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A SEQUENCE OF BEST PARABOLA THEOREMS FOR CONTINUED FRACTIONS

LISA JACOBSEN AND DAVID R. MASSON

In honor of W.J. Thron on his 70th birthday

1. Introduction. The history of the parabola theorems dates back to 1940 when Scott and Wall published the first, simple version [**5**]. Our paper is based on the following beautiful generalization by Thron from 1958 [**6**]:

Theorem A. *Let* $-\pi/2 < \alpha < \pi/2$ *be a fixed number,* $\{g_n\}_{n=0}^{\infty}$ *be a fixed sequence with* $0 < g_0 \leq 1$*,* $0 < g_n < 1$ *for* $n \geq 1$ *and*

$$\sum_{k=0}^{\infty} \prod_{n=1}^{k} \left(\frac{1}{g_n} - 1\right) = \infty, \tag{1.1}$$

and let

$$P_{\alpha,n} = \{z \in \mathbf{C} : |z| - \operatorname{Re}(ze^{-i2\alpha}) \leq 2g_{n-1}(1-g_n)\cos^2\alpha\} \tag{1.2}$$

for $n = 1, 2, 3, \ldots$ *. Finally, let* $K(a_n/1)$ *be a continued fraction with*

$$0 \neq a_n \in P_{\alpha,n} \text{ for all } n \in \mathbf{N}. \tag{1.3}$$

Then $K(a_n/1)$ *converges if and only if*

$$\sum_{n=1}^{\infty} \prod_{k=1}^{n} |a_k|^{(-1)^{n+k+1}} = \infty. \tag{1.4}$$

Comment 1. The conclusion in Theorem A is really just one of several proved in [**6**]. For instance, if (1.3) holds, then

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(i) all the approximants f_n of $K(a_n/1)$ are finite and contained in the half plane V_0 given by

$$(1.5)\quad V_n = \{z \in \mathbf{C}; \operatorname{Re}(ze^{-i\alpha}) \geq -g_n \cos\alpha\} \quad \text{for } n \in \mathbf{N}_0 = \mathbf{N} \cup \{0\},$$

(ii) the even and the odd approximants of $K(a_n/1)$ both converge to finite limits,

(iii) its approximants f_n satisfy

$$(1.6)\qquad |f_n - f_{n+m}| \leq \frac{|a_1|}{1 - g_1 \cos\alpha} \Big/ \prod_{j=2}^{n} \left(1 + r_{j-1} \frac{|c_j|}{|a_j|} \cos^2\alpha\right)$$

for all $m, n \in \mathbf{N}$, where

$$(1.7)\quad c_j = -g_{j-1}(1-g_j), \quad r_j = Q_j \Big/ \sum_{k=0}^{j-1} Q_k, \quad Q_n = \prod_{k=1}^{n} \left(\frac{1}{g_k} - 1\right).$$

Comment 2. If $\{g_n\}$ is bounded away from 0 and $a_n \in P_{\alpha,n}$ for all n, but $a_N = 0$, then $K(a_n/1)$ converges to $f_{N-1} = f_N = f_{N+1} = \cdots$. (This follows by a simple argument based on the fact that $\liminf\{|w+1| : w \in V_n, n \in \mathbf{N}\} > 0$.) Clearly (1.4) holds in this case, so that the condition $a_n \neq 0$ in (1.3) may be removed.

The simplicity and generality of such parabola theorems really meant a breakthrough in the convergence theory of continued fractions. Simple choices for $\{g_n\}$ are all $g_n = 1/2$ or $g_n = 1/2 + 1/(2(2n+1))$ for all n [**6**]. This corresponds to $c_n = -1/4$ or $c_n = -1/4 - 1/(4(4n^2-1))$. Further, (1.4) holds if $\{a_n\}$ is bounded. However, Theorem A implies convergence of larger classes of continued fractions than those indicated by these examples. In this paper we shall describe some of these classes and show that they are largest possible in a certain well-defined sense.

