ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 1, Winter 1991

ORTHOGONAL MOMENTS

ERIK HENDRIKSEN AND HERMAN VAN ROSSUM

ABSTRACT. Let $(c_n(t))_{n=0}^\infty$ be a sequence of orthogonal polynomials (OPS), and let $(y_n(t;z))_{n=0}^\infty$ be the OPS with respect to $(c_n(t))_{n=0}^\infty$ as a moment sequence. For $(c_n(t))_{n=0}^\infty$ we take successively the Laguerre, Hermite, ultraspherical, Charlier, and Meixner polynomial sequences and determine the newly generated OPSs. As an application, results on the Turanians of the special systems mentioned above, are found in a way completely different from the method used by Karlin and Szegö [**4**].

Introduction. Laguerre developed a method for obtaining continued fractions of a certain type for a class of complex functions [**6**, pp. 322–343]. Here we consider one of Laguerre's examples in a slightly different form, i.e.,

$$f(z) = \frac{1}{2t}\left(\left(\frac{z+1}{z-1}\right)^t - 1\right) \sim \sum_{n=0}^\infty c_n(t) z^{-n-1}$$

($t \in \mathbf{R}\backslash\mathbf{Z}; z \in \mathbf{C}\backslash[-1,1]$, branch with $f(\infty)=0$). Then

$$c_n = c_n(t) = \frac{1}{2t}\cdot\frac{(-1)^{n+1}(c)_n}{(n+1)!}\,{}_2F_1(-n-1,t;t-n;-1), \quad n=0,1,\dots.$$

The n-th approximants $U_n(z)/V_n(z)$ of the J-fraction for f match the power series in z^{-1}:

$$f(z) - \frac{U_n(z)}{V_n(z)} = O(z^{-2n-1}).$$

Hence, the polynomials V_n are orthogonal with respect to the moment sequence $(c_n(t))_{n=0}^\infty$.

AMS *Mathematics Subject Classification.* Primary 33A65.
Key words and phrases. Orthogonal polynomials, moments, Turanians.
Received by the editors on October 7, 1988, and in revised form on January 6, 1989.

From

$$(z^2-1)f'(z)+2tf(z)+1=0 \tag{1.1}$$

we obtain

$$(n+1)c_n(t)-2tc_{n-1}(t)-(n-1)c_{n-2}(t)=0, \quad n=2,3,\dots,$$

while

$$c_0(t)=1 \quad \text{and} \quad c_1(t)=t.$$

By Favard's theorem, the polynomials $c_n(t)$ in t form an orthogonal polynomial system (OPS). Laguerre's method applied to (1.1) yields the differential equation (d.e.)

$$(z^2-1)V_n''(z)+2(z-t)V_n'-n(n+1)V_n(z)=0,$$

from which we see that

$$V_n(z)=P_n^{(-t,t)}(z),$$

where $P_n^{(-t,t)}(z)$ denotes the Jacobi polynomial of degree n. Hence, $(P_n^{(-t,t)}(z))_{n=0}^{\infty}$ is an OPS whose moments $c_n(t)$ also form an OPS.

Taking this as our cue, we want to develop a theory of OPSs with orthogonal moments. In the present paper we restrict ourselves to moment sequences of Laguerre, Hermite, ultraspherical, Charlier, and Meixner polynomials.

In these cases an important result obtained by H.L. Krall [**5**], recorded here in Proposition I, immediately leads from the recurrence relations satisfied by the moments of a specific moment sequence to linear second order differential equations satisfied by the orthogonal polynomials belonging to the OPS generated by these moments. As corollaries we get results on Hankel determinants with orthogonal polynomial elements, called "Turanians" by Karlin and Szegö (see [**4**]). The Hankel determinant $H_n^{(0)}=\det(c_{i+j}(t))_{i,j=0}^{n-1}$ for a moment sequence $(c_n(t))_{n=0}^{\infty}$ is sometimes denoted by $T(c_0(t),c_1(t),\dots,c_{n-1}(t))$. In the following, $H_n^{(0)}$ is also referred to as the Hankel determinant (of order n) belonging to the J-fraction

$$\frac{\alpha_0}{z+\beta_1}+\frac{-\alpha_1}{z+\beta_2}+\cdots+\frac{-\alpha_{n-1}}{z+\beta_n}+\cdots,$$

with $\sum_{n=0}^{\infty} c_n(t)z^{-n-1}$ as its associated power series. We will normalize our moment sequences $(c_n(t))_{n=0}^{\infty}$ such that $c_0(t) = 1$. We notice that

$$H_n^{(0)} = \alpha_1^{n-1}\alpha_2^{n-2}\cdots\alpha_{n-2}^2\alpha_{n-1}.$$

Proposition 1. *Let* $(y_n(z))_{n=0}^{\infty}$ *be an* OPS *with* $(c_n(t))_{n=0}^{\infty}$ *as its moment sequence. Then the following are equivalent.*

