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THE DIAGONAL ENTRIES OF A HILBERT SPACE OPERATOR

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1. Introduction. Let T be a (bounded linear) operator acting on a complex, separable, infinite dimensional Hilbert space $\mathcal{H}$. For each orthonormal basis (ONB) $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{H}$, T admits a unique matrix representation of the form

$$T = \begin{pmatrix} t_{11} & t_{12} & \cdot & \cdot & \cdot & t_{1n} & \cdot & \cdot \\ t_{21} & t_{22} & \cdot & \cdot & \cdot & t_{2n} & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & & \\ \cdot & \cdot & & \cdot & & \cdot & & \\ \cdot & \cdot & & & \cdot & \cdot & & \\ t_{n1} & t_{n2} & \cdot & \cdot & \cdot & t_{nn} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_n \\ \cdot \\ \cdot \end{matrix}.$$

Let $\operatorname{diag}(T) = \{t_{11}, t_{22}, \ldots, t_{nn}, \ldots\}$ denote the *diagonal sequence* of T with respect to this basis. The diagonal entry t_{nn} is equal to $\langle Te_n, e_n\rangle$, and therefore it belongs to the *numerical range* of T,

$$W(t) = \{\langle Tx, x\rangle : x \in \mathcal{S}_1\},$$

where $\mathcal{S}_1 = \{x \in \mathcal{H} : ||x|| = 1\}$; moreover, if t is a limit point of the sequence $\{t_{nn}\}_{n=1}^{\infty}$, then t belongs to the *essential numerical range* of T,

$$W_e(T) = \cap\{W(T+K)^- : K \text{ is a compact operator}\}.$$

(See [**3**,**6**] for properties of $W(T)$ and $W_e(T)$. For instance, the well-known Toeplitz-Hausdorff theorem guarantees that $W(T)$ and $W_e(T)$ are convex sets.)

For which (necessarily bounded) sequences $\{a_n\}_{n=1}^{\infty}$ is it possible to find an ONB $\{e_n\}_{n=1}^{\infty}$ such that $\operatorname{diag}(T) = \{a_n\}_{n=1}^{\infty}$ with respect to this basis?

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We have the following partial answers.

Theorem. (i) *If* $\{a_n\}_{n=1}^{\infty} \subset W_e(T)^0$ (*the interior of* $W_e(T)$) *and* $\{a_n\}_{n=1}^{\infty}$ *has a limit point* a *in* $W_e(T)^0$, *then there exists an* ONB $\{e_n\}_{n=1}^{\infty}$ *such that* $\operatorname{diag}(T) = \{a_n\}_{n=1}^{\infty}$ *with respect to this basis.*

(ii) *If all the limit points of the sequence* $\{a_n\}_{n=1}^{\infty}$ *belong to* $W_e(T)$, *then there exist a compact operator* K *and an* ONB $\{e_n\}_{n=1}^{\infty}$ *such that* $\operatorname{diag}(T+K) = \{a_n\}_{n=1}^{\infty}$; *furthermore, if* $a_n \in W_e(T)$ (*for all* n), *then* K *can be chosen of arbitrarily small norm.*

(iii) *If* $\operatorname{dist}[a_n, W_e(T)] \to 0$ $(n \to \infty)$, *then there exist a sequence* $\{a'_n\}_{n=1}^{\infty}$ *and an* ONB $\{e_n\}_{n=1}^{\infty}$ *such that* $\operatorname{diag}(T) = \{a'_n\}_{n=1}^{\infty}$ *with respect to this basis, and* $|a_n - a'_n| \to 0$ $(n \to \infty)$.

These results will be proved in the next section. Section 3 is devoted to the analysis of several examples that show that the results are the best possible.

2. Proofs of the main results.

Lemma 1. [**5**] *If the operator* B (*on* $\mathbf{C}^n$) *has a matrix representation* $(b_{ij})_{i,j=1}^n$ *and* $a = \frac{1}{n}\sum_{j=1}^n b_{jj}$, *then there exists an* ONB $\{e_j\}_{j=1}^n$ *with respect to which* $\operatorname{diag}(B) = \{a, a, a, \dots, a\}$.

Lemma 2. [**6**, Problem 166, **3**] (i) *If* $\mathcal{M}$ *is a finite dimensional subspace of* $\mathcal{H}$, *and*

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{H} \ominus \mathcal{M} \end{matrix},$$

then $W_e(T_{22}) = W_e(T)$.

(ii) *If* $a \in W_e(T)^0$, *then* $a \in W_e(T_{22})$ *for all* $\mathcal{M}$ *as in* (i).