2. The choice of $\{\mathbf{g_n}\}$. The boundary $\partial P_{\alpha,n}$ of $P_{\alpha,n}$ is a parabola with focus at the origin, axis along the ray $\arg(z) = 2\alpha$, vertex at $e^{i2\alpha} c_n \cos^2\alpha$ and intersecting the negative real axis at c_n. $P_{\alpha,n}$ and c_n are as given by (1.2) and (1.7). Hence, if $|\tilde{c}_n| < |c_n|$, then the corresponding parabolic regions satisfy $\tilde{P}_{\alpha,n} \subset P_{\alpha,n}$. For our purpose we therefore want to choose $\{g_n\}$ such that "all $|c_n|$ are large."

We shall restrict ourselves to the case where $|c_n| \downarrow 1/4$ monotonically. The monotonicity represents the restriction. That c_n approaches the value—1/4 is a consequence of our wish to get $|c_n|$ as large as possible. The key to our investigations is the following result from [**2**, Theorem 1]:

Theorem B. *Let $T \in \mathbf{N}_0$ and $p_j > 0$ for $j = 0, 1, \dots, T-1$ be fixed numbers such that*

$$c_n = -\frac{1}{4} - \frac{1}{16}\sum_{k=1}^{T}\prod_{j=0}^{k-1}\Big(\log_{(j)}(n+p_j)\Big)^{-2} \neq \infty \quad \textit{for all } n \in \mathbf{N}. \tag{2.1}$$

Then $K(c_n/1)$ converges and $K(e_n/1)$ diverges, if

$$e_n = c_n - \varepsilon \prod_{j=0}^{T-1}\Big(\log_{(j)}(n+p_j)\Big)^{-2} + O(n^{-3}) \neq 0 \tag{2.2}$$

for all $n \in \mathbf{N}$ and $\varepsilon > 0$. (Notation: $\log_{(0)} k = k, \log_{(1)} k = \log k, \log_{(2)} k = \log(\log k)$, etc., where $\log k$ denotes the natural logarithm for $k > 0$.)

We recognize that $T = 0$ gives $c_n = -1/4$, i.e., all $g_n = 1/2$, and that $T = 1$ and $p_0 = 0$ give $c_n = -1/4 - 1/(16n^2)$, i.e., essentially as Thron suggested in [**6**]. For these values of T and p_0 we also have $e_n = -1/4 - \varepsilon + O(n^{-3})$ or $e_n = -1/4 - (1+\mu)/(16n^2) + O(n^{-3})$, which also previously were known to give divergent continued fractions $K(e_n/1)$ if $\varepsilon > 0$ and $\mu > 0$ [**3**, p. 47–48], [**1**]. Increasing T means increasing $|c_n|$ for n sufficiently large, as long as $\log_{(j)}(n+p_j) > 0$ for $j = 1, 2, \dots, T-1$.

By use of this result we get the following sequence ($T = 0, 1, 2, \dots$) of parabola theorems.

Theorem 1. *Let $-\pi/2 < \alpha < \pi/2$ and $T \in \mathbf{N}_0$ be fixed numbers, $\{c_n\}$ be given by* (2.1), *and*

$$P_{\alpha,n} = \{z \in \mathbf{C}; |z| - \operatorname{Re}(ze^{-i2\alpha}) \leq 2|c_n|\cos^2\alpha\} \quad \textit{for } n = 1, 2, 3, \dots. \tag{2.3}$$

Then $K(a_n/1)$ *with all* $0 \neq a_n \in P_{\alpha,n}$ *converges in* $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ *if and only if* (1.4) *holds.*

(As always, $K(a_n/1)$, $a_n \neq 0$, converges if some tail, $K(a_{m+n}/1)$, converges. Hence, it really suffices that $a_{n+m} \in P_{\alpha,n}$ from some n on, for a fixed $m \in \mathbf{N}$, if $a_n \neq 0$ for $n = 1, 2, \dots, m$.)