(a) $y_n(z)$ *satisfies the d.e.*

$$(l_{22}z^2 + l_{21}z + l_{20})y_n'' + (l_{11}z + l_{10})y_n' = \lambda_n y_n(z) \tag{1.2}$$

with $\lambda_n = l_{11}n + l_{22}n(n-1)$.

(b) *The moments* c_n *satisfy the recurrence relations*

$$(l_{11} + l_{22}(n-1))c_n + (l_{10} + l_{21}(n-1))c_{n-1} + l_{20}(n-1)c_{n-2} = 0 \tag{1.3}$$

and $H_n^{(0)} \neq 0$, $n = 1, 2, \dots$.

The l_{ij} *may depend on* t *but they do not depend on* n.

2. Laguerre moments. For the Laguerre polynomials $L_n^{(\alpha)}$, we adopt Szegö's definition

$$L_n^{(\alpha)}(t) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-t)^k}{k!}.$$

Here we consider normalized Laguerre polynomials

$$c_n(t) = \frac{L_n^{(\alpha)}(t)}{L_n^{(\alpha)}(0)} \quad \text{where} \quad L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}, \quad n = 0, 1, \dots .$$

The recurrence relations are

$$(n+\alpha)c_n + [t - (\alpha + 2n - 1)]c_{n-1} + (n-1)c_{n-2} = 0, \quad n = 2, 3, \dots ,$$
$$c_0 = 1, \quad c_1 = 1 - \frac{t}{\alpha+1}.$$

Comparison to (1.3) yields $l_{10} = t - \alpha - 1$, $l_{11} = \alpha + 1$, $l_{20} = l_{22} = 1$, $l_{21} = -2$.

Let $y_n = y_n(z; t; \alpha)$ denote the polynomial of degree n belonging to the OPS with (suitably normalized) Laguerre moments. Then Proposition I yields the d.e.

$$(2.1) \qquad (z-1)^2 y_n'' + [(\alpha+1)(z-1)+t]y_n' = n(n+\alpha)y_n.$$

The differentiation is with respect to z. For simplicity, we assume that the status of t now is that of a parameter allowed to assume positive real values and that $\alpha > -1$ throughout.

Theorem 2.1. *The members of the* OPS *generated by the normalized Laguerre polynomial moments c_n are (fixed) first degree transformations of the Bessel polynomials. The Turanian $T(c_0, c_1, \dots, c_{n-1})$ of the Laguerre moments is equal to the $(n \times n)$-Hankel determinant of the sequence $((\alpha+1)_n)^{-1}$ multiplied by $t^{n(n-1)}$.*

Proof. The Bessel polynomial of degree n,

$$R_n(\alpha; z) = (-1)^n ((\alpha+n)_n)^{-1} {}_2F_0(-n, \alpha+n; z)$$

is a polynomial solution of the d.e.

$$(2.2) \qquad z^2 y''(z) - [1-(\alpha+1)z]y'(z) - n(n+\alpha)y(z) = 0$$

and also the denominator of the n-th approximant of the J-fraction

$$(2.3) \qquad \frac{\alpha_0}{z+\beta_1} + \frac{-\alpha_1}{z+\beta_2} + \cdots + \frac{-\alpha_{n-1}}{z+\beta_n} + \cdots,$$

where $\alpha_0 = c_0 = 1$ and

(2.4)
$$\alpha_1 = \frac{-1}{(\alpha+1)(\alpha+2)}, \quad \alpha_n = \frac{-n(\alpha+n-1)}{(\alpha+2n-2)(\alpha+2n-1)^2(\alpha+2n)}, \quad n = 2, 3, \dots .$$

The successive transformations $z \mapsto -z/t$ and $z \mapsto z-1$ transform the d.e. in (2.2) into the d.e. in (2.1) so the OPS generated by the Laguerre moments is, apart from a first-degree transformation, the Bessel OPS.

