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ITERATION OF MÖBIUS TRANSFORMS AND CONTINUED FRACTIONS

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Dedicated to Professor W.J. Thron
on the occasion of his 70th birthday.

1. Introduction. In this paper we study sequences of complex points obtained by successive application of Möbius transforms to a starting point. Lately a large literature dealing with iteration of simple maps, notably quadratic polynomials, has appeared ([2] and references therein). It might seem to be of interest to do similar studies for Möbius transforms. However, the dynamics of Möbius transforms is much simpler (cf. Section 2), largely due to the fact that composition of two Möbius transforms is again a Möbius transform, while the composition of a quadratic polynomial with itself is a fourth degree polynomial. This latter fact is crucial for the period doubling scenario of the quadratic maps.

A good reason for studying the effect of repeated applications of Möbius transforms is that it relates to continued fractions. To be a bit more precise, let s_k be Möbius transforms of the special type $s_k(z) = a_k/(1+z)$; then the *convergents* of the continued fraction

$$\mathop{K}_{k=1}^{\infty}(a_k/1) = \cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\ddots}}}$$

are $S_n(0) = s_1 \circ s_2 \circ \cdots \circ s_n(0)$.

It has become fashionable to consider so-called *modified* continued fractions where the truncation 0 is replaced by a modifying factor z_n, i.e., to study $S_n(z_n)$ ([**24,5,6,10**] and many other papers). This has been applied mostly for convergence acceleration but has been studied

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for its intrinsic interest for the theory of continued fractions [**23** and **17**], the idea being that while for series it makes perfect sense to set the tails equal to zero in approximating the value of a convergent series, the tails of a continued fraction, $K_{k=N}^{\infty}(a_k/1)$ are hardly ever close to zero.

We will look at the convergence and divergence of (generalized) iterates

$$\tilde{S}_n(z) = s_n \circ s_{n-1} \circ \cdots \circ s_1(z),$$

where $z = z_0$ is the initial value and $z_{n+1} = \tilde{S}_n(z_0)$ satisfies $z_{n+1} = s_n(z_n)$, $n = 0, 1, 2, \ldots$. We will compare and contrast their behavior with continued fractions in simple cases where the a_k are in some way close to being constant. To us this way of approaching continued fractions via "discrete dynamics" ideas has been a revelation. We feel that this paper is a very good port of entry to the theory of continued fractions for the beginner. We also feel that this approach to the subject makes one pose problems slightly differently. For instance, it becomes very natural to describe and classify divergence behavior. While most of the results of this paper are well known to workers in continued fractions, we think some proofs are new and hope that this way of presenting the material will be interesting to old hands as well.

In the simple cases considered here it transpires that it is possible to study the behavior of the modified continued fractions $S_n(z)$ by looking at the generalized iterates $\tilde{S}_n(z)$. This is interesting because the $\tilde{S}_n$ are easy to compute recursively, while with the S_n you have to start all over again with every value of n. Now those who know continued fractions know that there is a recursive scheme for calculating successive approximants, called *forward recursion*, which gives three term recurrences for both numerators and denominators (and which is responsible for the connection with orthogonal polynomials). However, as a computational method this is not known to be stable, and, theoretically, it is much less transparent than the composition of Möbius transforms (*backward recursion*) that we just described. It should be noted that the same argument of computational ease can be made for the study of *tail sequences* t_n defined by $t_n = s_{n+1}(t_{n+1})$, so

$$t_n = s_n^{-1} \circ s_{n-1}^{-1} \circ \cdots \circ s_1^{-1}(t_0).$$

FIGURE 1. Iterates of $y_{n+1} = a/(1 + y_n)$, $-0.5 < a < 0$, $-2 < y < 2$.

These were originally studied in connection with modifications for convergence acceleration (the right tails) and analytic continuation (the wrong tails) of continued fractions with coefficients depending on a complex variable. In the simple cases studied in this paper, there is not a great deal of difference between the properties of s_k and s_k^{-1}, so it is just as easy to study generalized iterates of the ones as the others. However, tail sequences have proven their worth in more complicated settings [**13,14**].