Proof. In [**2**, Theorem 3.1 and its proof] it was found that a minimal solution $\{Y_n^{(1)}\}$ of the recurrence relation $Y_n = Y_{n-1} + c_n Y_{n-2}$ is given by

$$Y_n^{(1)} = X_n^{(T-1),-}(1 + O((\log_{(T)} n)^{-\varepsilon})) \tag{2.4}$$

where $X_n^{(T),-}$ has the large n behavior

$$X_n^{(T),-} = C(1/2)^n \left(\prod_{k=0}^{T} (\log_{(k)} n)^{1/2} \right) (1 + O(1/\log_{(T)} n)). \tag{2.5}$$

Hence, $g_n = Y_n^{(1)}/Y_{n-1}^{(1)}$ satisfies $c_n = -g_{n-1}(1 - g_n)$. Further, either by straightforward computation, or by use of the main result in [**8**], we find that this choice of $\{g_n\}$ also satisfies

$$\sum_{k=0}^{\infty} \prod_{n=1}^{k} (1/g_n - 1) = \infty. \tag{2.6}$$

Since $g_n \to 1/2$, we thus have that the sequence $\{g_{N+n}\}$ satisfies (1.1) for N sufficiently large. Hence, $K_{n=N+1}^{\infty}(a_n/1)$ converges by Theorem A, and thus $K(a_n/1)$ converges. □

These parabola theorems are best in the sense that $\{c_n\}$ cannot be replaced by $\{e_n\}$ which is slightly larger in modulus, at least from some n on. From (2.4)–(2.5) it follows that the $\{g_n\}$ corresponding to (2.1) has the form

$$g_n = \frac{1}{2} + \frac{1}{4} \sum_{k=1}^{T} \prod_{j=0}^{k-1} 1/\log_{(j)}(n + p_j) + O(n^{-2}). \tag{2.7}$$

To ascertain that $\{g_n\}$ satisfies (1.1) if $T \geq 1$, such that also the properties of Comment 1 are valid, one can adjust p_j to obtain $g_0 = 1$. For instance, one can choose

(2.8)
$$g_n = \frac{1}{2} + \frac{1}{4(n+T/2)} \sum_{k=1}^{T} \prod_{j=1}^{k-1} 1/\log_{(j)}(n+p_j) \quad \text{for } n = 0, 1, 2, \dots,$$

where

$$p_1 = e, \quad p_2 = e^e, \quad p_3 = e^{e^e}, \dots . \tag{2.9}$$

This gives

(2.10)
$$c_n = -g_{n-1}(1-g_n) = \\ -\frac{1}{4} - \frac{1}{16(n+T/2)(n+T/2-1)} \sum_{k=1}^{T} \prod_{j=1}^{k-1} \Big(\log_{(j)}(n+p_j)\Big)^{-2} + O(n^{-3}),$$

and the truncation error estimate (1.6) becomes

$$|f_n - f_{n+m}| \leq M_n = O(1/\log_{(T)} n) \quad \text{for all } n \text{ and } m \text{ in } \mathbf{N}. \tag{2.11}$$

3. The condition (1.4). From the Seidel-Stern-Stolz criterion (see for instance [**3**, Theorem 4.19, p. 79 and Theorem 4.28, p. 87]) it follows that (1.4) holds if and only if $K(|a_n|/1)$ converges. Further, by [**4**, Satz 2.11, p. 47] it follows that $K(|a_n|/1)$ converges if $\sum |a_n|^{-1/2} = \infty$. Clearly, then, it suffices that $\{a_n\}$ has a subsequence $\{a_{n(k)}\}$ such that $\sum |a_{n(k)}|^{-1/2} = \infty$. We thus have

Theorem 2. *The sequence* $\{|a_n|\}$ *satisfies* (1.4) *if there exist a subsequence* $\{|a_{n(k)}|\}$*, an* $M > 0$ *and a* $T \in \mathbf{N}$ *such that*

$$|a_{n(k)}| \leq d_k = \left(M \prod_{j=0}^{T-1} \log_{(j)} k\right)^2 \quad \textit{from some } k \textit{ on.} \tag{3.1}$$

This is a best result in the sense that $K(\tilde{d}_k/1)$ *diverges if*

$$\tilde{d}_k = d_k (\log_{(T-1)} k)^{\varepsilon} + O(k) \quad \textit{from some } k \textit{ on, where } \varepsilon > 0. \tag{3.2}$$

