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SOME ASPECTS OF GENERALIZED T-FRACTIONS

MARCEL G. DE BRUIN

Dedicated to Wolf Thron on the occasion of his 70th birthday

ABSTRACT. The purpose of this paper is to generalize the concept of a T-fraction into the direction of simultaneous approximation with rational functions having a common denominator. The generalization includes aspects of correspondence with formal power series in z and $1/z$, the notion of orthogonality w.r.t. a linear functional and convergence.

1. Introduction. Using the (by now) standard notation for a continued fraction, the following can be taken from the valuable source [**4**]:

A T-fraction is a continued fraction of the form

$$\underset{k=1}{\overset{\infty}{K}} \frac{F_k z}{1+G_k z}, \tag{1}$$

where F_n, G_n are complex numbers.

The k-th approximant $f_k(z) = P_k(z)/Q_k(z)$ is a rational function for which both numerator and denominator satisfy the same three-term recurrence relation

$$X_k(z) = (1+G_k z)X_{k-1}(z) + F_k z X_{k-2}(z), \quad k \geq 1, \tag{2}$$

with initial values

$$P_{-1}(z) = 1, \quad P_0(z) = 0; \qquad Q_{-1}(z) = 0, \quad Q_0(z) = 1. \tag{3}$$

For the sake of simplicity, a term of the form $\sum_{k=1}^{\mu} c_k^* z^k + \sum_{k=-\nu}^{0} c_k z^k$, $\mu, \nu \geq 0$, has been omitted here (and in the sequel).

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The main importance of this type of continuous fraction lies in its correspondence properties:

A. If $F_k \neq 0$, there exists a unique formal power series (fps) $f(z) = \sum_{k=1}^{\infty} c_k z^k$ such that (1) corresponds to f at $z = 0$:

$$f(z) - f_k(z) = O(z^{k+1}), \quad \text{as } z \to 0, \quad k \geq 0.$$

B. If $G_k \neq 0$, there exists a unique fps $g(z) = \sum_{k=0}^{\infty} d_k z^{-k}$ such that (1) corresponds to g at $z = \infty$:

$$g(z) - f_k(z) = O(z^{-k}), \quad \text{as } z \to \infty, \quad k \geq 0.$$

Combination of the conditions A, B for $k \geq 0$ leads to continued fractions which can be recovered from the so-called two-point Padé table for f, g (cf. [**7, 8**]). Also, conditions can be given under which, starting from two power series f, g as given in A, B, there exists a T-fraction that corresponds to the pair of functions described in the manner given above. The conditions can be expressed in terms of the nonvanishing of certain Hankel matrix determinants in which the doubly infinite sequence of moments constructed from the coefficients $(c_k)_{k\geq 1}$, $(d_k)_{k\geq 0}$ plays a role. This moment sequence can also be used to derive the P_k, Q_k as solutions of an orthogonalization problem (cf. [**1**]), using the method from [**5**]—relaxing the conditions given there—and introducing linear functionals on the space of formal Laurent series.

The contents of the paper are now arranged as follows. In Section 2 the so-called T-n-fraction, generalizing (1) will be introduced, followed by results on correspondence, determinantal expressions and orthogonality in Section 3. The results reduce to known results on ordinary T-fractions (1) on taking $n = 1$.

In Section 4 a convergence result will be given, for sake of simplicity for $n = 2$ only, that resembles a result for T-fractions in [**4**, pp. 141–143]. Finally, Section 5 will be devoted to the proofs.

This paper must be seen as an introduction of the concept of a T-n-fraction; it will be followed later on by a more detailed study of the connection with the Padé-n-table and numerical behavior of the approximating n-tuples.