We have restricted our attention to continued fractions $K(a_n/1)$. This is mostly to keep the dimension of some pictures down. Most of what we (and others) say generalizes easily to more general Möbius transforms, and, indeed, some of our results are formulated more generally. Gill [**7,8**] has even done some work on the analog of limit periodic continued fractions ($s_k \to s$) when s_k are not even Möbius.

FIGURE 2. $s = -0.4/(1+z)$.

All our Möbius transforms are assumed to be nonsingular. Our numbers are numbers in the extended complex number system S^2 and we admit convergence to infinity.

2. The constant case. Let's begin with the simplest case $s_k(z) = s(z) = a/(1+z)$, $a \neq 0$, for all k. In this case there is no difference between S_n and $\tilde{S}_n$. Following the tradition in discrete dynamics, consider Figure 1. In the x direction we have different values of a and in the y direction there are, for every a, 100 iterates $y_{n+1} = a/(1+y_n)$

FIGURE 3. $s = (-0.4 + 0.02i)/(1 + z)$.

with random real initial value y_0. We see convergence to a fixed point for $a \geq -1/4$ and divergence for other a. Since this is supposed to be about complex numbers, let's see what happens in the complex plane. Figure 2 shows 100 iterates with various complex initial points for an a on the real axis to the left of $-1/4$ and Figure 3 shows what happens if we move a slightly off the axis. In Figure 2 the different circles correspond to different initial values. In Figure 3, regardless of the initial value, the iterates spiral toward a fixed point. The spiral arms are not the paths followed. Rather, the arms are visited in sequence while the iterates spiral inwards. The arm structure is an artifact of a stability phenomenon; iterates with different initial values tend to get close to one another.

It turns out there is nothing more to be seen. We state a bit of terminology: If $a \neq -1/4$, $s(z)$ has two fixed points (points such that $s(z) = z$), one attractive (meaning that $|s'(z)| < 1$ in that point) and one repulsive ($|s'(z)| > 1$ in that point), or both indifferent ($|s'(z)| = 1$). If $a = -1/4$, $s(z)$ has one fixed point $z = -1/2$ which is indifferent.

Theorem 1. *If $a < -1/4$, then, for every initial value in S^2, different from the fixed points of s, the iterates of $s(z) = a/(1+z)$ lie on a circle such that the two fixed points of $a/(1+z)$ are conjugate with respect to that circle. (If the initial value is real, this means that the circle is the real axis.) If $a = -1/4$, the iterates converge to the single fixed point of s. For all other a the iterates converge to the attractive fixed point of $a/(1+z)$ for all initial values in S^2 except the repulsive fixed point.*

We will formulate a slightly more general

Lemma 2. *Let s be any nonsingular Möbius transform. Then either the iterates converge for every initial value in S^2 or lie on circles as described above.*

Proof. Suppose that the fixed points of $s(z)$ are distinct: z_1 and z_2. The transformation $\varphi(z) = w = (z - z_1)/(z - z_2)$ moves the fixed points to 0 and ∞, and s will transform to $T = \varphi \circ S \circ \varphi^{-1}$ with fixed points 0 and ∞. Such a Möbius transform must be $T(w) = \alpha w$. Now, either $|\alpha| = 1$ in which case the iterates in the w-plane run around in circles, which transform to the circles described in the z-plane, or $|\alpha| \neq 1$ in which case we have convergence to 0 or ∞ in the w-plane. The remaining cases are analogous or obvious. □

Proof of Theorem 1. If $s(z) = a/(1+z)$ the fixed points are $z_{1,2} = -1/2 \pm \sqrt{1/4 + a}$, which are distinct unless $a = -1/4$. Thus, the lemma tells us that, for $a = -1/4$, the iterates converge to the single fixed point and, for all other a, we either run around in the circles described or converge to the attractive fixed point. It only remains to determine for which a $|\alpha| = 1$. For such an a the bisecting normal to the line segment between z_1 and z_2 must be invariant under s and s^{-1}. This normal can be parameterized by

$$z(t) = \frac{z_1 + z_2}{2} + it(z_1 - z_2) = -1/2 + 2it\sqrt{1/4 + a}.$$

FIGURE 4. $\tilde{S}_n$ for $s_a = -0.67/(1+z)$, $s_b = -0.4/(1+z)$.