Only the transformation $z \mapsto z/t$ affects the partial numerators α_n, $n = 1, 2, \dots$. The J-fraction in (2.3) becomes

$$\frac{1}{z + b_1 t} + \frac{-\alpha_1 t^2}{z + b_2 t} + \cdots + \frac{-\alpha_{n-1} t^2}{z + b_n t} + \cdots . \tag{2.5}$$

The Hankel determinant $H_n^{(0)}$ belonging to the J-fraction in (2.3) is given by

$$H_n^{(0)} = \alpha_1^{n-1} \alpha_2^{n-2} \cdots \alpha_{n-2}^2 \alpha_{n-1}.$$

Since $T(c_0, c_1, \dots, c_{n-1})$ is the $(n \times n)$-Hankel determinant belonging to the J-fraction in (2.5) we find

$$T(c_0, c_1, \dots, c_{n-1}) = t^{n(n-1)} \alpha_1^{n-1} \alpha_2^{n-2} \cdots \alpha_{n-2}^2 \alpha_{n-1}$$

with α_j as given in (2.4). $\square$

Remark 2.1. ${}_1F_1(1; \alpha; z)$ is the generating function of the moments of the generalized Bessel polynomials. It has been shown by De Bruin [**1**] that, only if $\alpha = 1$ (i.e., the case of $\exp(z)$), the moment sequence is totally positive. Hence, the sequence of normalized Laguerre polynomials $(c_n)_{n=0}^{\infty}$, $t \neq 0$, is totally positive if and only if $\alpha = 1$.

Another result is Karlin's [**3**], i.e.,

$$\left(\frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + n + 1)} L_n^{(\alpha)}(-t) \right)_{n=0}^{\infty}, \quad t > 0,$$

is a totally positive sequence (is a one-sided Pólya frequency sequence).

3. Hermite moments. The Hermite polynomials $H_n(t)$ satisfy

$$H_n(t) = 2t H_{n-1}(t) - 2(n-1) H_{n-2}(t), \quad n = 2, 3, \dots ,$$
$$H_0(t) = 1, \quad H_1(t) = 2t, \quad t \in \mathbf{R}.$$

Applying Proposition I we find, for the polynomials $y_n = y_n(z; t)$ belonging to the OPS with moment sequence $(H_n(t))_{n=0}^{\infty}$, the d.e.

$$2y_n'' + (z - 2t) y_n' = n y_n. \tag{3.1}$$

The d.e. for the Hermite polynomial $H_n(z)$ is

$$y'' - 2zy' = -2ny. \tag{3.2}$$

The special J-fraction in $2s$,

$$\frac{1}{2s} + \frac{-2}{2s} + \frac{-4}{2s} + \cdots + \frac{-2(n-1)}{2s} + \cdots \tag{3.3}$$

has denominators $H_n(s)$. The Hankel determinant $H_n^{(0)}$ belonging to this fraction is equal to

$$2^{\frac{1}{2}n(n-1)} 1!2! \cdots (n-1)!$$

Comparing (3.1) and (3.2) suggests the substitution $2s = i(z-2t)$ in (3.3). After an equivalence transformation we obtain the J-fraction (written in normalized form)

$$\frac{1}{z+-2t} + \frac{2}{z+-2t} + \cdots + \frac{2(n-1)}{z+-2t} + \cdots .$$

The n-th denominator is $y_n(-2is+2t) = (-i)^n H_n(s)$. Hence, we finally obtain

Theorem 3.1. *The members of the* OPS *generated by the Hermite polynomial moments are (fixed) first-degree transformations of the Hermite polynomials. Moreover,*

$$T(H_0(t), H_1(t), \dots, H_{n-1}(t)) = (-2)^{\frac{1}{2}n(n-1)} 1!2! \cdots (n-1)!$$

Remark 3.1. From the sign pattern of the sequence of Turanians, namely $+--++--++--\cdots$, it follows that the Hermite moment sequence $(H_n(t))_{n=0}^{\infty}$ is totally positive for all values of t.