2. T-n-fractions. Let $n \geq 1$ be a fixed integer and consider the n-fraction (cf. [**2**]) given by the notation

$$\underset{k=1}{\overset{\infty}{K}} \begin{bmatrix} a_{1,k} z^{\min(n,k)} \\ a_{2,k} z^{\min(n-1,k)} \\ \vdots \\ a_{n,k} z \\ 1 + b_k z \end{bmatrix}, \tag{4}$$

where $(a_{i,k})$, $1 \leq i \leq n$ and (b_k) are sequences of complex numbers satisfying

$$a_{1,k} \neq 0, \quad b_k \neq 0, \qquad k \geq 1. \tag{5}$$

An n-fraction of the form (4) is called a T-n-fraction; its sequence of approximant n-tuples $(f_k^{(i)}(z))_{i=1}^n$, $k \geq 0$, consists of rational functions with a common denominator

$$f_k^{(i)}(z) = P_k^{(i)}(z)/P_k^{(0)}(z), \quad 1 \leq i \leq n; \quad k \geq 0, \tag{6}$$

where the polynomials $P_k^{(i)}$, $0 \leq i \leq n$, all satisfy the same $(n+2)$-term recurrence relation

(7)
$$X_k(z) = \\ (1 + b_k z) X_{k-1}(z) + a_{n,k} z X_{k-2}(z) + \cdots + a_{1,k} z^{\min(n,k)} X_{k-n-1}(z), \\ k \geq 1,$$

with initial values

$$\begin{aligned} P_{-k}^{(i)} &= \begin{cases} 0, & i + k \neq n + 1 \\ 1, & i + k = n + 1, \end{cases} \quad 1 \leq i \leq n, \quad 0 \leq k \leq n, \\ P_{-k}^{(0)} &= \begin{cases} 0, & 1 \leq k \leq n, \\ 1, & k = 0. \end{cases} \end{aligned} \tag{8}$$

Remarks. 1. The condition (5) has been added for the sake of simplicity.

2. From the initial values (8), the recurrence (7) and $a_{1,k} \neq 0$, it follows easily that $\deg P_k^{(i)} \leq k$, $1 \leq i \leq n$, $k \geq 1$, $\deg P_k^{(0)} = k$, $k \geq 1$, and $P_k^{(0)}(0) = 1$, $P_k^{(i)}(0) = 0$, $1 \leq i \leq n$, for $k \geq 0$.

3. For $n = 1$, the T-n-fraction reduces to an ordinary T-fraction (1)–(3).

3. Correspondence, determinants and orthogonality. In this section the main results on correspondence will be given, along with a determinantal expression for the numerators and denominators of the n-tuples of approximants in the case that a given pair of n-tuples of power series (in z and in $1/z$) give rise to a T-n-fraction (4)–(7). Without stating so each time, it must be pointed out that, for $n = 1$, the results reduce to known results for ordinary T-fractions.

Theorem 1. *Consider a T-n-fraction* (4). *If the condition* (5) *is satisfied, there exist unique fps*

$$f^{(i)}(z) = \sum_{k=1}^{\infty} c_k^{(i)} z^k, \qquad g^{(i)}(z) = \sum_{k=0}^{\infty} d_k^{(i)} z^{-k}, \quad 1 \le i \le n, \tag{9}$$

such that the T-n-fraction corresponds to the pair of n-tuples at $z = 0$ and $z = \infty$ in the following manner:

(10a)
$$f^{(i)}(z) - f_k^{(i)}(z) = O(z^{k+1}), \quad 1 \le i \le n \quad \text{as } z \to 0, \quad k \ge 0,$$
(10b)
$$g^{(i)}(z) - f_k^{(i)}(z) = O_-(z^{-[(k-i)/n]-1}), \quad 1 \le i \le n, \quad \text{as } z \to \infty, \quad k \ge 0.$$

([$\cdot$] denotes the greatest integer function and O_- denotes descending powers of z.) □

We now consider an inverse problem: given two n-tuples of fps as in (9), can one find a T-n-fraction that corresponds to these fps as described in Theorem 1? To this end, the following moment sequences are introduced

$$\delta_k^{(i)} = \begin{cases} c_k^{(i)}, & k \ge 1 \\ -d_{-k}^{(i)}, & k \le 0 \end{cases}, \quad 1 \le i \le n, \tag{11}$$

and—for notational convenience—the following columns of moments:

$$\Gamma_k = (\delta_k^{(1)}, \dots, \delta_k^{(n)})^T, \qquad \Gamma_k^{(r)} = (\delta_k^{(1)}, \dots, \delta_k^{(r)})^T, \qquad 1 \le r \le n-1, \ k \ge 1. \tag{12}$$

Then we have the conditions (reminiscent of [**3**]) given in

Theorem 2. *Consider two n-tuples of fps as in* (9) *and the associated moment sequences as in* (11).