Now, since ∞ is on the normal, $s^{-1}(\infty) = -1$ must be on the normal too. Thus, there is a real t such that $-1 = -1/2+2it\sqrt{1/4+a}$ which is possible only if $\sqrt{1/4+a}$ is purely imaginary. Conversely, if $a < -1/4$, the real axis is the bisecting normal above and the real axis is also invariant under s. □

Remark 1. This is all well known [**4**, Chapter 1], [**19**, Section 3.2], but we include the proofs to make this paper easier to understand. We also want to put special emphasis on the divergence behaviors.

Remark 2. As we have seen in the proof, the iterates will be uniformly distributed on circles in the w-plane in the divergent cases (unless $\arg(\alpha)$ is a rational multiple of π). When we transform back to the z-plane, this distribution changes in the usual way.

FIGURE 5. S_n for $s_a = -0.67/(1+z)$, $s_b = -0.4/(1+z)$.

Remark 3. If we want to treat tail sequences, s^{-1} will transform to w/α in the w-plane so the only difference is that when $|\alpha| \neq 1$ the fixed points interchange their roles. If $S_n(z) = \tilde{S}_n(z) \to z_2$, the tails will all converge to z_1 unless $t_0 = z_2$ (the right tails).

3. The (two-)periodic case. The periodic case will mean that $s_k = a_k/(1+z)$ and a_k is a periodic sequence. We'll begin by looking at a sequence with period two and call $a_1 = a$, $a_2 = b$, $a \neq b$, $s_1 = s_a$, $s_2 = s_b$. Then

$$S_n = \begin{cases} s_a \circ s_b \cdots \circ s_b & \text{if } n \text{ is even,} \\ s_a \circ s_b \cdots \circ s_a & \text{if } n \text{ is odd} \end{cases}$$

and

FIGURE 6. $\tilde{S}_n$ for $s_a = (-0.67 + 0.01i)/(1+z)$, $S_b = -0.4/(1+z)$.

$$\tilde{S}_n = \begin{cases} s_b \circ \cdots \circ s_b \circ s_a & \text{if } n \text{ is even,} \\ s_a \circ s_b \circ \cdots \circ s_a & \text{if } n \text{ is odd.} \end{cases}$$

It is thus clear that the key to the behavior of S_n and $\tilde{S}_n$ lies in $T = s_a \circ s_b$ and $\tilde{T} = s_b \circ s_a$. S_n will behave as iterates of T with different initial values for odd and even n while $\tilde{S}_n$ will be iterates of $\tilde{T}$ with an extra s_a acting if n is odd. Thus, even if the iterates of $\tilde{T}$ converge to a fixed point, $\tilde{S}_n$ will not converge. Theorem 3 will tell us, however, that T and $\tilde{T}$ have the same behavior (both converge or both diverge) under iteration, so it is possible to decide the convergence of S_n by looking at $\tilde{S}_n$. In Figure 4 we see $\tilde{S}_n$ jumping between two circles when $\tilde{T}$ is of the diverging kind and in Figure 5 we see the corresponding S_n also jumping between circles but now with the same conjugate points. In both Figures 4 and 5 there are two different initial

FIGURE 7. S_n for $s_a = (-0.67 + 0.01i)/(1+z)$, $s_b = -0.4/(1+z)$.

values. Figures 6 and 7 show the same situation when $\tilde{T}$ and T are both of the converging kind. In Figure 6 we get an asymptotic limit cycle of length 2 and, in Figure 7, convergence.

Theorem 3. *Let a_k be periodic with period p and let $s_k(z) = a_k/(1+z)$. Call $s_1 \circ s_2 \circ \cdots \circ s_p = T$ and $s_p \circ \cdots \circ s_1 = \tilde{T}$. Further, suppose T has two distinct fixed points. Then the iterates of T and $\tilde{T}$ either both converge or both diverge as in Lemma* 2.

Theorem 3 is essentially a reformulation of the essential fact about so-called *dual continued fractions* [**19**, Section 3.3]. However, it rests on an interesting lemma that establishes a general relation between convergence of the iterates $\tilde{S}_n$ and the tails S_n^{-1}.