Proof. The first part follows trivially from the considerations above. To see that $K(\tilde{d}_k/1)$ diverges, we let $N \in \mathbf{N}$ be so large that $\log_{(T-1)}(2N-1) > 0$. Then

$$\sum_{k=N+1}^{\infty} \frac{\tilde{d}_{2N+1}\tilde{d}_{2N+3}\cdots\tilde{d}_{2k-1}}{\tilde{d}_{2N+2}\tilde{d}_{2N+4}\cdots\tilde{d}_{2k}}$$
$$= \sum_{k=N+1}^{\infty} \prod_{n=N+1}^{k} \left(\frac{2n-1}{2n}\frac{\log(2n-1)}{\log(2n)}\cdots\frac{\log_{(T-2)}(2n-1)}{\log_{(T-2)}(2n)}\right)^2 \cdot \left(\frac{\log_{(T-1)}(2n-1)}{\log_{(T-1)}(2n)}\right)^{2+\varepsilon}(1+O(n^{-1-\varepsilon}))$$
$$= \sum_{k=N+1}^{\infty}\left(\prod_{n=N+1}^{k}\frac{n-1}{n}\frac{\log(n-1)}{\log n}\cdots\frac{\log_{(T-2)}(n-1)}{\log_{(T-2)} n} \cdot \left(\frac{\log_{(T-1)}(n-1)}{\log_{(T-1)} n}\right)^{1+\varepsilon/2}(1+O(n^{-1-\varepsilon})+O(n^{-2}))\right)$$

which converges to a finite number. Similarly,

$$\sum_{k=N+2}^{\infty}\frac{\tilde{d}_{2N}\tilde{d}_{2N+2}\cdots\tilde{d}_{2k}}{\tilde{d}_{2N+1}\tilde{d}_{2N+3}\cdots\tilde{d}_{2k+1}} < \infty. \quad \square$$

4. Limit periodic continued fractions of parabolic type. A continued fraction $K(a_n/1)$ is said to be limit p-periodic of parabolic type for a $p \in \mathbf{N}$ if

$$\lim_{n\to\infty} a_{pn+r} = c_r \in \mathbf{C} - \{0\}, \quad \text{for } r = 1, 2, \dots, p, \tag{4.1}$$

and the linear fractional transformation

$$S_p(w) = \frac{c_1}{1} + \frac{c_2}{1} + \cdots + \frac{c_p}{1+w} \tag{4.2}$$

is of parabolic type, i.e., S_p has coinciding fixed points. If $c_r = 0$ or ∞, then special requirements are needed. For $p = 1$ this means that

$a_n \to -1/4$. From the parabola theorems one obtains convergence criteria for such continued fractions. In particular, we get

Corollary 3. *$K(a_n/1)$ converges in $\hat{\mathbf{C}}$ if*

$$|a_{n+m} + 1/4| \le \frac{1}{16} \sum_{k=0}^{T-1} \prod_{j=0}^{k} \left(\log_{(j)} n\right)^{-2} \tag{4.3}$$

from some n on, for some m and T in N_0. Its rate of convergence is $O(1/\log_{(T)} n)$.

Proof. The disk $|z + 1/4| \le R_n$, where R_n is given by the bound in (4.3), is contained in $P_{0,n}$ given by (2.4). □

This extends a result from [**7**]. The basis of the proof was that $-1/4 \in \partial P_{0,\infty}$. Indeed, $-1/4 \in \partial P_{\alpha,\infty}$ for every permissible α. One of the beautiful properties of $P_{\alpha,\infty}$ is that its boundary consists of pairs (c_1, c_2) such that $S_2(w)$ is of parabolic type. Better still, to every such pair (c_1, c_2) we can find a corresponding parabola:

Theorem 4. *Let $S_2(w)$ given by (4.2) be of parabolic type. Then*

$$|c_r| - \operatorname{Re}(c_r e^{-i2\alpha}) = \frac{1}{2}\cos^2\alpha \quad \textit{for } r = 1, 2, \tag{4.4}$$

where

$$\alpha = \operatorname{Arctan} \frac{\operatorname{Im}(\sqrt{c_1}) \pm 1/2}{\operatorname{Re}(\sqrt{c_1})} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{4.5}$$

(Here $\operatorname{Re}(\sqrt{c_1}) > 0$, and the minus sign in (4.5) is to be chosen if and only if $\operatorname{Im}(\sqrt{c_1}) > \operatorname{Im}(\sqrt{c_2})$. Statement (4.4) is equivalent to $c_r \in \partial P_{\alpha,\infty}$.)