Proof of the Theorem. (i) We use Lemmas 1 and 2 as in [**1**]. Let $\{g_n\}_{n=1}^{\infty}$ be an ONB of $\mathcal{H}$. Suppose $\{\lambda \in \mathbf{C} : |\lambda - a| < \varepsilon\} \subset W_e(T)^0$. If $a_n = a$ for infinitely many a's, we can find $b \in W_e(T)^0$, $|a-b| < \varepsilon$, and m_1 such that $(m_1 - 1)b + \langle Tg_1, g_1\rangle = m_1 a$. By Lemma 2(ii) we can find an orthonormal system (ONS) $\{h_j\}_{j=2}^{m_1}$ such that $\langle Th_j, h_j\rangle = b$ for all $j = 2, 3, \dots, m_1$. By Lemma 1, the linear span of g_1 and $\{h_j\}_{j=2}^{m_1}$, $\{g_1\} \vee \{h_j\}_{j=2}^{m_1}$, admits an ONB $\{e'_j\}_{j=1}^{m_1}$ such that $\langle Te'_j, e'_j \rangle = a$ for all $j = 1, 2, \dots, m_1$. Since $a_n = a$ for infinitely many n's, we can find m_1 distinct indices $n(1), n(2), \dots, n(m_1)$ such that $a_{n(j)} = a$, $j = 1, 2, \dots, m_1$.

If $a_n = a$ for only finitely many indices, then by continuity we can find $\{h_j\}_{j=2}^{m_1}$ and $\{e'_j\}_{j=1}^{m_1}$ as above such that $g_1 \in \vee\{e'_j\}_{j=1}^{m_1}$ and $\langle Te'_j, e'_j \rangle = a_{n(j)}$, where $|a - a_{n(j)}| < \varepsilon$ $(j = 1, 2, \dots, m_1)$.

Since $a_n \in W_e(T)^0$, in either case we can use Lemma 2 in order to extend $\{e'_j\}_{j=1}^{m_1}$ to an ONS $\{e_n\}_{n=1}^{n(m_1)}$, with $e_{n(j)} = e'_j$ $(j = 1, 2, \dots, m_1)$ such that

$$\langle Te_n, e_n \rangle = a_n \quad \text{for } n = 1, 2, \dots, n(m_1).$$

Let $g'_2 \in \mathcal{S}_1 \cap [\vee\{e_n\}_{n=1}^{n(m_1)}]^\perp$ be a vector such that $g_2 \in \{g'_2\}$ $\vee\{e_n\}_{n=1}^{n(m_1)}$. By a formal repetition of the same argument, we can extend $\{e_n\}_{n=1}^{n(m_1)}$ to an ONS $\{e_n\}_{n=1}^{n(m_2)}$ such that

$$\bigvee\{g_1, g_2\} \subset \bigvee\{e_n\}_{n=1}^{n(m_2)}$$

and

$$\langle Te_n, e_n \rangle = a_n \quad \text{for } n = 1, 2, \dots, n(m_2).$$

By induction, we can construct an ONS $\{e_n\}_{n=1}^{\infty}$ such that $\langle Te_n, e_n \rangle = a_n$ for all $n = 1, 2, \dots,$ and

$$\bigvee\{e_n\}_{n=1}^{\infty} \supset \bigvee\{g_n\}_{n=1}^{\infty} = \mathcal{H};$$

that is, $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$, and it is completely apparent that $\operatorname{diag}(T) = \{a_n\}_{n=1}^{\infty}$ with respect to this basis.

(ii) Let $a'_n \in W_e(T)$ be any point such that $\operatorname{dist}[a_n, W_e(T)] = |a'_n - a_n|$. Since $\{a_n\}_{n=1}^{\infty}$ only accumulates on $W_e(T)$, it readily follows that $|a_n - a'_n| \to 0$ $(n \to \infty)$, and therefore the operator

$$K = \begin{pmatrix} a_1 - a'_1 & & & & & & & \\ & a_2 - a'_2 & & & & & & \\ & & \cdot & & \mathbf{O} & & & \\ & & & \cdot & & & & \\ & & & & \cdot & & & \\ & & \mathbf{O} & & & a_n - a'_n & & \\ & & & & & & \cdot & \\ & & & & & & & \cdot \\ & & & & & & & & \cdot \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_n \\ \cdot \\ \cdot \\ \cdot \end{matrix}$$

is compact (for any possible ONB $\{e_n\}_{n=1}^{\infty}$).