4. Ultraspherical moments. The ultraspherical polynomials $P_n^{(a)}(t)$, $a > -1/2, t \in \mathbf{R}$ satisfy the recurrence relations

$$nP_n^{(a)}(t) = 2(n+a-1)tP_{n-1}^{(a)}(t) - (n+2a-2)P_{n-2}^{(a)}(t), \quad n = 2, 3, \dots ,$$
$$P_0^{(a)}(t) = 1, \quad P_1^{(a)}(t) = 2at.$$

We introduce the normalized ultraspherical polynomials $c_n = c_n^{(a)}(t)$ defined by

$$c_n^{(a)}(t) = \frac{P_n^{(a)}(t)}{P_n^{(a)}(1)},$$

where

$$P_n^{(a)}(1) = \binom{n+2a-1}{n} = \frac{(2a)_n}{n!}.$$

The recurrence relations are

$$(2a+n-1)c_n - 2(n+a-1)tc_{n-1} + (n-1)c_{n-2} = 0, \quad n = 2, 3, \dots,$$
$$c_0 = 1, \quad c_1 = t.$$

In this case Proposition I applies. Let $(y_n)_{n=0}^{\infty} = (y_n(z;t;a))_{n=0}^{\infty}$ denote the OPS generated by the moment sequence $(c_n)_{n=0}^{\infty}$. Then the y_n satisfy the d.e.

$$(4.1) \qquad (z^2 - 2tz + 1)y_n'' + 2a(z-t)y_n' = n(n+2a-1)y_n.$$

The d.e. satisfied by $c_n = c_n^{(a)}(t)$ is

$$(4.2) \qquad (t^2-1)y'' + (2a+1)ty' = n(n+2a)y.$$

Putting $z - t = x$ in the d.e., (4.1) followed by the substitution $x = s\sqrt{t^2-1}$ yields the d.e.

$$(4.3) \qquad (s^2-1)v''(s) + 2av'(s) = n(2a-1+n)v(s),$$

satisfied by $v(s) = y_n(t + s\sqrt{t^2-1})$. For $\lambda = a - 1/2$, the d.e. in (4.3) is the d.e. for the ultraspherical polynomial $P_n^{(\lambda)}$. Hence,

$$y_n(z;t;a) = P_n^{(a-\frac{1}{2})}\left(\frac{z-t}{\sqrt{t^2-1}}\right).$$

We have

Theorem 4.1. *The orthogonal polynomials generated by the sequence of ultraspherical moments* $(c_n^{(a)}(t))_{n=0}^{\infty}$ *are first-degree transformations of the ultraspherical polynomials* $P_n^{(a-1/2)}$. *Moreover,*

$$T(c_0^{(a)}(t), c_1^{(a)}(t), \dots, c_{n-1}^{(a)}(t)) = (t^2-1)^{\frac{1}{2}n(n-1)} H_n^{(0)},$$

where $H_n^{(0)}$ is the $(n \times n)$-Hankel determinant of the moment sequence corresponding to the polynomials $P_n^{(a-\frac{1}{2})}$.

Remark 4.1. The sequence $(c_n^{(a)}(t))_{n=0}^{\infty}$ of ultraspherical moments with $|t| > 1$ and $a > 0$ is positive definite.

5. Charlier moments. The Charlier polynomials, also called Poisson-Charlier polynomials, $c_n(t; a)$ are defined as follows (see Chihara [**2**])

$$c_n(t;a) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k!}{a^k} \binom{t}{k}, \quad n = 0, 1, \dots .$$

Here t denotes a real variable and a is a nonzero real number. The recurrence relations are

$$ac_n(t;a) + (t - n + 1 - a)c_{n-1}(t;a) + (n-1)c_{n-2}(t;a) = 0, \\ n = 2, 3 \dots , \\ c_0(t;a) = 1, \quad c_1(t;a) = 1 - \frac{t}{a}.$$

Let $(y_n(z;t;a))_{n=0}^{\infty}$ be the OPS generated by the Charlier moment sequence $(c_n(t;a))_{n=0}^{\infty}$. Then it follows from Proposition I that the polynomials $y_n = y_n(z;t;a)$ satisfy the d.e.

$$(1 - z)y_n'' + (az + t - a)y_n' = any_n. \tag{5.1}$$

The Laguerre polynomial $L_n^{(\alpha)}(z)$ satisfies the d.e.

$$zy'' + (\alpha + 1 - z)y' + ny = 0. \tag{5.2}$$

The transformations $z \mapsto z - 1$ and $z \mapsto az$ transform the d.e. (5.1) into the d.e. (5.2), with $\alpha = -t - 1$. Hence,