A. *There exist unique polynomials* $\{P_k^{(i)}\}_{i=0}^n$, $k \ge 1$, $\deg P_k^{(i)} \le k$, $1 \le i \le n$, $\deg P_k^{(0)} = k$, $P_k^{(i)}(0) = 0$, $1 \le i \le n$, $P_k^{(0)}(0) = 1$, *with rational functions* $f_k^{(i)} = P_k^{(i)}/P_k^{(0)}$, $1 \le i \le n$, *that satisfy the correspondence formulae* (10a,b) *under the conditions*

(13a)

$$\begin{vmatrix} \Gamma_{k-v} & \Gamma_{k-v-1} & \cdots & \Gamma_{-v+1} \\ \Gamma_{k-v+1} & \Gamma_{k-v} & \cdots & \Gamma_{-v+2} \\ \vdots & \vdots & & \vdots \\ \Gamma_{k-1} & \Gamma_{k-2} & \cdots & \Gamma_0 \end{vmatrix} \neq 0, \quad \begin{vmatrix} \Gamma_{k-v+1} & \Gamma_{k-v} & \cdots & \Gamma_{-v+2} \\ \Gamma_{k-v+2} & \Gamma_{k-v+1} & \cdots & \Gamma_{-v+3} \\ \vdots & \vdots & & \vdots \\ \Gamma_{k} & \Gamma_{k-1} & \cdots & \Gamma_1 \end{vmatrix} \neq 0,$$

$$k = nv, \quad v \ge 1,$$

(13b)

$$\begin{vmatrix} \Gamma_{k-v-1}^{(r)} & \Gamma_{k-v-2}^{(r)} & \cdots & \Gamma_{-v}^{(r)} \\ \Gamma_{k-v} & \Gamma_{k-v-1} & \cdots & \Gamma_{-v+1} \\ \vdots & \vdots & & \vdots \\ \Gamma_{k-1} & \Gamma_{k-2} & \cdots & \Gamma_0 \end{vmatrix} \neq 0 \quad \begin{vmatrix} \Gamma_{k-v}^{(r)} & \Gamma_{k-v-1}^{(r)} & \cdots & \Gamma_{-v+1}^{(r)} \\ \Gamma_{k-v+1} & \Gamma_{k-v} & \cdots & \Gamma_{-v+2} \\ \vdots & \vdots & & \vdots \\ \Gamma_{k} & \Gamma_{k-1} & \cdots & \Gamma_1 \end{vmatrix} \neq 0$$

$$k = nv + r, \quad v \ge 0, \quad 1 \le r \le n-1$$

B. *Under the conditions* (13a,b), *the polynomials* $P_k^{(i)}$ *can be given explicitly by the following determinantal representation.*

1. *For* $k = nv$, $v \ge 1$

$$P_k^{(0)}(z) = \begin{vmatrix} 1 & z & \cdots & z^k \\ \Gamma_{k-v+1} & \Gamma_{k-v} & \cdots & \Gamma_{-v+1} \\ \vdots & \vdots & & \vdots \\ \Gamma_k & \Gamma_{k-1} & \cdots & \Gamma_0 \end{vmatrix} \Bigg/ \begin{vmatrix} \Gamma_{k-v} & \cdots & \Gamma_{-v+1} \\ \vdots & & \vdots \\ \Gamma_{k-1} & \cdots & \Gamma_0 \end{vmatrix},$$

