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SOLUTIONS OF A NONLINEAR BOUNDARY LAYER PROBLEM ARISING IN PHYSICAL OCEANOGRAPHY

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ABSTRACT. We investigate a mathematical model for large scale ocean circulation. Under reasonable assumptions the partial differential equations reduce to the third order ordinary differential equation $\phi''' + \lambda(\phi\phi'' - (\phi')^2) + 1 - \phi = 0$ with either "no-slip" initial conditions $\phi(0) = 0$, $\phi'(0) = 0$ or "stress-free" initial conditional $\phi(0) = 0$, $\phi''(0) = 0$. The appropriate boundary condition in each case is $\phi(\infty) = 1$. We prove that for each $\lambda \geq (27/4)^{1/3}$, the no slip problem and the stress free problem each has at least one solution.

I. Introduction. We investigate the existence of solutions of the equation

$$\phi''' + \lambda(\phi\phi'' - (\phi')^2) + 1 - \phi = 0 \tag{1}$$

which satisfy either of the initial conditions

$$\phi(0) = \phi'(0) = 0 \tag{2}$$

or

$$\phi(0) = \phi''(0) = 0, \tag{3}$$

and subject to the boundary condition

$$\phi(\infty) = 1. \tag{4}$$

Equation (1) arises in the theory of physical oceanography and was developed by Ierley and Ruehr [**2**]. They derived a two-dimensional, one layer model for large scale ocean circulation with particular emphasis on the gulf stream. They assume that the steady state vorticity equation holds and restrict x and y to a rectangular region. Taking into account the east-west flow of the wind, they observe that a boundary layer

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is formed on the westernmost edge of the region. As is typical with boundary layer problems, they then introduce the similarity function ϕ and assume that the stream function is proportional to ϕ. Subsequently, they employ an appropriate matched asymptotic expansion procedure and arrive at the boundary value problem for ϕ, namely (1)–(2)–(4) or (1)–(3)–(4). Conditions (2) and (3) correspond to "no-slip" and "stress-free" conditions, respectively. Condition (4) is arrived at by matching the solutions across the edge of the boundary layer. The complete derivation of the problem (1)–(4) is given in [**2**]. A summary of results on similar models is found in [**1**].

The numerical studies [**2**] predict that solutions of the no-slip problem (1)–(2)–(3) behave as follows:

(i) for $\lambda \geq 0$ there is a unique solution,

(ii) for $-.7913 < \lambda < 0$ two solutions exist, and

(iii) for $\lambda < -.7913$ no solution exists.

Further, their computations show that solutions of the stress free problem (1)–(2)–(4) satisfy

(i) if $\lambda > 0$ then exactly two solutions exist,

(ii) if $\lambda = 0$ one solution exists,

(iii) if $-.29657 < \lambda < 0$ there are two solutions, and

(iv) if $\lambda < -.29657$ then no solution exists.

Recently, Lu and Troy [**3**] have rigorously investigated the existence of solutions of these problems. The results of that study are summarized in the following two theorems.

Theorem 1. *There is an* $\varepsilon_0 > 0$ *such that if* $|\lambda| < \varepsilon_0$, *then* (1)–(2)–(3) *has at least one solution and* (1)–(2)–(4) *has at least one solution. Furthermore, these solutions have the following asymptotic behavior*:

(i) $(\phi(x) - 1)e^{x/8} \to 0$ *as* $x \to \infty$

(ii) $|\phi'(x)e^{x/8}| \leq 3$ *and* $|\phi''(x)e^{x/8}| \leq 3$ *for all* $x \geq 0$,

(iii) *the solution of* (1)–(2)–(3) *satisfies* $\lim_{\lambda\to 0} |\phi(x) - \phi_0(x)| = 0$ *uniformly for* $x \in [0, \infty)$, *and* $\phi_0 = 1 - (e^{-x/2}/\sqrt{3}) \sin(\sqrt{3}x/2) - e^{-x/2} \cos(\sqrt{3}x/2)$;

(iv) *the solution of* (1)–(2)–(4) *satisfies* $\lim_{\lambda\to 0} |\phi(x) - \phi_1(x)| = 0$ *uniformly for* $0 \leq x \leq \infty$, $\phi_1 = 1 + e^{-x/2}((1/\sqrt{3})\sin(\sqrt{3}/2x) - \cos(\sqrt{3}x/2))$.