Theorem 5.1. *The orthogonal polynomials generated by the sequence of Charlier moments $(c_n(t;a))_{n=0}^{\infty}$ are first-degree transformations of* *the Laguerre polynomials $L_n^{(\alpha)}$ with $\alpha = -t-1$. For the Turanian of the Charlier polynomials we find*

$$T(c_0(t;a), c_1(t;a), \dots, c_{n-1}(t;a))$$
$$= a^{-n(n-1)} 1!2! \cdots (n-1)! \cdot (-t)^{n-1}(1-t)^{n-2} \cdots (n-3-t)^2(n-2-t).$$

(Compare [**4**, p. 73 ff.]).

Remark 5.1. The sequence $(c_n(t;a))_{n=0}^{\infty}$ of Charlier polynomials is positive definite if $a \neq 0$ and $t < 0$.

6. Meixner moments. *Meixner* I *polynomials.* The Meixner I polynomials are defined by (see [**2**])

$$m_n(t) = m_n(t;b;c) = (-1)^n n! \sum_{k=0}^{n} \binom{t}{k} \binom{-t-b}{n-k} c^{-k},$$

$c \neq 0, 1$ and $b \neq 0, -1, -2, \dots$. Hence,

$$m_n(0) = (-1)^n n! \binom{-b}{n} = (b)_n.$$

Normalized Meixner I polynomials are defined by

$$c_n(t) = \frac{m_n(t)}{(b)_n}, \quad n = 0, 1, \dots,$$

where b and c are assumed to be real, $b > 0$, from now on. The recurrence relations are

$$c(b+n-1)c_n(t) - [(c-1)t + (c+1)(n-1) + cb]c_{n-1}(t)$$
$$+ (n-1)c_{n-2}(t) = 0, \qquad n = 2, 3, \dots,$$
$$c_0(t) = 1, \quad c_1(t) = \frac{c-1}{bc}t + 1.$$

Proposition I leads to the d.e.

(6.1)
$$(cz^2 - (c+1)z)y_n''(z) + [cb(z-1) - (c-1)t]y_n'(z) = cn(b+n-1)y_n(z).$$

Here $(y_n(z))_{n=0}^{\infty} = (y_n(z;t;b;c))_{n=0}^{\infty}$ denotes the OPS generated by the Meixner I moments. First-degree transformations lead from (6.1) to

$$s(1-s)v_n''(s) - (bs+t)v_n'(s) + n(b-1+n)v_n(s) = 0, \tag{6.2}$$

where $v_n(s) = y_n(1-((c-1)/c)s)$.

We put $b = q+1$ and $t = -p-1$. Then the d.e. in (6.2) becomes

$$s(1-s)v_n''(s) + [p+1-(q+1)s]v_n'(s) + n(q+n)v_n(s) = 0.$$

Obviously, $v_n(s) = J_n(p,q;s)$, the Jacobi polynomial of degree n on the interval $[0,1]$. Moreover, $y_n(z) = J_n(p,q;\gamma(1-z))$, where $\gamma = c/(c-1)$. If the y_n are normalized such that they are monic, then we have the recurrence relations

$$y_n(z) = \frac{\gamma(1-z)+\beta_n}{-\gamma} y_{n-1}(z) - \frac{\alpha_{n-1}}{\gamma^2} y_{n-2}(z).$$

For the Turanian $T(c_0(t), c_1(t), \dots, c_{n-1}(t))$ of the normalized Meixner I polynomials we find

$$T(c_0(t), c_1(t), \dots, c_{n-1}(t)) = \gamma^{-n(n-1)} \cdot \alpha_1^{n-1} \alpha_2^{n-2} \cdots \alpha_{n-2}^2 \alpha_{n-1},$$

where

$$\alpha_k = \frac{k(p+k)(q+k-1)(q-p+k-1)}{(q+2k-2)(q+2k-1)^2(q+2k)}, \quad k = 1, 2, \dots . \tag{6.3}$$

Hence, we have

Theorem 6.1. *The orthogonal polynomials generated by the sequence of Meixner I moments are first-degree transformations of the Jacobi polynomials. For the Turanian of the Meixner I polynomials we find*

$$T(c_0(t), c_1(t), \dots, c_{n-1}(t)) = \gamma^{-n(n-1)} \cdot \alpha_1^{n-1} \alpha_2^{n-2} \cdots \alpha_{n-2}^2 \alpha_{n-1},$$

where α_k *is as in* (6.3).