$$P_k^{(i)}(z) = \sum_{j=1}^{k} \begin{vmatrix} c_j^{(i)} & c_{j-1}^{(i)} & \cdots & c_{j-k}^{(i)} \\ \Gamma_{k-v+1} & \Gamma_{k-v} & \cdots & \Gamma_{-v+1} \\ \vdots & \vdots & & \vdots \\ \Gamma_k & \Gamma_{k-1} & \cdots & \Gamma_0 \end{vmatrix} z^j \Bigg/ \begin{vmatrix} \Gamma_{k-v} & \cdots & \Gamma_{-v+1} \\ \vdots & & \vdots \\ \Gamma_{k-1} & \cdots & \Gamma_0 \end{vmatrix},$$

$$1 \le i \le n$$

2. *For $k = nv + r$, $v \geq 0$, $1 \leq r \leq n-1$,*

$$P_k^{(0)}(z) = \begin{vmatrix} 1 & z & \cdots & z^k \\ \Gamma_{k-v}^{(r)} & \Gamma_{k-v-1}^{(r)} & \cdots & \Gamma^{(r)} \\ \Gamma_{k-v+1} & \Gamma_{k-v} & \cdots & \Gamma_{-v+1} \\ \vdots & \vdots & & \vdots \\ \Gamma_k & \Gamma_{k-1} & \cdots & \Gamma_0 \end{vmatrix} \Bigg/ \begin{vmatrix} \Gamma_{k-v-1}^{(r)} & \cdots & \Gamma_{-v}^{(r)} \\ \Gamma_{k-v} & \cdots & \Gamma_{-v+1} \\ \vdots & & \vdots \\ \Gamma_{k-1} & \cdots & \Gamma_0 \end{vmatrix},$$

$$P_k^{(i)}(z) = \sum_{j=1}^{k} \begin{vmatrix} c_j^{(i)} & c_{j-1}^{(i)} & \cdots & c_{j-k}^{(i)} \\ \Gamma_{k-v}^{(r)} & \Gamma_{k-v-1}^{(r)} & \cdots & \Gamma_{-v}^{(r)} \\ \Gamma_{k-v+1} & \Gamma_{k-v} & \cdots & \Gamma_{-v+1} \\ \vdots & \vdots & & \vdots \\ \Gamma_k & \Gamma_{k-1} & \cdots & \Gamma_0 \end{vmatrix} z^j \Bigg/ \begin{vmatrix} \Gamma_{k-v-1}^{(r)} & \cdots & \Gamma_{-v}^{(r)} \\ \Gamma_{k-v} & \cdots & \Gamma_{-v+1} \\ \vdots & & \vdots \\ \Gamma_{k-1} & \cdots & \Gamma_0 \end{vmatrix},$$

$$1 \leq i \leq n. \square$$

Finally, the construction of the approximants will be given from the viewpoint of (formal) orthogonality, using the moment sequences (11). Introduce functionals $\Omega^{(i)}$, $1 \leq i \leq n$, and columns Ω by

$$\Omega^{(i)}(z^j) = \delta_j^{(i)}, \quad 1 \leq i \leq n, \quad j \in \mathbf{Z}, \tag{14a}$$

$$\Omega(z^j) = (\Omega^{(1)}(z^j), \ldots, \Omega^{(n)}(z^j))^T, \quad j \in \mathbf{Z}. \tag{14b}$$

Extend these to linear functionals on the space of formal Laurent series of the form $\sum_{k=-\infty}^{+\infty} \alpha_k z^k$. Then we have

Theorem 3. *Let the moment sequences* (11) *satisfy the conditions* (13a,b). *Then there exists a unique monic polynomial $Q_k(z)$ of exact degree k and with $Q_k(0) \neq 0$ satisfying*

$$\begin{gathered} \Omega(z^{-j} Q_k(z)) = 0, \quad 0 \leq j \leq \left[\frac{k}{n}\right] - 1, \\ \Omega^{(i)}(z^{-[k/n]} Q_k(z)) = 0, \quad 1 \leq i \leq k - n\left[\frac{k}{n}\right]. \end{gathered} \tag{15}$$

Moreover, the denominator polynomial $P_k^{(0)}$ from Theorem 2 satisfies

$$P_k^{(0)}(z) = z^k Q_k(z^{-1}). \square \tag{16}$$

Remark. In the case that k is divisible by n, the last set of conditions in (15) has to be omitted.