Lemma 4. *Let* $s_k = a_k/(1+z)$, $a_k \neq 0$. *Then* $S_n^{-1}(-z) = -1 - \tilde{S}_n(z-1)$.

Proof of the Lemma [**19**, proof of Theorem 3.4].

$$\begin{aligned}
s_k^{-1}(z) &= -1 + a_k/z \\
s_k^{-1}(-z) &= -1 - s_k(z-1) \\
s_2^{-1} \circ s_1^{-1}(-z) &= -1 - s_2(-s_1^{-1}(-z) - 1) \\
&= -1 - s_2(1 + s_1(z-1) - 1) \\
&= -1 - s_2 \circ s_1(z-1), \text{ etc.} \quad \square
\end{aligned}$$

Now the theorem follows by observing that tail sequences are connected with iterates of T^{-1}, and, as in Lemma 2, T can be transformed into αw and T^{-1} into w/α so iterates of T and T^{-1} converge or circle simultaneously. The lemma then gives that the same is true for T^{-1} and $\tilde{T}$.

Remark. In the two-periodic case it is easy to see that if z_1 and z_2 are the fixed points of T, then $s_b(z_k)$ are the fixed points of $\tilde{T}$, so the limit cycle of Figure 6 is actually the attractive fixed points of T and $\tilde{T}$.

4. The limit periodic case. A lesson to be learned from the preceding section is that $\tilde{S}_n$ will not converge unless $\lim s_n = s$. In this section we will look at $s_n(z) = a_n/(1+z)$, $a_n \to a$. A full analysis of this will involve delicate interplay between the rate of convergence of a_n and the initial values. We will do the case when the limit a corresponds to an α with $|\alpha| \neq 1$ which turns out to be easy to formulate and restrict ourselves to examples indicating the things that can happen in other cases.

Theorem 5. *Let* $a_n \to a$ *where* $a \notin (-\infty, -1/4]$, a_n, $a \neq 0$. *Then* $\tilde{S}_n(z) = s_n \circ \cdots \circ s_1(z)$ *converges to the attractive fixed point of* $s(z) = a/(1+z)$ *for all* $z \in S^2$ *except possibly for one initial value when it converges to the repulsive fixed point of* s.

FIGURE 8. $a_n = -0.4 + 0.1i + 1/(n+50)$.

We will need the following lemma [**22**, Theorem 1].

Lemma 6. *Let T_n be a sequence of nonsingular Möbius transforms such that $T_n(z)$ converges for all $z \in S^2$. Suppose that there exist two points $z_1, z_2, z_1 \neq z_2$, and a w_0 such that $T_n(z_1) \to w_0$, $T_n(z_2) \to w_0$. Then $T_n(z) \to w_0$ for all z except possibly for one z_0.*

Proof. By transforming T_n to $\varphi \circ T_n \circ \psi^{-1}$ where φ is a nonsingular Möbius transform mapping w_0 to 0 and ψ one mapping z_1 to 0 and z_2 to ∞, we realize that we may assume that $z_1 = 0$, $z_2 = \infty$, and $w_0 = 0$.

Let $T_n(z) = (a_n z + b_n)/(c_n z + d_n)$, $a_n d_n \neq b_n c_n$, $T_n(0) \to 0$, $T_n(\infty) \to 0$ and $T_n(z)$ converges for all z. By the assumptions $a_n/c_n \to 0$ and $b_n/d_n \to 0$ and one would expect $T_n(z) \to 0$ except possibly when $c_n z + d_n \to 0$. Verifying this is a tedious but not hard

FIGURE 9. $a_n = -0.4 + i/(n^2 + 100)$. The term 100 is to make sure that the perturbation is fairly small for small n. If it is not included, we get rapid convergence of the first few iterates toward the real axis and the picture does not look too interesting.

examination of different cases. □

Remark. This lemma is related to the fact that continued fractions often converge to the same thing no matter how they are modified, the "general convergence" of Jacobsen [**14,17,3**].

