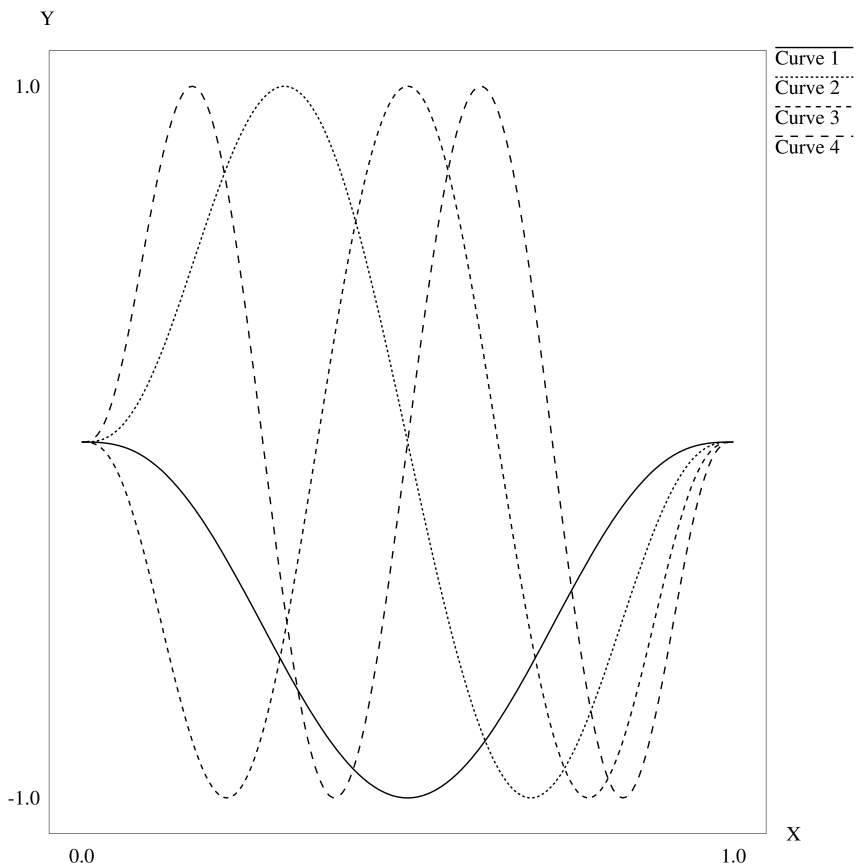

FIGURE 1d. $C_m^{(\alpha)}(s)/E_m^{(\alpha)}, m = 6, 7, 8, 9$

where

$$\sigma_m^{(\alpha)} = 2^{1-2\alpha}\pi[\Gamma(\alpha)]^{-2}\frac{\Gamma(m+2\alpha)}{(m+\alpha)\Gamma(m+1)}, \tag{4.2}$$
$$\Gamma_m^{(\alpha)} = 2^m\binom{m+\alpha-1}{m}.$$

A three-term recurrence relation is available for evaluating the ultraspherical polynomials. Alternatively, the coefficients can be generated and stored, offering modest computational savings. When $\alpha = .5$, the ultraspherical polynomials are the classical Legendre polynomials $P_m(x)$ whose recurrence relation is given by

$$\begin{aligned} P_0(x) &= 1, \qquad P_1(x) = x, \\ mP_m(x) &= (2m-1)xP_{m-1}(x) - (m-1)P_{m-2}(x), \qquad m \geq 2. \end{aligned} \tag{4.3}$$

Because of our specific continuity requirements, we are most interested in the values $\alpha = 2k + .5$, $k = 1, 2, 3$. Starting with the Legendre polynomials, and using the differential recurrence relation

$$2\alpha P_{m-1}^{(\alpha+1)}(x) = \frac{d}{dx} P_m^{(\alpha)}(x), \tag{4.4}$$

the other ultraspherical polynomials can easily be tabulated.

