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THE ONE-DIMENSIONAL DISPLACEMENT IN AN ISOTHERMAL VISCOUS COMPRESSIBLE FLUID WITH A NONMONOTONE EQUATION OF STATE

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The one-dimensional conservation laws of volume and momentum [**10**] may be written as

$$\frac{\partial V}{\partial t} = \frac{\partial v}{\partial x}, \qquad \frac{\partial v}{\partial t} = \frac{-\partial S}{\partial x} \tag{1.1}$$

where V is a specific volume, v the velocity and S the stress. In this paper it will be assumed that $S = P + q$ where P is the pressure and q is the part of the stress due to viscosity. It is also assumed that the internal energy is constant. Thus, it seems reasonable that

$$P = P(V), \qquad q = -\alpha(V)\frac{\partial v}{\partial x}, \qquad \alpha(V) > 0, \tag{1.2}$$

where $\alpha(V)$ is a coefficient of viscosity. Define the displacement, U, by

$$U(t,x) = \int_0^t v(s,x)\,ds. \tag{1.3}$$

Thus $U_t = v$, $U_{tt} = v_t$, $U_{tx} = v_x$, and

$$U_x(t,x) = \int_0^t v_x(s,x)\,ds = \int_0^t V_t(s,x)\,ds = V(t,x) - V_0(x). \tag{1.4}$$

Substituting this into the second of the equations of (1.1) and allowing for a body force yields the equation for U,

$$U_{tt}(t,x) + (P(V(t,x)))_x - (\alpha(V(t,x))U_{tx}(t,x))_x = g(t,x), \tag{1.5.1}$$

where $V(t,x) = V_0(x) + U_x(t,x)$ and g comes from the body force. Letting $U_1(x) = U_t(0,x)$ (1.3) yields the initial conditions

$$U(0,x) = 0, \tag{1.5.2}$$

$$U_t(0,x) = U_1(x). \tag{1.5.3}$$

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To this, we add boundary conditions of the form

$$U(t,1) = 0, \qquad P(V(t,0)) - \alpha(V(t,0))U_{xt}(t,0) = k_0(t), \tag{1.5.4}$$

or

$$P(V(t,1)) - \alpha(V(t,1))U_{xt}(t,1) = k_1(t),$$

$$P(V(t,0)) - \alpha(V(t,0))U_{xt}(t,0) = k_0(t), \tag{1.5.5}$$

where k_i is assumed to be a C^1 function satisfying

$$k_i(t) \geq \eta > 0. \tag{1.6}$$

For α, we have in mind functions of the form $\alpha(V) = CV^{-1}$, i.e., the Navier-Stokes viscosity, but the paper is developed for a much more general class of functions. It can be shown [**7**] that if the initial specific volume is Holder continuous with exponent $1/2$, and if the body force, g, is identically zero and the k_i are positive constants, then if P is nonincreasing, there exists a constant, C, which does not depend on time such that

$$|V(t,x) - V(t,y)| \leq C|x-y|^{1/2}, \tag{1.7}$$

for all $t > 0$ and $V(t,x)$ is bounded away from zero and infinity by constants that are independent of time. This effectively precludes the development of discontinuities in the specific volume. To avoid this situation and allow for phase changes in the material which correspond to discontinuities in the specific volume, we do not assume that P is monotone and we consider weak solutions to (1.5) in which, U_x and the initial specific volume, V_0, are not required to be continuous.

The purpose of this paper is to present some existence and uniqueness theorems for global weak solutions to problem (1.5). Such theorems should form the basis for the study of a partial differential equation but the theory of well posedness of equations like (1.5), especially when $\lim_{V\to\infty} \alpha(V) = 0$ and when the initial data is quite rough is presently not well developed [**10**]. The time dependent boundary conditions (1.5.4) and (1.5.5) create special difficulties and there are very few papers in the available literature that deal with them. Actually, it may be more interesting to let k_i depend on the velocity as well as t.

This follows naturally from an assumption that whatever produces the force acting on the ends of the material should have finite power. This possibly more interesting and more realistic case will be considered later.