4. Convergence. For the sake of simplicity, we now restrict ourselves to the case $n = 2$ and consider

$$\text{(17)} \qquad \operatorname*{K}_{k=1}^{\infty} \begin{bmatrix} a_{1,k} z^{\min(2,k)} \\ a_{2,k} z \\ 1 + b_k z \end{bmatrix} \quad \text{with} \quad a_{1,k} \neq 0,\ b_k \neq 0, \quad k \geq 1.$$

Then the following convergence result can be proved (cf. [**4**, p. 141–143]):

Theorem 4.

A. *If $|b_k| \leq b$, $|a_{i,k}| \leq a_i$, $i = 1, 2$, for all k, (thus, $a_1 > 0$, $a_2 \geq 0$ and $b > 0$), the T-2-fraction* (17) *converges (to a pair of functions) on*

$$G = \{z : |z| \leq (\{((1 + a_2)^2 + 4(b + a_1))^{1/2} - (1 + a_2)\}/\{2(b + a_1)\})^2\}.$$

The limit functions are analytic on the interior of G.

B. *If $|b_k| \geq b$, $|a_{i,k}| \leq a_i$, $i = 1, 2$, for all k, with $a_1 > 0$, $a_2 \geq 0$ and $b > 0$, the T-2-fraction* (17) *converges (to a pair of functions) on $G = \{z : |z| \geq \rho^3\}$, ρ the positive real root of $b\rho^3 - (1 + a_1)\rho^2 - a_2\rho - 1 = 0$. The limit functions are analytic on the interior of G. Furthermore, the same result holds if G is replaced by $H = \{z : |z| \geq \rho^4\}$ where ρ is the positive real root of $b\rho^4 - \rho^3 - a_1\rho^2 - a_2\rho - 1 = 0$.*

Remark. For general n, the bounds in A, respectively B, can be replaced by τ^2, where τ is the positive real root of $a_1\tau^n + a_2\tau^{n-1} + \cdots + a_{n-2}\tau^3 + (b + a_{n-1})\tau^2 + (1 + a_n)\tau - 1 = 0$, respectively ρ_0^{n+1} with ρ_0 the positive real root of $b\rho^{n+1} - (1 + a_1)\rho^n - a_2\rho^{n-1} - \cdots - a_n\rho - 1 = 0$.

5. Proofs. Since large parts of the proofs use elementary linear algebra and induction, they will not be given in full detail; the parts left out, however, are easily filled in.

Proof of Theorem 1. Using the form of the exponents of z in the recurrence relation (7)—specifically the way they increase—it is a matter of tedious but straightforward calculations, taking the initial values (8) into account, to prove that the forms

$$\Delta_{i,k}(z) = f_k^{(i)}(z) - f_{k-1}^{(i)}(z), \quad 1 \le i \le n, \quad k \ge 1,$$

satisfy the order relations

$$\begin{aligned} \Delta_{i,k}(z) &= O(z^k) \quad \text{as } z \to 0, \\ \Delta_{i,k}(z) &= O_-(z^{-[(i-k)/n]}) \quad \text{as } z \to \infty, \end{aligned} \tag{18}$$

for $1 \le i \le n$, $k = 1, 2, \dots, n$.