Proof of Theorem 5. Let s have fixed points z_1 and z_2 and transform the problem to the w-plane by introducing

$$w = \varphi(z) = \frac{z - z_1}{z - z_2}, \quad t_n = \varphi \circ s_n \circ \varphi^{-1}, \quad \text{and} \quad t = \varphi \circ s \circ \varphi^{-1}.$$

Instead of $\tilde{S}_n$, we study $\tilde{T}_n$ defined by $\tilde{T}_n = t_n \circ \cdots \circ t_1$. Then $t(w) = \alpha w$, α constant. We assume that z_1 is the attractive fixed point of s and so

FIGURE 10. $a_n = -0.4 + i/(n+100)$.

$|\alpha| < 1$. The idea of the proof is to use that t attracts every $w \neq \infty$ towards 0 and that t_n does the same for large n since $a_n \to a$. To take care of the technical details we insert two lemmas.

Lemma 7. *Given* $\varepsilon > 0$ *and* $M > 0$ *there exists an* $n(\varepsilon, M) = n(\varepsilon, M, \alpha)$ *such that*

$$|t_n(w)| \leq \varepsilon + (|\alpha| + \varepsilon)|w| \quad \textit{if } |w| \leq M \textit{ and } n \geq n(\varepsilon, M).$$

Proof of Lemma 7. Since $a_n \to a$, direct calculation shows that $t_n = \varphi \circ s_n \circ \varphi^{-1}$ has the form

$$t_n(w) = \frac{A_n w + B_n}{C_n w + D_n}$$

FIGURE 11. $a_n = -0.4 + 1/(n+10)$.

where $A_n \to \alpha$, $B_n \to 0$, $C_n \to 0$ and $D_n \to 1$. Now we fix ε and M and consider the expression

$$t_n(w) - \alpha w = \frac{B_n + (A_n - \alpha D_n)w - \alpha C_n w^2}{C_n w + D_n}.$$

For every $\delta > 0$ there exists an $n(\delta)$ such that, for $n \geq n(\delta)$, $|A_n - \alpha| < \delta$, $|B_n| < \delta$, $|C_n| < \delta$ and $|D_n - 1| < \delta$. For these n and $|w| \leq M$ the expression gives

$$|t_n(w) - \alpha w| \leq \frac{1}{1 - \delta - M\delta}[\delta + (\delta + \alpha\delta)|w| + \alpha\delta M|w|]$$

which is less than $\varepsilon + \varepsilon|w|$ if δ is small. This proves Lemma 7. ◻

Lemma 8. *Fix ε, M and $n(\varepsilon, M)$ according to Lemma 7 so that*

$$|\alpha| + \varepsilon < 1 \quad \textit{and} \quad \frac{\varepsilon}{1 - (|\alpha| + \varepsilon)} + (|\alpha| + \varepsilon)M \leq M.$$

FIGURE 12. $a_n = -0.4 + 1/(n+10) + 0.001i$.

Then, for all $m \geq 0$ *and all* $n \geq n(\varepsilon, M)$,

$$|t_{n+m} \circ t_{n+m-1} \circ \cdots \circ t_n(w)| \leq \frac{\varepsilon}{1-(|\alpha|+\varepsilon)} + (|\alpha|+\varepsilon)^{m+1}|w|,$$

if $|w| \leq M$.

Proof of Lemma 8. We make the induction assumption that, for $n \geq n(\varepsilon, M)$ and a fixed m, $|t_{n+m} \circ \cdots \circ t_n(w)| \leq \varepsilon \sum_{j=0}^{m} (|\alpha|+\varepsilon)^j + (|\alpha|+\varepsilon)^{m+1}|w|$, for $|w| \leq M$. By Lemma 7 this inequality holds for $m = 0$. Since, for $|w| \leq M$, the right-hand side of this inequality is less than $\varepsilon/(1-(|\alpha|+\varepsilon)) + (|\alpha|+\varepsilon)M$, which by our assumption is less than M, we can use Lemma 7 and our induction assumption to show that our induction assumption in fact holds for all $m \geq 0$. This proves Lemma 8. □

We now return to the proof of Theorem 5. In the w-plane we study $\tilde{T}_n = t_n \circ \cdots \circ t_1$. We observe that $\tilde{S}_n = \varphi^{-1} \circ \tilde{T}_n \circ \varphi$. Now, consider an initial value $w = \varphi(z)$. If $\tilde{T}_n(w) \to \infty$, then $\tilde{S}_n(z) \to z_2$. If $\tilde{T}_n(w)$ does not converge to infinity, then there exists an $M > 0$ such that $|\tilde{T}_k(w)| \leq M$ for certain arbitrarily large $k : s$. We now use Lemma 8 with this M and with w in the lemma equal to $\tilde{T}_k(w)$ for one of these $k : s$. Since ε in the lemma may be chosen arbitrarily small, we conclude that $\tilde{T}_n(w) \to 0$ which means that $\tilde{S}_n(z) \to z_1$.