Theorem 2. *If* $\lambda \leq -9$, *then* (1)–(2)–(3) *has no solution. If* $\lambda \leq -(2)^{1/3}$, *then* (1)–(2)–(4) *has no solution.*

In Theorem 2, we have obtained results concerning the nonexistence of solutions of (1)–(2)–(3) and (1)–(2)–(4) for large negative λ. As described by Ierley (private communication) the nonexistence of solutions for large negative λ may play a role in the explanation why separation of the Gulf Stream occurs at a point considerably south of the observed "zero of wind stress curl."

In this paper we investigate the existence of solutions of (1)–(2)–(4) and (1)–(3)–(4) for large positive values of λ. In this range the fixed point argument used in Theorem 1 fails and we resort to a topological shooting argument. Our results are summarized in the following two theorems:

Theorem 3. (no-slip). *For each* $\lambda \geq (27/4)^{1/3}$ *there is at least one solution for the problem* (1)–(2)–(4).

Theorem 4. (stress-free). *For each* $\lambda \geq (27/4)^{1/3}$ *there is at least one solution of the problem* (1)–(3)–(4).

Comments. We have restricted our attention to the parameter range $\lambda \geq (27/4)^{1/3}$. For these values of λ a stability analysis of the steady state shows that solutions of both (1)–(2)–(4) and (1)–(3)–(4) must eventually become monotonic as $n \to \infty$. This property allows us to reduce the original third order problem via a Ricatti transformation to a second order nonautonomous equation. We then develop our topological shooting argument to prove the existence of the solutions described in our theorems. We prove our theorems separately in sections two and three. It remains an open problem to prove the existence of the second branch of solutions of the stress-free problem.

II. Proof of Theorem 3 (no-slip conditions). For the proofs of both of our theorems we find it convenient to make the transformation $u = \phi - 1$. Then (1) becomes

$$u''' + \lambda u'' - u + \lambda(uu'' - (u')^2) = 0. \tag{5}$$

The no-slip boundary conditions are

$$u(0) = -1, \qquad u'(0) = 0 \tag{6}$$

and

$$u(\infty) = 0. \tag{7}$$

As stated in the previous section our method of proof is to use a topological shooting argument. For this we assume that $u''(0) = \beta$. The first step in our analysis is to determine the behavior of solutions of (5)–(6)–(7) for large $\beta > 0$. We do this in the following technical lemma.

Lemma 1. *Let $\lambda > 0$. If $\beta > 7/3$, then $u' > 0$ and $u'' > 0$ on $(0,1)$, and $u(1) > 0$.*

Proof. From (5) and (6) it follows that

$$u'''(0) = -1. \tag{8}$$

Differentiating (5) twice and using (6) and (8), we find that

$$u''''(0) = 0 \tag{9}$$

and u satisfies

$$u^{(5)} + \lambda(1+u)u^{(4)} = \lambda(u^{(2)})^2 + u^{(2)} \tag{10}$$

with

$$u^{(5)}(0) = \beta + \lambda\beta^2. \tag{11}$$

From (9)–(10)–(11) it follows that $u^{(4)} > 0$ for $0 < \eta < 1$ as long as $u'' > 0$. Thus $u''' > -1$ as long as $u'' > 0$ on (0,1). Integrating, we conclude that

$$u'' > \beta - \eta > \frac{7}{3} - \eta > 0 \quad \text{on } [0,1].$$

Thus,

$$u' > \frac{7\eta}{3} - \frac{\eta^2}{2} \quad \text{and} \quad u > \frac{7\eta^2}{6} - \frac{\eta^3}{6} - 1 \tag{12}$$

for $0 < \eta < 1$. From (12) it follows that $u' > 0$ on (0,1) and $u(1) > 0$, completing the proof of the lemma. □