As with the constrained Chebyshev polynomials, it will be convenient to introduce notation for the monic constrained ultraspherical polynomials on the interval $[0,1]$. Paralleling the notation of Section 3, we have

Definition 4.1. For each pair of nonnegative integers (m, α), with $m \geq 2\alpha$, the *constrained ultraspherical polynomials* are given by

$$\begin{aligned} U_m^{(\alpha)}(s) &= \frac{s^\alpha (s-1)^\alpha P_{m-2\alpha}^{(2\alpha+.5)}(2s-1)}{2^{m-2\alpha}\tau_{m-2\alpha}^{(2\alpha+.5)}} \\ &= s^\alpha (s-1)^\alpha (s^{m-2\alpha} + \cdots), \end{aligned} \tag{4.5}$$

$$[\varepsilon_m^{(\alpha)}]^2 = \int_0^1 [\mathrm{U}_m^{(\alpha)}(s)]^2\, ds = \frac{\sigma_{m-2\alpha}^{(2\alpha+.5)}}{2^{2m+1}[\tau_{m-2\alpha}^{(2\alpha+.5)}]^2} \tag{4.6}$$

The values m and α have the same interpretation as in the symmetric constrained Chebyshev polynomials; the values of σ and τ are defined in (4.2) above.

5. Constrained uniform approximation of curves. Consider a parametric curve $\mathbf{f}$. In practice, a polynomial approximation scheme constructs an approximate and then audits its performance relative to a given tolerance. If the fit is unsatisfactory, then $\mathbf{f}$ is partitioned into two or more pieces. Each of these is then fit individually by a polynomial and is in turn audited. In order to insure that the individual polynomials link up in a visually smooth manner, tangent and acceleration directions are controlled. One way to gain this control is to force the polynomials to interpolate the endpoints and derivatives of $\mathbf{f}$'s pieces. The constraints on each polynomial lead to

Problem 5.1. For each pair of positive integers (m, α), with $m \geq 2\alpha$, and for each vector-valued function $\mathbf{f} \in C^{\alpha-1}[0,1]^3$, determine

(5.1)
$$\underset{\mathbf{p}\in\pi_m^3}{\text{minimum}} \{||\mathbf{f}-\mathbf{p}||_{L_\infty[0,1]} : \mathbf{p}^{(i)}(\delta) = \mathbf{f}^{(i)}(\delta), \quad i = 0, \ldots, \alpha-1, \delta = 0, 1\}.$$

It is tempting to approach this problem component-wise. However, the following simple example shows that L_∞ problems over the Euclidean norm on an interval cannot, in general, be handled this way.

Example 5.1. Let $\mathbf{f}(s) = (s, s^2) \in C[-1,1]^2$. The best *component-wise* constant approximation to $\mathbf{f}$ is $(0, .5)$, with maximum derivation $\sqrt{5}/2$. The best *uniform* constant approximation which attains the minimum of $\max_{-1\leq s\leq 1} ||(s, s^2) - (a, b)||_E$ on the interval $[-1, 1]$ is $(0, 1)$ with uniform error 1.

In the case when $\mathbf{f}$ is a polynomial of degree $m + 1$, this author [**6**] showed that the minimum value of the expression (5.1) *can* be determined component-wise. In particular, if

(5.2)
$$\mathbf{f}(s) = \mathbf{a}_{m+1}s^{m+1} + \cdots \in \pi_{m+1}^3, \qquad \mathbf{a}_{m+1} \neq \mathbf{0}, \text{ then}$$
$$\mathbf{p}(s) = \mathbf{f}(s) - \mathbf{a}_{m+1}C_{m+1(s)}^{(\alpha)}$$

is the unique polynomial which attains the minimum in (5.1). The reader will note that this definition is simply Chebyshev economization where the constrained Chebyshev polynomials replace the traditional Chebyshev polynomials. The corresponding error is

$$||\mathbf{f}-\mathbf{p}||_{L_\infty[0,1]} = ||\mathbf{a}_{m+1}||_E E_{m+1}^{(\alpha)}.$$

The number $||\mathbf{a}_{m+1}||_E E_{m+1}^{(\alpha)}$ is the maximum distance between like parametric values of $\mathbf{f}$ and $\mathbf{p}$, and provides an a priori bound on the Hausdorff distance.

A degree reduction routine can easily be based upon this economization, simply by iterating to the desired degree. The errors at each stage accumulate and will possibly force the breaking up of $\mathbf{f}$. In this event, the composite approximation will be geometrically smooth, depending on the value of α. For a discussion of parametric versus geometric continuity, we refer the interested reader to Bartels, et al. [**1**, p. 293].

6. Constrained least square approximation of curves. From the previous discussion, Problem 5.1 for nonpolynomial $\mathbf{f}$ cannot, in general, be treated component-wise. If the least square counterpart to this problem is considered, as suggested by Goult [**4**], then this objection is circumvented. In this section we show that the constrained ultraspherical polynomials $\{U_i^{(\alpha)}(s)\}_{i=2\alpha}^{\infty}$ are the natural sequence for an orthogonal expansion of $\mathbf{f}$ once the interpolatory conditions are satisfied. First we restate Problem 5.1 in terms of the L_2 norm.