Hence, unless $\tilde{T}_n(w)$ converges to infinity, it converges to zero. It is also clear that there exists more than one initial value w that ends up in zero. For instance, for a given $M > 0$, choose $w = \tilde{T}_k^{-1}(\xi)$ where ξ is any number satisfying $|\xi| < M$ and k is large. As above, we conclude that $\tilde{T}_n(w) \to 0$. Thus, in the w-plane convergence to infinity is the exception rather than the rule and, by Lemma 6, we conclude that $\tilde{S}_n(z) \to z_1$ except possibly for one z where $\tilde{S}_n(z) \to z_2$. This completes the proof of Theorem 5. □

For the proof of the next theorem [**21**, Theorem 1], which deals with continued fractions, we use a variant of Lemma 8 which is proved in the same way as Lemma 8.

Lemma 8′. *Fix ε, M and $n(\varepsilon, M)$ as in Lemma 8. Then, for all $m \geq 0$ and all $n \geq n(\varepsilon, M) + m$,*

$$|t_{n-m} \circ t_{n-m+1} \circ \cdots \circ t_n(w)| \leq \frac{\varepsilon}{1 - (|\alpha| + \varepsilon)} + (|\alpha| + \varepsilon)^{m+1}|w|,$$

if $|w| \leq M$.

Theorem 9. *With the assumptions in Theorem 5, $S_n(z) = s_1 \circ \cdots \circ s_n(z)$ converges to the same value for all initial $z \in S^2 \backslash \{z_2\}$ where z_2 is the repulsive fixed point of s.*

Proof. As in the proof of Theorem 5, we introduce φ by $w = \varphi(z) = (z - z_1)/(z - z_2)$, where z_1 is the attractive fixed point of s, and $t_n = \varphi \circ s_n \circ \varphi^{-1}$. We fix $z_0 \neq z_2$, and, by that, $w_0 = \varphi(z_0) \neq \infty$. Put $M = |w_0|$ and choose $\delta > 0$. By Lemma 8′ there exist $m(\delta)$ and $n(\delta)$ (depending also on α and w_0) such that

$$|t_{n-m} \circ t_{n-m+1} \circ \cdots \circ t_n(w)| < \delta \quad \text{for } m = m(\delta),\ n \geq n(\delta) + m.$$

If V is any fixed small neighborhood of z_1, we can choose δ so small that $\varphi^{-1}(\{w : |w| < \delta\}) \subset V$. Since $s_{n-m}\circ\cdots\circ s_n = \varphi^{-1}\circ(t_{n-m}\circ\cdots\circ t_n)\circ\varphi$, the previous estimate means that

$$s_{n-m} \circ \cdots \circ s_n(z_0) \subset V \quad \text{for } m = m(\delta),\ n \geq n(\delta) + m.$$

This seems to indicate that $S_n(z_0)$ may converge and, indeed, a well-known type of argument from the theory of continued fractions can be used to show convergence. One reference for this argument is the proof of Theorem 12.3d and 12.3f in [**9**]. A more specific reference is formula (2.6) and (2.7) in [**12**] which we use to finish our proof. In fact, in the notation of [**12**], in (2.6) we choose a μ, $0 < \mu < D$, and a k so that (2.6) holds, not for all n, but for $n \geq k$. Let V above be chosen as V in (2.7) in [**12**]. By (2.7), $s_k \circ \cdots \circ s_n(z_0)$ converges, as $n \to \infty$, to a $u \in V$, which means that $S_n(z_0) \to s_1 \circ \cdots \circ s_{k-1}(u)$. Here u and k are independent of z_0 and the proof of Theorem 9 is complete. □

Remark 1. It is instructive to compare the proofs of Theorems 5 and 9 to gain an understanding of the difference between convergence of iterates and continued fractions.