Lemma 1 will be of use in the definition of our shooting set. Before defining this set, however, we need to obtain more technical information about the behavior of solutions for small values of β. To do this we make the transformation $r = u'/u$. Then r satisfies the equation

$$(r'' + (3r + \lambda + \lambda u)r' + r^3 + \lambda r^2 - 1)\, u = 0 \tag{13}$$

with

$$r(0) = 0 \quad \text{and} \quad r'(0) = -\beta. \tag{14}$$

A note of clarification is in order concerning the existence, uniqueness and continuity properties of solutions of (13)–(14). The function $u(\eta)$ is assumed to satisfy (5) together with the initial conditions

$$u(0) = -1, \qquad u'(0) = 0, \qquad u''(0) = \beta. \tag{15}$$

We restrict our attention to solutions of (13) which satisfy $r(0) = 0$, $r'(0) = -\beta$. Thus, there is a one-to-one correspondence between solutions of (5)–(15) and solutions of (13)–(14). Furthermore, $r = u'/u$ must be continuously dependent upon η and β as long as $u < 0$ and u, u' and u'' are continuous in η and β. Similarly, the uniqueness of solutions of (13) is guaranteed over any interval $I \subseteq [0, \infty)$ over which $u < 0$. In order to proceed with our shooting arguments and analyze solutions of (5), we find it necessary to analyze (13) and make use of some special solutions as described below. An elementary analysis shows that the equation

$$r^3 + \lambda r^2 - 1 = 0 \tag{16}$$

has three real roots for each $\lambda \geq (27/4)^{1/3}$. One of the roots, r_0, is positive. The other two, r_1 and r_2, are negative and we assume that $r_2 \leq r_1 < 0$. Furthermore, we observe that $u_1(n) \equiv -e^{r_1 \eta}$ is a solution of the full nonlinear equation (5).

We are now prepared to proceed with our shooting arguments. We assume that $\lambda \geq (27/4)^{1/3}$ and define the set

$$A_1 = \{\beta > 0 \mid \exists\, \eta_\beta > 0 \text{ with } r(\eta_\beta) = r_1, \text{ and } r' < 0 \text{ for } 0 \leq \eta \leq \eta_\beta\}.$$

We summarize the properties of this set in the following.

Lemma 2. *The set A_1 is nonempty, open and bounded below with*

$$\beta_1 = \inf A_1 > 0.$$

Proof. Recall from Lemma 1 that if $\beta > 7/3$, then there exists $\eta_0 \in (0,1)$ such that $u' > 0$, $u'' > 0$ on $(0, \eta_0)$ and $u(\eta_0) = 0$. Therefore, since $u'' = (r' + r^2)u$, it follows that $r(\eta) < 0$ on $(0, \eta_0)$. Also, since $u(\eta_0) = 0$, there is a first $\bar{\eta} \in (0, \eta_0)$ such that $u(\bar{\eta}) = u_1(\bar{\eta})$, hence $u'(\bar{\eta}) \geq u_1'(\bar{\eta})$. That is, $r(\bar{\eta}) \leq r_1$. Thus, there is a first $\eta_1 \in (0, \bar{\eta}]$ such that $r(\eta_1) = r_1$ and $r'(\eta_1) \leq 0$. The uniqueness of solutions of (14) implies that $r'(\eta_1) < 0$. From these observations and (15) it follows that $A_1 \supseteq (7/3, \infty)$, and continuity implies that A_1 is open. Finally, we consider the case $\beta = 0$. Then $u'''(0) = -1$ and there is an interval $(0, \tilde{\eta})$ on which $u' < 0$ and $u < -1$. Thus, $r > 0$ on $(0, \tilde{\eta})$. From this, (14) and continuity it follows that there is a value $\beta_0 > 0$ such that if $0 < \beta < \beta_0$, then $r' = 0$ at some first $\hat{\eta} \in (0, \tilde{\eta})$, $r(\hat{\eta}) > r_1$ and $r''(\hat{\eta}) > 0$. Thus $(0, \beta_0) \cap A_1 = \phi$ and the lemma follows. □

The proof of Theorem 3 is completed in the following lemma.