Remark 6.1. The sequence of normalized Meixner I polynomials $c_n(t) = c_n(t;b;c)$ is positive definite if $b+t>0$ and $t<0$.

Remark 6.2. Let $P_n^{(\alpha,\beta)}$ be the Jacobi polynomial of degree n on the interval $[-1,1]$. Then we have

$$y_n\left(1-\frac{1+z}{2\gamma}\right)=P_n^{(t-1+b,-t-1)}(z).$$

If $b=2$, then

$$y_n\left(1-\frac{1+z}{2\gamma}\right)=P_n^{(t+1,-t-1)}(z),$$

and we are back to the example in our introduction.

Meixner II *polynomials.* The Meixner II polynomials $M_n(t) = M_n(t;\delta;\eta)$ satisfy

$$M_n(t)=[t-(2(n-1)+\eta)\delta]M_{n-1}(t)-(\delta^2+1)(n-1)(n+\eta-2)M_{n-2},$$
$$n=2,3,\dots,$$
$$M_0(t)=1,\quad M_1(t)=t-\eta\delta.$$

The normalized Meixner II polynomials $c_n(t)=(\eta)_n^{-1}M_n(t)$ satisfy

$$(\eta+n-1)c_n-[t-(2(n-1)+\eta)\delta]c_{n-1}+(\delta^2+1)(n-1)c_{n-2}=0.$$

This leads to the d.e.

$$(z^2+2\delta z+\delta^2+1)y_n''(z)+(\eta z-t+\eta\delta)y_n'(z)-(n\eta+n(n-1)y_n(z)=0,$$

where $y_n(z)$ belongs to the OPS generated by the sequence $(c_n(t))_{n=0}^{\infty}$. If we substitute $z=is-\delta$, we obtain the d.e.

$$(1-s^2)v_n''(s)+(-\eta s-it))v_n'(s)+n(\eta+n-1))v_n(s)=0,$$

for $v_n(s) = y_n(is-\delta)$. Comparing this to the d.e. for the Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$, i.e.,

$$(1-z^2)y''(z)+[(\beta-\alpha)-(\alpha+\beta+2)z]y'(z)+n(n+\alpha+\beta+1)y(z)=0,$$

we see

$$v_n(s)=P_n^{(\alpha,\beta)}(s)=\text{const.}\,{}_2F_1\left(-n,n+\alpha+\beta+1;\beta+1;\frac{1+z}{2}\right),$$

where

$$\alpha = \frac{\eta - 2}{2} + \frac{1}{2}it,\ \beta = \frac{n-2}{2} - \frac{1}{2}it.$$

The computation of the Turanian is similar to the Meixner I case.

7. Concluding remarks. 1. We have generated all classical OPSs, including the Bessel OPS, from moment sequences consisting of Laguerre, Hermite, ultraspherical, Charlier, or Meixner polynomials. Consider the following table:

Moment sequence	Generated OPS
Laguerre	Bessel
Hermite	Hermite
Ultraspherical	Ultraspherical
Charlier	Lagauerre
Meixner I,II	Jacobi

2. The recurrence relations satisfied by the Jacobi polynomials and the Bessel polynomials preclude application of Proposition I.

REFERENCES

1. M.G. de Bruin, *Convergence in the Padé table for* ${}_1F_1(1;c;x)$, Kon. Ned. Akad. v. Wet. Ser. A, **79**(5) = Indag. Math. **38**(5) (1976), 408–418.

2. T.S. Chihara, *An introduction to orthogonal polynomials*, Gordon and Breach, New York (1978).

3. S. Karlin, *Total positivity*, Vol. I, Stanford University Press, Stanford, CA, 1968.

4. S. Karlin and G. Szegö, *On certain determinants whose elements are orthogonal polynomials*, J. d'Anal. Math. **8** (1961), 1–157.

5. H.L. Krall, *On derivatives of orthogonal polynomials* II, Bull. Amer. Math. Soc. **47** (1941), 261–264.

6. E.N. Laguerre, *Œuvres de Laguerre*, Tome I, Chelsea Publishing Company, Bronx, NY, 1972.

Department of Mathematics, University of Amsterdam, Roeters-straat 15, 1018 WB Amsterdam, The Netherlands