Throughout the paper, V_0 will be a measurable function with

$$0 < a_0 \le V_0(x) \le b_0 < \infty, \tag{1.8}$$

$U_1 \in L^2(0,1)$, and $g \in L^2((0,T)\times(0,1))$. Throughout the paper C^1 is the restriction to $[0,T]$ of functions in $C^1(-\infty,\infty)$ and derivatives are weak.

The approximate problem. It is an exercise in integration by parts to verify that an appropriate variational formulation to the problem (1.5) is

$$\begin{aligned} -&\int_0^T\int_0^1 U_t(t,x)\psi_t(t,x)\,dx\,dt \\ -&\int_0^T\int_0^1 [P(V(t,x)) - b(t,x)]\psi_x(t,x)\,dx\,dt \\ +&\int_0^T\int_0^1 \alpha(V(t,x))U_{xt}(t,x)\psi_x(t,x)\,dx\,dt = \\ &\quad\int_0^T\int_0^1 (g(t,x) - b_x(t,x))\psi(t,x)\,dx\,dt, \end{aligned} \tag{2.1.1}$$

$$\begin{aligned} &U, U_t, U_x, U_{xt} \quad \text{are in } L^2((0,T)\times(0,1)), \\ &U_x(t,x) = \int_0^t U_{xt}(s,x)\,ds, \end{aligned} \tag{2.1.2}$$

where $b(t,x) = k_0(t)(1-x) + k_1(t)x$ and (2.1) holds for all $\psi \in C_0^\infty(0,T;E)$ for E a closed subspace of $H^1(0,1)$ containing $C_0^\infty(0,1)$. Here E will be either $H^1(0,1)$ to give (1.5.5) or $E = \{U \in H^1(0,1) : U(1) = 0\}$ to obtain (1.5.4). The initial condition is given by

$$U(t,\cdot) = v_0(t), \qquad U_t(t,\cdot) = v_1(t) \quad \text{a.e.,} \tag{2.1.3}$$

$$v_0 \in C(0,T;E), \qquad v_1 \in C(0,T;H), \tag{2.1.4}$$

$$\lim_{t\to 0} |v_1(t) - U_1|_H + ||v_0(t)||_E = 0, \tag{2.1.5}$$

where $H = L^2(0,1)$.

This section will deal with problem (2.1) under the assumptions

(2.2.1) $0 < \delta \leq \alpha(V) \leq M, \quad \alpha$ continuous,

(2.2.2) P is bounded and globally Lipschitz.

There exist constants a_1, b_1 with $0 < a_1 < 1 < b_1$ such that

(2.2.3)
$$\begin{aligned} &P(V) - \eta < 0 \quad \text{if } V > b_1, \\ &P(V) - (||k_0||_\infty + ||k_1||_\infty) > 0 \quad \text{if } V < a_1, \end{aligned}$$

where η is the constant of (1.6). Also,

(2.2.4) $U_1 \in H, g \in L^2((0,T) \times (0,1))$.

Theorem 1. *If* (2.2) *holds, there exists a unique solution to* (2.1).

This theorem is proved in [**8**]. In fact, it is not necessary to have $k_i \in C^1$. It suffices to take k_i bounded and measurable.

Lemma 1. *Define*

$$W(t,x,V) = -\int_1^V [P(s) - b(t,x)]\, ds. \tag{2.3}$$

Then there exists a constant J independent of t and x such that

$$W(t,x,V) \geq -J. \tag{2.4}$$

Proof. $\partial W/\partial V = -P(V) + b(t,x)$. The result follows from (2.2.3) because $\partial W/\partial V > 0$ if $V > b_1$ and $\partial W/\partial V < 0$ if $V < a_1$.

Corollary 1. *Let $\{P_m\}$ be a collection of functions satisfying* (2.2.3) *and let $W_m(t,x,V)$ be given by* (2.3) *with P_m in place of P. Also assume $P_m(V) = P(V)$ for all $V \in [a_1, b_1]$. Then $W_m(t,x,V) \geq -J$ where J is independent of m,t,x.*