Proof. It is well known and easy to ascertain that, writing $a_n = v_n^2$, we have $a_n \in P_{\alpha,n}$ if and only if v_n is contained in the strip

$$Q_{\alpha,n} = \{z \in \mathbf{C} : |\operatorname{Im}(ze^{-i\alpha})| \le \sqrt{g_{n-1}(1-g_n)}\cos\alpha\}. \tag{4.6}$$

$Q_{\alpha,\infty}$ is a strip symmetric about its axis $z = te^{i\alpha}$, $t \in \mathbf{R}$, with constant vertical width 1, whose boundary contains the points $\pm i/2$. Since $S_2(w)$ is parabolic, we have that $c_1 c_2 \neq 0$ and $c_1 = (i \pm \sqrt{c_2})^2$. Without loss of generality, we thus write $c_r = u_r^2$, where $u_1 = u_2 \pm i$ with $\mathrm{Re}\,(u_1) > 0$. Hence, u_1 and u_2 are elements of $\partial Q_{\alpha,\infty}$ if

$$\tan\alpha = \left(\mathrm{Im}\,(u_1) \pm \frac{1}{2}\right) / \mathrm{Re}\,(u_1). \quad \square$$

Therefore, we also get convergence results for limit 2-periodic continued fractions of the parabolic type from the parabola theorems, in particular

Theorem 5. *Let c_1, c_2 and α be as in Theorem 4, and let T and m be fixed nonnegative integers. Then $K(a_n/1)$ with $a_{2n+r} = c_r + \delta_{2n+r}$ for all n and $r = 1, 2$ converges if*

$$(4.7) \quad |\delta_{n+2m}| \leq \frac{\sqrt{|c_r|}\cos\alpha}{8} \sum_{k=0}^{T-1} \prod_{j=0}^{k} \left(\log_{(j)} n\right)^{-2} \quad \textit{from some } n \textit{ on},$$

where $r = 1$ if n is odd, and $r = 2$ if n is even. Its rate of convergence is $O(1/\log_{(T)} n)$.

Proof. Let $a_n = v_n^2$ and $c_r = u_r^2$ for $r = 1, 2$, such that $\mathrm{Re}\,(u_1) > 0$ and $u_2 = u_1 \pm i$, and let g_n be given by (2.6). Then v_{n+2m} is contained in the strip $Q_{\alpha,n}$ if $|v_{n+2m} - u_r| \leq (\sqrt{g_{n-1}(1-g_n)} - 1/2)\cos\alpha$, since $u_r \in \partial Q_{\alpha,\infty}$. We have

$$\begin{aligned} |\delta_{n+2m}| &= |a_{n+2m} - c_r| = |v_{n+2m} + u_r||v_{n+2m} - u_r| \\ &= 2|v_{n+2m} - u_r||u_r|(1 + O(n^{-2})) \\ &\leq \frac{|u_r|\cos\alpha}{8} \sum_{k=0}^{T-1} \prod_{j=0}^{k} (\log_{(j)} n)^{-2} \end{aligned}$$

from some n on, and thus

$$\begin{aligned} |v_{n+2m} - u_r| &\leq \frac{\cos\alpha}{16} \sum_{k=0}^{T-1} \prod_{j=0}^{k} (\log_{(j)} n)^{-2} + O(n^{-4}) \\ &= (\sqrt{g_{n-1}(1-g_n)} - 1/2)\cos\alpha + O(n^{-3}). \end{aligned}$$

(See (2.10).) Hence, possibly by increasing m, we have $v_{n+2m} \in Q_{\alpha,n}$ from some n on, and the result follows from (2.11).

For period lengths $p > 2$, we can no longer use the parabola theorem directly, but by considering contractions of $K(a_n/1)$ one can obtain similar results.

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Department of Mathematics, University of Trondheim, N-7034 Trondheim, Norway

Department of Mathematics, University of Toronto, Toronto, M5S 1A1, Canada