Important in the proof of (18) is the fact that the contribution of $P_j^{(i)}$ with negative j is restricted; fix i, $k \in \{1, \dots, n\}$, then

$$\Delta_{i,k} = \sum_{j=1}^{n} a_{n+1-j,k} z^{\min(j,k)} (P_{k-1-j}^{(i)} P_{k-1}^{(0)} - P_{k-1}^{(i)} P_{k-1-j}^{(0)}) / (P_k^{(0)} P_{k-1}^{(0)}),$$

and this sum splits into two sums. The first with terms with $1 \le j \le k-1$:

$$a_{n+1-j,k}\, z^j \frac{P_{k-j-1}^{(0)}}{P_k^{(0)}} \left(\frac{P_{k-1-j}^{(i)}}{P_{k-1-j}^{(0)}} - \frac{P_{k-1}^{(i)}}{P_{k-1}^{(0)}} \right),$$

which all behave like $z^j \cdot z^{k-1-j} \cdot z^{-k} = z^{-1}$ or better near infinity (the form between parentheses satisfies (18) by induction hypothesis). The remaining terms have to be written (there $k \le j \le n$ and, thus, $P_{k-1-j}^{(0)} = 0$)

$$a_{n+1-j,k} z^k P_{k-i-j}^{(i)} / P_k^{(0)}.$$

Because of the initial values, the only contribution arises for $-(k-1-j)+i = n+1$, i.e., $j = n+k-i$. The conditions $k \le j \le n$ then translate into $k \le i$: in this case, that special term is of order $z^k \cdot z^{-k} = z^0$ near infinity. Therefore,

$$\Delta_{i,k} = O_-(z^0) \quad \text{for } 1 \le k \le i \le n.$$

For $i + 1 \le k \le n$, the second sum is empty and we at once find

$$\Delta_{i,k} = O_-(z^{-1}) \quad \text{for } i + 1 \le k \le i \le n.$$

Since $-[(i-k)/n]$ has the same values for $1 \le i,\ k \le n$, the form (18) is established.

With the aid of (18) and the following formula, which follows from the recurrence relation (7),

$$\Delta_{i,k}(z) = \sum_{j=1}^{n} a_{n+1-j,k} z^{\min(k,j)} \frac{P_{k-j-1}^{(0)}(z)}{P_k^{(0)}(z)} \left(\frac{P_{k-j-1}^{(i)}(z)}{P_{k-j-1}^{(0)}(z)} - \frac{P_{k-1}^{(i)}(z)}{P_{k-1}^{(0)}(z)} \right),$$

for $1 \le i \le n$, $k \ge n$, and the fact that $f_k^{(i)}(z) = \Delta_{i,k}(z) + \Delta_{i,k-1}(z) + \cdots + \Delta_{i,1}(z)$, it is easy to prove (18) for all $k \ge 1$. Using $O_-(z^j P_{k-j-1}^{(0)}/P_k^{(0)}) = -1$ and the identity $[(i-k-1)/n] = -[(k-i)/n] - 1$ (valid for $1 \le i \le n$, $k \ge 1$), the order O_- of $\Delta_{i,k}$ follows from the worst contribution in the summation, i.e.,

$$-1 - [(k-n-1-i)/n] - 1 = -1 - [(k-1-i)/n] = [(i-k)/n].$$

The final stage of the proof then follows by induction and the fact that

$$f^{(i)}(z) - f_0^{(i)}(z) = O(z^{0+1}) \quad \text{as } z \to 0,$$
$$g^{(i)}(z) - f_0^{(i)}(z) = O_-(z^{-[(0-i)/n]-1}) \quad \text{as } z \to \infty$$

$1 \le i \le n$, combined with the rule that $f_k^{(i)}$ can be used to define the fps $f^{(i)}$, respectively $g^{(i)}$, as far as the order of $\Delta_{i,k+1}$ permits. □

Proof of Theorem 2. The information on the degrees and constant coefficients is used to write

$$\begin{aligned} P_k^{(i)}(z) &= \sum_{j=1}^{k} p_j^{(i)} z^j, \quad 1 \le i \le n, \\ P_k^{(0)}(z) &= \sum_{j=0}^{k} q_j z^j, \quad k \ge 1. \end{aligned} \tag{19}$$

where$q_0 = 1$, $q_k \ne 0$.