Lemma 2. *Let q, q_t both be in $L^2((0,T)\times(0,1))$. Then there exists a measurable set, D, with $m(D)=0$ such that for $x \notin D$, $t \to q(t,x)$ is equal to a continuous function a.e. t and if $q(0,x)$ is the value of this continuous function at $t=0$,*

$$q(t,x) = q(0,x) + \int_0^t q_t(s,x)\,ds \quad \text{a.e. } t. \tag{2.5}$$

It is convenient to describe an abstract formulation of problem (2.1). To do this, identify H and H', so that $E \subseteq H = H' \subseteq E'$. For $U \in E$ and $t \in [0,T]$, let $Q(U)$ and $N(t,\cdot)$ be operators mapping E to E' given by

$$\langle Q(U)w, v\rangle = (\alpha(V)w_x, v_x)_H, \tag{2.6.1}$$

$$\langle N(t,U), v\rangle = -(P(V) - b(t,\cdot), v_x)_H, \tag{2.6.2}$$

where $V = U_x + V_0$. Let $\mathcal{V} = L^2(0,T;E)$ so $\mathcal{V}' = L^2(0,T;E')$ [**4**]. For $U \in \mathcal{V}$, let $Q(U)$ and N be operators mapping $\mathcal{V}$ to $\mathcal{V}'$ defined by

$$(Q(U)W)(t) = Q(U(t))W(t), \tag{2.7.1}$$

$$NW(t) = N(t, W(t)). \tag{2.7.2}$$

Also, for $h \in L^1(0,T;E')$, h' may be defined as an E' valued distribution according to the rule

$$h'(\psi) = -\int_0^T h(t)\psi'(t)\,dt \tag{2.8}$$

for all $\psi \in C_0^\infty(0,T)$. The following theorem is known [**8**].

Theorem 2. *Suppose U is the solution of (2.1) and let $v(t) = U(t,\cdot)$. Then*

$$v'(t) = U_t(t,\cdot) \quad \text{a.e.}, \tag{2.9.1}$$

$$v'' \in \mathcal{V}', \qquad v, v' \in \mathcal{V}, \tag{2.9.2}$$

$$v'' + Nv + Q(v)v' = f, \tag{2.9.3}$$

where $f(t) = g(t,\cdot) - b_x(t,\cdot)$.

$$v(0) = 0, \qquad v'(0) = U_1. \tag{2.9.4}$$

In fact, Theorem 1 was proved [**6,8**] by taking a measurable representative of the solution of (2.9.3), (2.9.4).

Lemma 3. *Let* v *solve* (2.9) *and let* $L_0 = ||k_0||_\infty + ||k_1||_\infty$, $L_1 = ||k_0'||_\infty + ||k_1'||_\infty$. *Then*

$$|v'(t)|_H^2 \le C_1^2 \tag{2.10}$$

where

(2.11)
$$\begin{aligned} C_1^2 = &e^T(|U_1|_H^2 + ||f||^2_{L^2(0,T;H)} + 2\int_0^1 W(0,x,V_0(x))\,dx + 2J \\ &+ \min\left(\frac{2TL_0^2}{\delta} + 4L_0||V_0||_{L^1(0,1)}, \frac{T^3}{2\delta}L_1^2 + 2TL_1||V_0||_{L^1(0,1)}\right)\end{aligned}$$

Proof. Multiply (2.9.3) by v' and take $\int_0^t$ of both sides. This yields

(2.12)
$$\begin{aligned} &|v'(t)|_H^2 + 2\int_0^t\int_0^1 \delta|V_t(s,x)|^2\,dx\,ds - 2\int_0^t\int_0^1 b_t(s,x)V(s,x)\,dx\,ds \le \\ &2J + 2\int_0^1 W(0,x,V_0(x))\,dx + ||f||_{L^2(0,T;H)} + \int_0^t |v'(s)|_H^2\,ds + |U_1|_H^2.\end{aligned}$$

Now $V(s,x) = V_0(x) + \int_0^s V_t(\tau,x)\,d\tau$ a.e. Substituting this into the third term of (2.12), using Fubini's theorem, $(V(t,x) = U_x(t,x) + V_0(x))$ and the definition of $b(t,x)$, it follows that the sum of the second and third terms in (2.12) is bounded below by

$$\max\left(\frac{-T^3}{2\delta}L_1^2 - 2TL_1||V_0||_{L^1(0,1)}, \frac{-2T}{\delta}L_0^2 - 4L_0||V_0||_{L^1(0,1)}\right).$$

Gronwall's inequality yields (2.10) and (2.11).