Lemma 3. *Let $u''(0) = \beta_1$. Then $u' > 0$ for all $\eta > 0$ and $u(\infty) = 0$.*

Proof. Since $r'(0) = -\beta_1$ and $\beta_1 > 0$, then r is initially decreasing. Our goal is to prove that $r' < 0 \ \forall \eta > 0$. Suppose that $r'(\eta_0) = 0$ at some first $\eta_0 > 0$ with $r(\eta_0) \geq r_1$. If $r(\eta_0) = r_1$, then uniqueness of solutions implies that $r'(\eta_0) < 0$. From this and (14) we conclude that $r_1 < r(\eta_0) < 0$ and $r''(\eta_0) > 0$. Thus, by continuity it follows that if $\beta - \beta_1 > 0$ is sufficiently small then $r' = 0$ before $r = r_1$, contradicting the definition of β_1. Therefore, $r' < 0$ as long as $r > r_1$. As noted above, if $r(\eta_1) = r_1$ at some first $\eta_1 > 0$, then $r'(\eta_1) < 0$. Again, from continuity we conclude that if $\beta_1 - \beta > 0$ is sufficiently small, then $r(\eta_1) = r_1$ at some first $\eta_1 > 0$ with $r' < 0$ on $[0, \eta_1]$, contradicting the definition of β_1. Therefore, it follows that $r > r_1$ and $r' < 0 \ \forall \eta > 0$. Let $\gamma = r(1)$. Then $r_1 < u'/u < \gamma$ on $(1, \infty)$. An integration shows that $u(1)e^{\gamma(n-1)} < u(\eta) < u(1)e^{r_1(\eta-1)} \ \forall \eta \geq 1$ and the lemma follows. □

III. Proof of Theorem 4 (stress-free conditions). The stress free boundary value problem consists of the equation

$$u''' + \lambda u'' - u + \lambda(uu'' - (u')^2) = 0 \tag{17}$$

where

$$u(0) = -1, \qquad u''(0) = 0, \qquad u(\infty) = 0. \tag{18}$$

We set $u'(0) = \beta$. As in the proof of Theorem 3 we need to determine the behavior of solutions of (17)–(18) for large and small values of $\beta > 0$. Again we let $r = u'/u$. For each $\lambda \geq (27/4)^{1/3}$ define the set

$$A_2 = \{\beta \geq 0 \mid \ni \eta_\beta > 0 \quad \text{with } r(\eta_\beta) = r_1 \quad \text{and } r' < 0 \text{ on } [0, \eta_\beta]\}.$$

The crucial properties of this set are summarized in

Lemma 5. *The set A_2 is open, nonempty, and bounded below with* $\beta_2 = \inf A_2 > 0$.

Proof. Recall that $u(0) = -1$, $u'(0) = \beta$, $u''(0) = 0$. Thus, $r(0) = -\beta$, $r'(0) = -\beta^2$. In particular, at $\beta = -r_1$ we have $r(0) = r_1$, $r'(0) = -r_1^2$. Therefore, by continuity, if $\beta \in (-r_1 - \varepsilon, -r_1)$ and $\varepsilon > 0$

is sufficiently small, then $r' < 0$ until $r < r_1$. Thus, $(-r_1 - \varepsilon, -r_1) \subseteq A_2$ for sufficiently small $\varepsilon > 0$. It follows from uniqueness of solutions and continuity that A_2 is open. Next, we consider the value $\beta = 0$. Here $u'(0) = u''(0) = 0$ and $u'''(0) = -1$. Therefore, $r(0) = r'(0) = 0$, $r''(0) > 0$, hence $r(\eta) > 0$ on a small interval $(0, \delta)$. From this and continuity it follows that if $\beta > 0$ is sufficiently small, then $r' = 0$ at some first $\eta_1 > 0$, $r(\eta_1) > r_1$ and $r''(\eta_1) > 0$. Therefore $\beta \in A_2$ for $\beta > 0$ sufficiently small, hence $\inf A_2 > 0$ which completes the proof of the lemma. □

To complete the proof of Theorem 4 we consider the particular solution of (17) with initial conditions

$$u(0) = -1, \qquad u'(0) = \beta_2, \qquad u''(0) = 0. \tag{19}$$

For this solution we observe that $r_1 < r(0) = -\beta_2^2 < 0$ and $r'(0) = -\beta_2^2 < 0$ it follows exactly as in the proof of Lemma 3 that $r' < 0$ and $r > r_1 \ \forall \eta > 0$. Thus, $r_1 < r(\eta) = u'/u < \gamma = r(1) \ \forall \eta > \eta_1$. An integration shows that u satisfies

$$u(1)e^{\gamma(n-1)} < u(\eta) < u(1)e^{r_1(\eta-1)} \qquad \forall \eta > \eta_1.$$

Thus, $\lim_{n\to\infty} u(\eta) = 0$. This completes the proof of Theorem 2. □

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