The correspondence formula (10a) can be translated into the condition that $z^1, z^2, \dots, z^k$ have to cancel from the expression $P_k^{(0)}(z) f^{(i)}(z) - P_k^{(i)}(z)$, $1 \le i \le n$—here the fact that $q_0 \ne 0$ is used to prevent reduction of the order of contact at the origin—and we get the equations

$$q_0 c_j^{(i)} + q_1 c_{j-1}^{(i)} + \cdots + q_{j-1} c_1^{(i)} = p_j^{(i)}, \quad 1 \le j \le k, \quad 1 \le i \le n. \tag{20a}$$

To handle the correspondence (10b) we first have to remind ourselves of the fact that

$$f_k^{(i)}(z) = P_k^{(i)}(z)/P_k^{(0)}(z) = (P_k^{(i)}(z)/z^k)/(P_k^{(0)}(z)/z^k),$$

showing that a type of reversed polynomials play a role: (10b) shows that in $\sum_{j=0}^{k} q_{k-j} z^{-j} \sum_{m=0}^{\infty} d_m^{(i)} z^{-m} - \sum_{j=0}^{k-1} p_{k-j}^{(i)} z^{-j}$ the terms with $z^0, z^{-1}, \dots, z^{-[(k-i)/n]}$ have to cancel (here we use $q_k \ne 0$ to prevent order reduction). The equations are

$$\begin{gathered} q_{k-j} d_0^{(i)} + q_{k-j+1} d_1^{(i)} + \cdots + q_k d_j^{(i)} = p_{k-j}^{(i)}, \\ 0 \le j \le [(k-i)/n], \quad 1 \le i \le n. \end{gathered} \tag{20b}$$

The overlap in the equation (20a,b) leads to homogeneous linear equations in the unknowns $q_0, q_1, \dots, q_k$. The number of these equations is

$$\sum_{i=1}^{n} \left(\left[\frac{k-i}{n} \right] + 1 \right) = k - n + n = k,$$

while the number of unknowns also is k ($q_0 = 1!$). The equations in (20a) are used to calculate the $p_j^{(i)}$ once the q_j are known.

Now it is necessary to distinguish two cases: $k = nv$, $v \ge 1$ or $k = nv + r$, $v \ge 0$, $1 \le r \le n-1$. As the treatment of these cases is quite similar, we will only look into $k = nv$, $v \ge 1$.

Rewriting the overlap between (20a) and (20b) as equations in the unknowns $q_1, \dots, q_k$, we find

(21)

$$\begin{bmatrix} c_{k-v}^{(i)} & c_{k-v-1}^{(i)} & \cdots & c_1^{(i)} & -d_0^{(i)} & & \cdots & -d_{v-1}^{(i)} \\ c_{k-v+1}^{(i)} & c_{k-v}^{(i)} & \cdots & c_2^{(i)} & c_1^{(i)} & -d_0^{(i)} & \cdots & -d_{v-2}^{(i)} \\ \vdots & & & & \ddots & & \ddots & \vdots \\ c_{k-1}^{(i)} & c_{k-2}^{(i)} & & \cdots & & c_1^{(i)} & & -d_0^{(i)} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{bmatrix}$$

$$= - \begin{bmatrix} c_{k-v+1}^{(i)} \\ c_{k-v+2}^{(i)} \\ \vdots \\ c_k^{(i)} \end{bmatrix}, \quad 1 \le i \le n.$$

Putting the n $(v \times k)$-matrices together in one $k \times k$ matrix, $k = nv$, rearranging the rows to have $c_\cdot^{(i)}$'s and $d_\cdot^{(i)}$'s with the same index together, we see that the first condition in (13a) ensures the existence of a unique solution, while the second condition takes care of $q_k \neq 0$ (Cramer's rule).