Corollary 2. *For U the solution of* (2.1),

$$|U_t(t,\cdot)|_H^2 \le C_1^2 \quad a.e.\ t. \tag{2.13}$$

From now on, assume

$$E \supseteq \{U \in H^1(0,1) : U(1) = 0\}. \tag{2.14}$$

Define

(2.15.1)

$$\begin{aligned} q(t,x) = \int_0^x U_t(t,z)\,dz - \beta(V(t,x)) \\ - \int_0^t k_0(s)\,ds + \eta t - \int_0^t \int_0^x g(s,z)\,dz\,ds, \end{aligned}$$

where

$$\beta(V) = \int_1^V \alpha(s)\,ds. \tag{2.15.2}$$

Lemma 4. *If U is the solution of* (2.1),

$$q_t(t,x) = \eta - P(V(t,x)), \tag{2.16}$$

where the derivative is taken in the sense of distributions.

Lemma 5. *Let*

$$\begin{aligned} C_2 = 2(C_1 + ||k_0||_\infty T + \eta T + ||g||_{L^2((0,T)\times(0,1))} \\ + |U_1|_H + |\beta(a_0)| + |\beta(b_0)| + |\beta(a_1)| + |\beta(b_1)|). \end{aligned} \tag{2.17}$$

Then there exists a set of measure zero, D, such that for $x \notin D$

$$|\beta(V(t,x))| < \frac{3C_2}{2} \quad \text{for all } t \in [0,T]. \tag{2.18}$$

(*Here* $V(t,x) = V_0(x) + U_x(t,x)$ *with* U *the solution of* (2.1).)

Proof. From Lemma 2 there exists a set of measure zero, D, such that for $x \notin D$,

(2.19.1)
$$q(t,x) = q(0,x) + \int_0^t [\eta - P(V(s,x))]\, ds \quad \text{a.e. } t,$$

(2.19.2)
$$q(0,x) = \int_0^x U_1(z)\, dz - \beta(V_0(x)).$$

Fix $x \notin D$ and define

(2.20)
$$\tilde{q}(t,x) = q(0,x) + \int_0^t [\eta - P(V(s,x))]\, ds, \quad \text{all } t \in [0,T].$$

It follows from Corollary 2 and (2.15) that for a.e. t,

(2.21.1)
$$|\tilde{q}(t,x) + \beta(V(t,x))| \le C_1 + ||k_0||_\infty T + \eta T + ||g|| \le \frac{C_2}{2},$$

(2.21.2)
$$|q(0,x)| \le |U_1|_H + |\beta(a_0)| + |\beta(b_0)| < \frac{C_2}{2}.$$

If $\tilde{q}(t,x) > C_2$ for some t, then thanks to (2.21.2), there exists an interval $[a, a+\epsilon]$ such that $\tilde{q}(a,x) = C_2$ but $\tilde{q}(t,x) > C_2$ for all $t \in (a, a+\epsilon)$. Thus

(2.22)
$$\tilde{q}(t,x) = C_2 + \int_a^t [\eta - P(V(s,x))]\, ds.$$

It follows that in a subset of $[a, a+\epsilon]$ having positive measure,

(2.23.1)
$$\eta - P(V(t,x)) > 0,$$

and (2.21.1) holds. This implies that in this subset of $[a, a+\epsilon]$,

(2.23.2)
$$\beta(V(t,x)) < -\frac{C_2}{2} \le \beta(a_1).$$

Hence $V(t,x) < a_1$ and so from (2.2.3), $\eta - P(V(t,x)) < 0$, contradicting (2.23.1). It follows that $\tilde{q}(t,x) \leq C_2$ for all $t \in [0,T]$. A similar argument shows $\tilde{q}(t,x) \geq -C_2$ for all $t \in [0,T]$. Since $|\tilde{q}(t,x)| \leq C_2$ for all $t \in [0,T]$, (2.21.1) implies

$$|\beta(V(t,x))| < \frac{3C_2}{2} \quad \text{a.e. } t. \tag{2.24}$$

Thanks to (2.1.2), (2.24) holds for all $t \in [0,T]$.