Problem 6.1. For each pair of positive integers (m, α), with $m \geq 2\alpha$, and, for each vector-valued function $\mathbf{f} \in C^{\alpha-1}[0,1]^3$, determine

$$\underset{\mathbf{p}\in\pi_m^3}{\text{minimum}}\{||\mathbf{f}-\mathbf{p}||_{L_2[0,1]} : \mathbf{p}^{(i)}(\delta) = \mathbf{f}^{(i)}(\delta),\quad i = 0, \ldots, \alpha-1, \delta = 0, 1\}. \tag{6.1}$$

To determine the extremal polynomial for this problem, it is convenient to consider the Bernstein representation for $\mathbf{p}(s)$: $\mathbf{p}(s) = \sum_{i=0}^{m} \mathbf{v}_i B_{i,m}(s)$. According to (2.1), the first and last α of its coefficients are completely determined by the interpolation conditions. That is, 2α of the $\mathbf{v}_i$ are given by

$$\begin{aligned} [\mathbf{v}_0, \ldots, \mathbf{v}_{\alpha-1}] &= [\mathbf{f}^{(0)}(0), \ldots, \mathbf{f}^{(\alpha-1)}(0)]D^{-1}U^{-1} \\ [\mathbf{v}_m, \ldots, \mathbf{v}_{m-\alpha+1}] &= [\mathbf{f}^{(0)}(1), \ldots, \mathbf{f}^{(\alpha-1)}(1)]SD^{-1}U^{-1}, \end{aligned} \tag{6.2}$$

where D and U are given in (2.2), $S = [(-1)^{i+1}\delta_{ij}]$, leaving $m - 2\alpha + 1$ as yet undetermined coefficients. Using the coefficients from (6.2), we define an m^{th} degree *Hermite* interpolating polynomial $\mathbf{h}_m^{(\alpha)}(s)$ by

$$\begin{aligned}\mathbf{h}_m^{(\alpha)}(s) = &\sum_{i=0}^{\alpha-1}[\mathbf{v}_i B_{i,m}(s) + \mathbf{v}_{m-i}B_{m-i,m}(s)] \\ &+ \sum_{i=\alpha}^{m-\alpha}\mathbf{w}_i B_{i,m}(s),\end{aligned} \tag{6.3}$$

where the $\mathbf{w}_i$, $\alpha \le i \le m-\alpha$, are fixed but arbitrary. Then the polynomial $\mathbf{p}(s)$ must take the form

$$\mathbf{p}(s) = \mathbf{h}_m^{(\alpha)}(s) + w_\alpha(s)\mathbf{q}(s) \tag{6.4}$$

where $w_\alpha(s) = [s(1-s)]^\alpha$ and $\mathbf{q} \in \pi^3_{m-2\alpha}$. The expression for which a minimum is sought is

$$||w_\alpha[\mathbf{g}_m^{(\alpha)} - \mathbf{q}]||_{L_2[0,1]},$$

where

$$\mathbf{g}_m^{(\alpha)}(s) = \frac{[\mathbf{f}(s) - \mathbf{h}_m^{(\alpha)}(s)]}{w_\alpha(s)}, \qquad 0 < s < 1.$$

The collection $\{P_m^{(2\alpha+.5)}(2s-1)\}_{m=0}^\infty$ forms an orthogonal collection on the interval $[0,1]$ with respect to the weight function $w_\alpha(s)^2$. Thus, the minimum is obtained by taking $\mathbf{q}(s) = \sum_{i=0}^{m-2\alpha}\mathbf{a}_i P_i^{(2\alpha+.5)}(2s-1)$, with

$$\mathbf{a}_i = \frac{\int_0^1 w_\alpha(s)^2\mathbf{g}_m^{(\alpha)}(s)P_i^{(2\alpha+.5)}(2s-1)\,ds}{\int_0^1 w_\alpha(s)^2[P_i^{(2\alpha+.5)}(2s-1)]^2\,ds}, \qquad i = 0,\dots,m-2\alpha.$$