The other case, $k = nv + r$ with $1 \le r \le n-1$ and $v \ge 0$, is proved along the same lines: the order conditions on $g^{(i)}$ then lead to some "extra" equations for $1 \le i \le r$. These appear as $\Gamma_\cdot^{(r)}$ in the conditions (13b). This proves part A of the Theorem. The part on the explicit determinantal forms for the polynomials $P_k^{(i)}$ follows easily: add to the equations (21) the equation $1 \cdot q_0 + z \cdot q_1 + \cdots + z^k \cdot q_k = P_k^{(0)}(z)$ and write the equations in the original (21) with 0 on the right-hand side. Then apply Cramer's rule to find $P_k^{(0)}(z)$.

The forms for the $P_k^{(i)}$, $1 \le i \le k$, are found from multiplication of the first line in the determinantal expression for $P_k^{(0)}$ by $f^{(i)}$, and using

$$O(P_k^{(0)}(z) f^{(i)}(z) - P_k^{(i)}(z)) \ge k + 1,$$

to change the sums

$$z^{r-1} \sum_{m=1}^{\infty} c_m^{(i)} z^m \quad \text{(in column number } r\text{)}$$

into

$$\sum_{j=1}^{k} c_{j-r+1}^{(i)} z^j,$$

for $r = 1, 2, \ldots, k$, (the other terms disappear) and taking the sum out of the determinant. □

Proof of Theorem 3. Write out the equations (15) for the coefficients of the unknown, Q_k, the determinantal conditions (13a,b) show the existence of a unique monic solution of exact degree k with $Q_k(0) \neq 0$.

Finally, inspection of the equations for the coefficients of $P_k^{(0)}$ and those for the reversed Q_k, leads to (16). □

Proof of Theorem 4. We can use a generalization of the Sleshinskii [**9**]-Pringsheim theorem on convergence of continued fractions due to P. Levrie [**6**]; first, the multiplication of the continued fraction (17) by $z^{-1/2}$ (for a very small neighborhood around the origin there is, evidently, convergence and we can later on piece the results together) leads to an equivalent 2-fraction (cf. [**2**]) with coefficients

$$z^{-1/2} + b_k z^{1/2},\ a_{2,k}, a_{1,k} z^{1/2}.$$

Writing $\zeta = z^{-1/2}$, the convergence condition is

$$|\zeta + b_k \zeta^{-1}| \geq 1 + |a_{2,k}| + |a_{1,k}\zeta^{-1}|,$$

which leads, using $|\zeta + b_k\zeta^{-1}| \geq |\zeta| - |b_k||\zeta^{-1}| \geq |\zeta| - b|\zeta|^{-1}$, to

$$|\zeta| \geq \frac{1}{2}(1 + a_2 + ((1 + a_2)^2 + 4(b + a_1))^{1/2}).$$

Inserting $|z| = |\zeta|^{-2}$, we find the condition in part A of Theorem 4. The second part of Theorem 4 uses a different auxiliary variable. First, transform the 2-fraction by multiplication with $z^{-2/3}$ into one with coefficients

$$z^{-2/3} + b_k z^{1/3}, \quad a_{2,k} z^{-1/3}, \quad a_{1,k}.$$

Then, write $\zeta = z^{1/3}$ and apply the Levrie criterion:

$$b|\zeta| - |\zeta|^{-2} \geq 1 + a_1 + a_2|\zeta|^{-1}.$$

This shows $|\zeta| \geq \rho$ where ρ satisfies the condition in part B; thus, $|z| = |\zeta|^3 \geq \rho^3$. To find the alternative domain (which might be better, depending upon the values of a_1, a_2 and b), we apply multiplication by $z^{-3/4}$ and [**6**]:

$$b|z|^{1/4} - |z|^{-3/4} \geq 1 + a_2|z|^{-1/2} + a_1|z|^{-1/4},$$

then multiply through by $|z|^{3/4}$ and consider the resulting condition as a polynomial condition in $|z|^{1/4}$. $\square$

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Delft University of Technology, Faculty of Technical Mathematics and Informatics, P.O. Box 356, 2600 AJ Delft, The Netherlands