Clearly, this reduces our problem to show that if $a_n \in W_e(T)$ for all $n = 1, 2, \ldots$, then given $\varepsilon > 0$ there exists a compact operator K, with $||K|| < \varepsilon$, and an ONB $\{e_n\}_{n=1}^{\infty}$ such that $\operatorname{diag}(T+K) = \{a_n\}_{n=1}^{\infty}$ with respect to this basis.

Let $\{g_n\}_{n=1}^{\infty}$ be an ONB of $\mathcal{H}$, and let $a \in W_e(T)$ be a limit point of $\{a_n\}_{n=1}^{\infty}$. By proceeding as in the proof of (i), we can find an ONS $\{e'_j\}_{j=1}^{m_1}$ such that $g_1 \in \vee\{e'_j\}_{j=1}^{m_1}$ and $\langle Te'_j, e'_j\rangle = b_j$, with $|a - b_j| < \varepsilon$ for all $j = 1, 2, \ldots, m_1$.

Since $a_n \in W_e(T)$, we can find an ONS $\{e_n\}_{n=1}^{n(m_1)}$ such that $\langle Te_n, e_n\rangle = a'_n, |a'_n - a_n| < \varepsilon$ $(1 \leq n \leq n(m_1))$, $e_{n(j)} = e'_j$ and $a'_{n(j)} = b_j$ $(j = 1, 2, \ldots, m_1)$.

Let $g'_2 \in \mathcal{S}_1 \cap [\vee\{e_n\}_{n=1}^{n(m_1)}]^{\perp}$ be a vector such that $g_2 \in \{g'_2\}$

$\vee\{e_n\}_{n=1}^{n(m_1)}$. By a formal repetition of the same argument, we can extend $\{e_n\}_{n=1}^{n(m_1)}$ to an ONS $\{e_n\}_{n=1}^{n(m_2)}$ such that

$$\bigvee\{g_1, g_2\} \subset \bigvee\{e_n\}_{n=1}^{n(m_2)},$$
$$|\langle Te_n, e_n\rangle - a_n| < \varepsilon \quad \text{for } 1 \leq n \leq n(m_1),$$

and

$$|\langle Te_n, e_n\rangle - a_n| < \varepsilon/2 \quad \text{for } n(m_1) < n \leq n(m_2).$$

By an obvious inductive argument, we obtain an ONB $\{e_n\}_{n=1}^{\infty}$ and an increasing sequence $\{n(m_k)\}_{k=1}^{\infty}$ $(n(m_0) = 0)$ such that $\langle Te_n, e_n\rangle = a'_n$

satisfies

$$|a_n - a'_n| < \varepsilon/2^k \quad \text{for } n(m_k) < n \le n(m_{k+1}),\ k = 1, 2, \dots .$$

Since $a_n - a'_n \to 0$ $(n \to \infty)$, the operator

$$K = \begin{pmatrix} a_1 - a'_1 & & & & & & \\ & a_2 - a'_2 & & & & & \\ & \cdot & & \mathbf{O} & & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & \mathbf{O} & & & a_n - a'_n & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_n \\ \cdot \\ \cdot \end{matrix}$$

is compact, and

$$\operatorname{diag}(T + K) = \{a_n\}_{n=1}^{\infty}$$

with respect to the ONB $\{e_n\}_{n=1}^{\infty}$.

The proof of (ii) is now complete.

Finally, observe that the proof of (iii) follows by a minor modification of the proof of (ii). □

3. Examples and applications. 1. If B is an operator acting on $\mathbf{C}^n$ and $b_1 \in W(B)$, then we can find a unit vector e_1 such that $\langle Be_1, e_1\rangle = b_1$; we have

$$B = \begin{pmatrix} b_1 & * \\ * & B_1 \end{pmatrix} \begin{matrix} e_1 \\ (e_1)^{\perp} \end{matrix} .$$

If $b_2 \in W(B_1)$, then we can find $e_2 \perp e_1$ such that $\langle B_1 e_2, e_2\rangle = \langle Be_2, e_2\rangle = b_2$, and therefore

$$B = \begin{pmatrix} b_1 & * & * \\ * & b_2 & * \\ * & * & B_2 \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ [\vee\{e_1, e_2\}]^{\perp} \end{matrix} .$$

By induction, we can arbitrarily fix the diagonal entries $b_1, b_2, b_3, \dots, b_{n-1}$, so that $\langle Be_j, e_j\rangle = b_j$ $(j = 1, 2, \dots, n)$, and $b_{j+1} \in W(B_j)$, where B_j is the compression of B to the subspace spanned by $\{e_i\}_{i=1}^{j}$ $(j = 1, 2, \ldots, n-1)$. We obviously have

$$b_n = \operatorname{trace}(B) - \sum_{j=1}^{n-1} b_j.$$

Even for $n = 3$, it is very difficult to give a complete description of the possible diagonals of B.