Reviewing (2.17) and (2.11), it is easily seen that $|\beta(V(t,x))|$ is no larger than

(2.25)

$$\begin{aligned} &||k_0||_\infty + C_3 + \min((C_4L_0)^{1/2} + C_5L_0\delta^{-1/2}, (C_6L_1)^{1/2} + C_7L_1\delta^{-1/2}) \\ &\qquad = M(L_0, L_1, \delta) + C_3 + ||k_0||_\infty, \end{aligned}$$

where $C_3 - C_7$ depend only on T, $|U_1|_H$, V_0, J, g, and η.

The exact problem. In this section existence and uniqueness to problem (2.1) is obtained under the assumptions:

(3.1.1) $\alpha(V) > 0, \quad \alpha$ is continuous on $(0,\infty)$,

(3.1.2) α is nonincreasing,

(3.1.3) $\displaystyle\int_0^1 \alpha(s)\,ds = \infty$,

(3.1.4) P is Lipschitz continuous on every finite interval bounded away from 0

(3.1.5) $P(V) - \eta < 0 \quad \text{if } V > b_1$,

(3.1.6) $P(V) - (||k_0||_\infty + ||k_1||_\infty) > 0 \quad \text{if } V < a_1$.

For β defined in (2.15.2) and $C_3 - C_7$ given in (2.25), assume also that there exists $b_2 \geq \max(b_0, b_1)$ with

$$||k_0||_\infty + C_3 + M(L_0, L_1, \alpha(b_2)) \leq \beta(b_2). \tag{3.2}$$

Clearly (3.2) holds if k_0 and k_1 are constants and $\lim_{b\to\infty} \beta(b) = \infty$.

For $0 < a < \min\,(a_0, a_1)$ and $b > \max\,(b_0, b_1)$ define

$$\alpha_{ab}(V) = \begin{cases} \alpha(V) & \text{if } V \in [a,b] \\ a(b) & \text{if } V > b \\ \alpha(a) & \text{if } V < a \end{cases} \tag{3.3.1}$$

$$P_{ab}(V) = \begin{cases} P(V) & \text{if } V \in [a,b] \\ P(b) & \text{if } V > b \\ P(a) & \text{if } V < a. \end{cases} \tag{3.3.2}$$

Now (3.1.5) and (3.1.6) hold for all P_{ab}. Let β_{ab} be defined by (2.15.2) with α_{ab} in place of α. Because of (3.1.3) there exists $a > 0$ such that

$$\beta_{ab_2}(a) < -C_3 - M(L_0, L_1, \alpha(b_2)) - ||k_0||_\infty. \tag{3.4}$$

Let a_2 be such that (3.4) holds.

Let U be the unique solution to problem (2.1) with α and P replaced with $\alpha_{a_2b_2}$ and $P_{a_2b_2}$, respectively. Then for $V(t,x) = V_0(x) + U_x(t,x)$, Lemma 5 implies that (2.25) holds with $\beta_{a_2b_2}$ in place of β. Now (3.4) and (3.2) imply that $a_2 < V(t,x) < b_2$ a.e. Therefore, U is a solution of problem (2.1). In summary, we have proved the following:

Theorem 3. *Let V_0 be a measurable function with $a_0 \le V_0(x) \le b_0$ and let $U_1 \in L^2(0,1)$, $g \in L^2((0,T) \times (0,1))$, k_0 and k_1 are C^1, and suppose α and P satisfy* (3.1) *and* (3.2) *with $0 < \eta \le \min(k_0, k_1)$. Let E be a closed subspace of $H^1(0,1)$ satisfying* (2.14). *Then there exists a unique solution to problem* (2.1) *having the property that the specific volume, $V_0(x) + U_x(t,x)$ is bounded away from* 0 *and* ∞.

It seems worth noting that (3.1.2) was included mainly for convenience. A simple modification of the above argument would give Theorem 3 with (3.1.2) replaced with: There exists $b > 0$ such that α is nonincreasing on (b, ∞) and $\alpha(b) = \inf\{\alpha(V) : V \in (0,b)\}$. There are likely many other generalizations possible, but (3.1.1)–(3.1.3) includes the Navier-Stokes type viscosity of the form $\alpha(V) = CV^{-1}$.

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