The integration of the vector-valued function $\mathbf{g}_m^{(\alpha)}(s)$ is understood to be component-wise. Alternatively,

$$\begin{aligned}\mathbf{p}(s) &= \mathbf{h}_m^{(\alpha)}(s) + w_\alpha(s)\mathbf{q}(s) \\ &= \sum_{i=0}^{\alpha-1}[\mathbf{v}_i B_{i,m}(s) + \mathbf{v}_{m-i}B_{m-i,m}(s)] \\ &\quad + \sum_{i=2\alpha}^{m}\mathbf{b}_i U_i^{(\alpha)}(s),\end{aligned} \tag{6.5}$$

where the $\mathbf{v}_i$ are given by (6.2) and

$$\mathbf{b}_i = \frac{\int_0^1 [\mathbf{f}(s) - \mathbf{h}_m^{(\alpha)}(s)] U_i^{(\alpha)}(s)\, ds}{[\varepsilon_i^{(\alpha)}]^2}, \qquad i = 2\alpha, \dots, m. \tag{6.6}$$

The determination of the coefficients $\mathbf{b}_i$ in (6.6) requires the definition of an interpolating polynomial $\mathbf{h}_m^{(\alpha)}(s)$ as an initial guess. In [**4**] the suggestion for selecting $\mathbf{h}_m^{(\alpha)}(s)$ seems to be that it be of degree $2\alpha - 1$. A better choice could be made by choosing $\mathbf{h}_m^{(\alpha)}(s)$ to be of degree m, taking the best *desired degree* polynomial approximate available and viewing the term $w_\alpha(s)\mathbf{q}(s)$ as a correction to $\mathbf{h}_m^{(\alpha)}(s)$.

There are at least two drawbacks to this least square solution: An a priori bound on the uniform error is not readily available, and the coefficients of the correction term must be obtained using numerical integration. These objections become more serious when the methods are extended to surfaces.

7. Implications for parametric surfaces. For a polynomial surface $\mathbf{P}(s,t) = \sum_{i=0}^m \sum_{j=0}^n \mathbf{A}_{ij} s^i t^j$, parametrized on the unit square $[0,1]^2$, constrained Chebyshev economization can be accomplished in one parametric direction or both parametric directions simultaneously [**6**]. Denoting the polynomial coefficients of s^i and t^j by

$$\mathbf{P}(s,t) = \sum_{i=0}^m \mathbf{P}_i(t) s^i = \sum_{j=0}^n \mathbf{P}^j(s) t^j, \tag{7.1}$$

economizations may be defined by either

$$\mathbf{Q}(\alpha, \cdot : s,t) = \mathbf{P}(s,t) - \mathbf{P}_m(t) C_m^{(\alpha)}(s) \tag{7.2}$$

or

$$\begin{aligned} \mathbf{R}(\alpha,\beta : s,t) = \mathbf{P}(s,t) &+ \mathbf{A}_{m,n} C_m^{(\alpha)}(s) C_n^{(\beta)}(t) \\ &- \mathbf{P}_m(t) C_m^{(\alpha)}(s) - \mathbf{P}^n(s) C_n^{(\beta)}(t), \end{aligned} \tag{7.3}$$

where α and β control the desired amount of continuity in each of the parametric directions s and t, respectively.

To develop polynomial approximations for a nonpolynomial surface $\mathbf{F}(s,t)$, we parallel the development used for nonpolynomial curves in Section 6. That is, we introduce a Hermite interpolating polynomial $\mathbf{H}(s,t) = \mathbf{H}_{m,n}^{(\alpha,\beta)}(s,t)$ which agrees with $\mathbf{F}(s,t)$ at the *corners* of the unit square (depending on the values of α and β); then we consider the weighted optimization problem of minimizing

$$||[\mathbf{F}-\mathbf{H}]-\mathbf{P}||_{L_2[0,1]^2} = ||w_\alpha w_\beta[\mathbf{G}-\mathbf{Q}]||_{L_2[0,1]^2}$$

where

$$\mathbf{G}(s,t) = \frac{\mathbf{F}(s,t)-\mathbf{H}(s,t)}{w_\alpha(s)w_\beta(t)}, \qquad (s,t)\in(0,1)^2$$

and

$$\begin{aligned}\mathbf{P}(s,t) &= w_\alpha(s)w_\beta(t)\mathbf{Q}(s,t)\\ &= \sum_{i=2\alpha}^{m}\sum_{j=2\beta}^{n}\mathbf{B}_{ij}U_i^{(\alpha)}(s)U_j^{(\beta)}(t).\end{aligned}$$

The coefficients $\mathbf{B}_{ij}$ in this case require iterated numerical integration, while the determination of $\mathbf{H}(s,t)$ is relatively straightforward.

8. Acknowledgment. The author wishes to thank L.P. Johnston for kindly preparing the plots in Figure 1.

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