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SHARP INEQUALITIES FOR THE PADÉ APPROXIMANT ERRORS IN THE STIELTJES CASE

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1. Introduction. For the first time, ten years ago, the inequalities in question were quietly used by the authors to prove the existence of valleys in the c-table [**6**, Formulae (29) and (31)]. It was so natural for us to consider that everything about the Padé approximants to the Stieltjes functions coming from Stieltjes' work was well known. However, in the literature [**1, 2, 3, 9**] we have not been able to find this!

Having discovered the above accident we proved the valley property in another way (not published) and two conjectured inequalities became "open problems" [**7**]. Today we can give a complete proof of these inequalities.

2. Main result. Let f be a nonrational Stieltjes function and A_m/B_n the $[m/n]$ Padé approximant to f. We call the differences $f - [m/n]$ "Padé approximant errors."

Theorem. *Let* $[m/n]$ *be a Padé approximant to the nonrational Stieltjes function* f:

$$f(z) = \int_0^{1/R} \frac{d\mu(t)}{1-tz}, \quad d\mu \geq 0,\ R > 0. \tag{1}$$

The following inequalities occur:

$\forall n \geq \max(0, -k)$ and $\forall x \in]0, R[$,

$$0 < f(x) - [n+k+1/n](x) < \frac{x}{R}\{f(x) - [n+k/n](x)\}, \quad k \geq -2, \tag{2}$$

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(3) $$0 < f(x) - [n+k/n+1](x) < \frac{x}{R}\{f(x) - [n+k/n](x)\}, \qquad k \geq 0,$$

(4) $$0 < f(x) - [n+k+1/n+1](x) < \left(\frac{x}{R}\right)^2 \{f(x) - [n+k/n](x)\}, \qquad k \geq -1.$$

Moreover, the x factor on the right-hand side optimizes the order of the corresponding series, i.e., the power of x cannot be increased.

Remarks.

On sharpness. For each given function f, the inequalities in question can be improved by multiplying the right-hand sides by some positive factor not greater than 1, which corresponds to the intermediate value mentioned in the proof of the inequality (2). The referee remarked that, for $k = -1$, the inequality (2) can be derived from Corollary 17.1, p. 243 in [**1**], where this factor can be bounded in terms of the denominators $B_{n-1,n}$ and $B_{n,n}$ of $[n-1/n]$ and $[n/n]$:

$$f(x) - [n/n](x) \leq \frac{B_{n,n}(R)B_{n-1,n}(x)x}{B_{n,n}(x)B_{n-1,n}(R)R}\{f(x) - [n-1/n](x)\}.$$

On the strict inequalities. Firstly, observe that the rational Stieltjes functions are reconstructed by infinitely many Padé approximants, and in this case the error is zero. The error vanishes also for $x = 0$. Without loss of generality, we eliminate these two cases to obtain the strict inequalities, more interesting than the weak ones.

Logical motivation. By the definition of Padé approximants we have

$$f(x) - [m/n](x) = O(x^{m+n+1}),$$

i.e., the error is of the order $(m+n+1)$. On any antidiagonal $(m+n = \text{const})$ of the Padé table, all errors are of the same order. Conversely, for instance, following a diagonal, the order of errors changes and the classical inequality

(5) $$f(x) - [n+k+1/n+1](x) < f(x) - [n+k/n](x), \quad \forall x \in \,]0, R[\,,$$

which is improved by (4), is no longer balanced with respect to the order of the two sides. It follows that if x tends to zero, the left-hand side tends to zero more rapidly than the right-hand side, and, consequently, this inequality is, in this sense, trivial. Indeed, our idea was to balance all inequalities with respect to the series order.

The absence of the optimal (sharp) inequalities in the work on Padé approximations can be explained by the fact that the inequalities like (5) are sufficient to prove the convergence properties of Padé sequences.

Proof of Theorem. It is well known [**5**, p. 242] that, in the Stieltjes case defined in (1), the Padé approximant denominators of the sequence

$$\{[n+k/n] = A_{n+k}/B_n\}_{n\geq \max(0,-k)}, \quad k \geq -1 \text{ and fixed,} \tag{6}$$

form a family of orthogonal polynomials $\{\overline{B}_n\}$,

$$\overline{B}_n(t) = t^n B_n\left(\frac{1}{t}\right), \quad n \geq \max(0,-k), \tag{7}$$

with respect to the measure $t^{k+1}\,d\mu(t)$:

$$\int_0^{1/R} \overline{B}_n(t)\overline{B}_{n'}(t)t^{k+1}\,d\mu(t) = \alpha_n \delta_{nn'}. \tag{8}$$

In the following we use the family $\{P_{n;k}\}$ of the normalized orthogonal polynomials:

$$P_{n;k}(t;x) = \overline{B}_n(t)/\overline{B}_n(1/x), \quad P_{n;k}(1/x;x) = 1. \tag{9}$$

In order to prove our theorem, we need two known lemmas.

Lemma 1. [**8**] *Let f be a Stieltjes function defined by* (1) *and $P_{n;k}(t;x)$ an orthogonal polynomial of degree n in t defined by* (9)*; then the Padé approximant error is given by*

$$f(x) - [n+k/n](x) = x^{k+1}\int_0^{1/R} [P_{n;k}(t;x)]^2 \frac{t^{k+1}}{1-tx}\,d\mu(t), \tag{10}$$
$$k \geq -1;\ n \geq \max(0,-k).$$

Proof. We begin with the proof of a weaker relation than (10), without the square in the right-hand term:

$$f(x) - [n+k/n](x) = x^{k+1} \int_0^{1/R} P_{n;k}(t;x) \frac{t^{k+1}}{1-tx}\, d\mu(t), \tag{11}$$
$$k \geq -1;\ n \geq \max(0, -k).$$

Firstly, we show that the function r defined by

$$r(x) = f(x) - x^{k+1} \int P_{n;k}(t;x) \frac{t^{k+1}}{1-tx}\, d\mu(t)$$

is rational. In fact, with (1), (7) and (9), we obtain

$$r(x) = \frac{1}{B_n(x)} \int \frac{B_n(x) - \overline{B}_n(t) t^{k+1} x^{n+k+1}}{1-tx}\, d\mu(t),$$

where the numerator of the integrand vanishes for $xt = 1$. Then the integral is a polynomial of degree $n+k$ in x and r is rational of type $\{n+k/n\}$.