2. In the infinite dimensional case, the above argument is doomed to failure: it can happen that the orthonormal system thus constructed is not an orthonormal basis!

For instance, if S is the unilateral shift, defined by $Se_n = e_{n+1}$ with respect to an ONB $\{e_n\}_{n=0}^{\infty}$ of $\mathcal{H}$, then $W(S')$ is the open unit disk for any compression S' of S to a subspace of finite codimension. If $\{a_n\}_{n=0}^{\infty}$ is a sequence of points in the unit disk such that

$$\sum_{n=0}^{\infty}(1 - |a_n|) < \infty,$$

then it is impossible to find an ONB $\{f_n\}_{n=1}^{\infty}$ such that $\operatorname{diag}(S) = \{f_n\}_{n=1}^{\infty}$ with respect to this basis.

REASON: If

$$S = \begin{pmatrix} a_1 & & & & & & \\ & a_2 & & & * & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & * & & & a_n & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{pmatrix} \begin{matrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_n \\ \cdot \\ \cdot \end{matrix},$$

then $Sf_n = a_nf_n + r_n$, where $r_n \perp f_n$, $||r_n|| = ||Sf_n - a_nf_n|| =$

$(1-|a_n|^2)^{1/2}$ (because S is an isometry), whence it readily follows that

$$K = S - \begin{pmatrix} a_1 & & & & & & \\ & a_2 & & & \mathbf{O} & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & \mathbf{O} & & & & a_n & \\ & & & & & & \cdot \\ & & & & & & & \cdot \end{pmatrix}$$

is a Hilbert-Schmidt operator and

$$\begin{aligned} ||K|| \le |K|_{H-S} = \left(\sum_{n=1}^{\infty} ||r_n||^2\right)^{1/2} &= \left(\sum_{n=1}^{\infty}[1-|a_n|^2]\right)^{1/2} \\ &< \sqrt{2}\left(\sum_{n=1}^{\infty}[1-|a_n|^2]\right)^{1/2} < \infty. \end{aligned}$$

It follows that

$$S = \text{ Diagonal (hence normal) } + K \text{ (compact)},$$

which is clearly impossible because S is a Fredholm operator of index -1.

On the other hand, it is very easy to check that if $\{a_n\}_{n=1}^{\infty}$ is any sequence of points in the open unit disk such that $\sum_{n=1}^{\infty}(1-|a_n|) = \infty$, then there exists an ONB $\{g_n\}_{n=1}^{\infty}$ such that S admits a lower triangular matrix with diag $(S) = \{a_n\}_{n=1}^{\infty}$ with respect to this basis.

3. If $\{a_n\}_{n=1}^{\infty}$ is a sequence of points in $W_e(T)$, and $\mathcal{J}$ is a normed ideal of compact operators strictly larger than the trace class, then given $\varepsilon > 0$ we can find $K \in \mathcal{J}$, with $|K|_{\mathcal{J}} < \varepsilon$, such that diag $(T+K) = \{a_n\}_{n=1}^{\infty}$ with respect to a suitable ONB of $\mathcal{H}$. (The proof follows by a refinement of the proof of (ii), as in [**2**].) However, the result is false if $\mathcal{J}$ is taken equal to the trace class: take $T = S$ (as in the above example), or T equal to a positive compact operator, not in the trace class, and $a_n = 0$ for all $n = 1, 2, \dots$ [**2**]. (The reader is referred to [**8**] for definition and properties of the normed ideals.)

4. The diagonal sequence of an operator has received some attention in the literature, beginning with the thesis of P. Fan [**1**] and [**2, 7**]. In [**4**], C.K. Fong answered a question of T.A. Gillespie by showing that for each bounded sequence $\{a_n\}_{n=1}^{\infty}$ there exists a quasinilpotent operator N such that $\operatorname{diag}(N) = \{a_n\}_{n=1}^{\infty}$ (with respect to a suitable ONB); furthermore, N can be chosen so that $N^4 = 0$.

Since the numerical range of the 2×2 complex matrix

$$Q = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

coincides with the closed unit disk, it readily follows from the first result of the theorem that N can actually be chosen equal to $rQ \otimes I$ for any $r > \sup_n |a_n|$. (Clearly, $(rQ \otimes I)^2 = 0$.)

Fong's result is based on a lemma of P. Fan [**1**]: if 0 is an interior point of $W_e(T)$, then $\operatorname{diag}(T) = \{0, 0, 0, \dots\}$ (with respect to a suitable ONB of $\mathcal{H}$). Thus, the first result is, essentially, an extension of this lemma.

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