Remark 2. Lemma 4 combined with Theorem 5 shows that tail sequences also converge with at most one exceptional initial value. Compare with [**1**, Lemma 2.1]. There are also explicit formulas relating tail sequences to the values of S_n for general Möbius s_k [**13**, Theorem 2.3]. This has been used to study more general forms of limit periodicity where a_k is asymptotically k-periodic [**12,15**]. Another approach to proving convergence of limit periodic continued fractions (Theorem 9) is to use the following, which is proved much as Lemma 6 but also using the explicit formula for the inverse of a Möbius transform.

Lemma 10. *Let $S_n(z)$ be a sequence of nonsingular Möbius transforms such that $S_n^{-1}(z) \to z_2$ for all $z \neq z_0$ and $S_n^{-1}(z_0) \to z_1$, $z_1 \neq z_2$. Then $S_n(z) \to z_0$ except for $z = z_2$.*

Theorem 5 is illustrated in Figure 8. The situation when the limit $a \in (-\infty, -1/4)$ is trickier. Figure 9 shows divergence of $\tilde{S}_n$ and Figure 10 shows convergence. The critical difference seems to be that,

FIGURE 13. $a_n = -0.25 - 1/8n$. The picture shows 2000 iterates with one initial value. The iterates spiral slowly, the loops of the spiral getting smaller and drifting towards the real axis.

in Figure 9, $\Sigma|a_k - a| < \infty$ and, in Figure 10, $\Sigma|a_k - a| = \infty$. Note, however, that even with $\Sigma|a_k - a| = \infty$ we can have divergence (Figure 11). In Figure 11 this is to be expected since the limiting s is approached through a sequence s_k, all of which show the divergent behavior. In the case of Figure 10 the perturbations are allowed to accumulate and spoil the divergence behavior. Figure 12 shows better what is happening in Figure 11 by perturbing the circles into spirals to display the movement on the circles to better effect.

Conditions on convergence of sums of perturbations appear in [**3**]. Results for limit periodic continued fractions in analogous situations are proved in [**21**, Theorem 2] and [**5**, Corollary 2].

FIGURE 14. $a_n = -0.25 + 1/8n$. Starting from a couple of different initial values we move rapidly towards the real axis and then creep slowly leftwards towards $z = -0.5$.

5. $\mathbf{a_n \to -1/4}$. The study of limit periodic continued fractions with the exceptional limit $-1/4$ (where the two fixed points coincide) has attracted considerable interest [**5,16,11,20**, p. 165 ff]. It is rather difficult to study because in the limit iterations of $(-1/4)/(1+z)$ do converge but very slowly. (Transforming the single fixed point to ∞, as in the proof of Lemma 2, gives $w_{n+1} = w_n + 2$.) Thus, experimental work becomes difficult. As an example we offer Figure 13 where we do not know if we have convergence or not. (The transformed iterates seem to diverge, however.) It is to be expected that convergence is easier to obtain if $-1/4$ is approached through values off the cut $(-\infty, -1/4)$; cf. Figure 14.

We can prove a few things but only for real iterates where one can use monotonicity to deduce convergence. First we transform: Let

$$z_{n+1} = a_n/(1+z_n)$$
$$w = \frac{1}{z+\frac{1}{2}}.$$

Then, $w_{n+1} = (w_n+2)/(1+2\varepsilon_n w_n)$ if $a_n = -1/4 + \varepsilon_n$, $\varepsilon_n \to 0$.

If we can prove this sequence to be increasing, it will converge (and must converge to infinity). This is easily shown for real ε_n if $|\varepsilon_n| < \beta n^{-\alpha}$, $\alpha > 2$. Then $|w_n| < n^{\alpha/2}/\beta^{1/2}$ for large enough n which suffices to prove w_n increasing. If $\varepsilon_n = \beta n^{-\alpha} > 0$ the same is true (estimate on w_n and monotonicity) for all $\alpha > 0$. The $\alpha = 2$ limit for $\varepsilon_n < 0$ is probably related to the rather precise results for continued fractions due to Jacobsen [**11**].

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