We estimate now the remainder $f - r$:

$$f(x) - r(x) \\ = \frac{x^{n+k+1}}{B_n(x)} \int \overline{B}_n(t)[1 + xt + \cdots + (xt)^{n-1} + (xt)^n + \cdots] t^{k+1}\, d\mu(t).$$

The contribution of $[(1 + xt + \cdots + (xt)^{n-1}]$ is zero because of orthogonality (8); then, after resuming the rest,

$$f(x) - r(x) = x^{2n+k+1} \frac{1}{B_n(x)} \int \frac{\overline{B}_n(t) t^{n+k+1}}{1-tx}\, d\mu(t).$$

Because $B_n(0) \neq 0$, the remainder is of order $(2n+k+1)$, which proves that r is a Padé approximant $[n+k/n]$. This ends the proof of the relation (11).

We now introduce the square:

$$\int P_{n;k}(t;x) \frac{t^{k+1}}{1-tx}\, d\mu(t) \\ = \int P_{n;k}(t;x)[P_{n;k}(t;x) + 1 - P_{n;k}(t;x)] \frac{t^{k+1}}{1-tx}\, d\mu(t) \\ = \int [P_{n;k}(t;x)]^2 \frac{t^{k+1}}{1-tx}\, d\mu(t) + \int \frac{P_{n;k}(t;x)[1 - P_{n;k}(t;x)]}{1-tx} t^{k+1}\, d\mu(t).$$

Following (9), the polynomial $1 - P_{n;k}$ vanishes for $xt = 1$ and so can be simplified by $(1 - xt)$. Consequently, by orthogonality, the second integral vanishes, which completes the proof of Lemma 1. □

The square in (10) was needed to prove the following extremal property.

Lemma 2. [**4**, p. 25] *In the set of all polynomials $Q_n(t)$ of degree n such that $Q_n(1/x) = 1$, the polynomial $P_{n;k}(t;x)$ minimizes the integral*

$$I = \int_0^{1/R} [Q_n(t)]^2 \frac{t^{k+1}}{1-tx}\, d\mu(t), \tag{12}$$

for all x belonging to $]0, R[$.

Proof. Each polynomial Q_n can be written as

$$Q_n(t) = P_{n;k}(t;x) + (1 - xt) q_m(t),$$

where $\deg(q_m) < n$. Introducing this in (12) obtains

$$\begin{aligned} I = \int [P_{n;k}(t;x)]^2 \frac{t^{k+1}}{1-tx}\, d\mu(t) + 2 \int P_{n;k}(t;x) q_m(t) t^{k+1}\, d\mu(t) \\ + \int (1-tx)^2 q_m(t) t^{k+1}\, d\mu(t). \end{aligned}$$

The second term vanishes by orthogonality, and the third term is strictly positive for x belonging to $]0, R[$ because the measure $d\mu$ is positive. □

Proof of Inequality (2). We apply Lemma 1 to the left-hand side error of (2):

$$\begin{aligned} f(x) - &[n + k + 1/n](x) \\ &= x^{k+2} \int_0^{1/R} [P_{n;k+1}(t;x)]^2 \frac{t^{k+2}}{1-tx}\, d\mu(t) \quad k \geq -2, \\ &\leq x^{k+2} \int_0^{1/R} [P_{n;k}(t;x)]^2 \frac{t^{k+2}}{1-tx}\, d\mu(t). \end{aligned}$$

where the last majoration is obtained following Lemma 2. Taking some intermediate value of t outside the integral and replacing it by the maximal one, i.e., $1/R$, we obtain exactly the inequality (2).

Proof of Inequality (3). In a similar way, we apply Lemma 1 to the left-hand side Padé error of (3) and, with the help of Lemma 2, obtain the required majoration:

$$\begin{aligned} f(x)-&[n+k/n+1](x) \\ &= x^k \int_0^{1/R} [P_{n+1;k-1}(t;x)]^2 \frac{t^k}{1-tx}\, d\mu(t), \quad k \geq 0, \\ &\leq x^k \int_0^{1/R} [txP_{n;k}(t;x)]^2 \frac{t^k}{1-tx}\, d\mu(t) \\ &\leq \left(\frac{x}{R}\right) x^{k+1} \int_0^{1/R} [P_{n;k}(t;x)]^2 \frac{t^{k+1}}{1-tx}\, d\mu(t) \\ &= \left(\frac{x}{R}\right) \{f(x) - [n+k/n](x)\}. \end{aligned}$$

Termination of the Proof. The inequality (4) can be proved in a similar way, but it is also the consequence of the two inequalities (2) and (3). The positivity of errors in the interval $]0, R[$ is, of course, classic [**5**]. □

Conclusion. The inequalities obtained in our theorem are sharp in the sense that the power of x cannot be increased in the right-hand side. They are interesting especially when x tends to zero. Note also that the improving factor x/R belongs to the interval $]0, 1[$ and the classical inequalities correspond precisely to the value 1.

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