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STABILITY OF INTERFACES WITH VELOCITY CORRECTION TERM

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ABSTRACT. We consider two-dimensional solidification problems in which both surface tension and dynamical undercooling are incorporated into the temperature condition at the interface. Our study indicates that the presence of a dynamical undercooling term does not alter the stability-instability spectrum. That is, a wave mode is unstable in the presence of the dynamical undercooling and surface tension terms if and only if it is unstable in the presence of the surface tension term alone. The *magnitude* of the exponential growth or decay, however, is strongly influenced by the presence of dynamical undercooling. Within the unstable mode regime, dynamical undercooling tends to decrease the magnitude of the instability. In this sense, then, it is a stabilizing influence. Within the stable mode region, the influence of the dynamical undercooling depends on the magnitude of the velocity.

1. Introduction. Mathematical models of solidification which include the Gibbs-Thomson equilibrium conditions have been studied for quite some time [**8,9**]. Here we study the shape stability of planar fronts in the context of a simplified nonequilibrium model which includes dynamical undercooling (interface attachment kinetics) [**1,4,6,11,12**, and for a more general model, **7**]. Specifically let the curve in $\mathbf{R}^2$, described by

$$x = R(y,t) \quad \text{or} \quad S(x,y,t) \equiv x - R(y,t) = 0 \tag{1.1}$$

be defined as the interface which separates the liquid and solid phases. With T denoting temperature (scaled so that it vanishes for a planar interface at equilibrium), $\hat{n} \equiv \overline{\nabla} s/|\nabla s|$ the unit normal (in the direction from solid to liquid), and ν the normal component of the velocity (positive if motion of the interface is toward the liquid side), one may

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write the system of equations as

$$(1.2) \qquad T_t = \Delta T, \qquad T_t = \Delta t$$

$$\nu = -q(T + \gamma\kappa(s)), \qquad \frac{R_t}{(1+R_y^2)^{1/2}} = -q\left[T - \frac{\gamma R_{yy}}{(1+R_y^2)^{3/2}}\right]$$

$$(1.3) \qquad \text{on } S = 0$$

$$(1.4) \quad \overline{\nabla} T \cdot \frac{\overline{\nabla}_S}{|\nabla_S|} = -L\nu, \qquad T_x - T_y R_y = -LR_t \qquad \text{on } S = 0.$$

Here L, γ, K are latent heat, surface tension and curvature, respectively. The constant q incorporates the surface tension and relaxation time constants and must be positive. These equations are subject to the initial and boundary conditions

$$(1.5) \qquad T(x,y,0) = T_0(x,y); \qquad T(x,y,t) \to T_\infty \quad \text{as } x \to \infty.$$

Thus, when $q \approx \infty$ then, in order to obtain finite normal velocities, one must have the boundary temperature maintained at the equilibrium, Gibbs-Thomson value $-\gamma\kappa(s)$. On the other hand, if $q < \infty$, then the boundary temperature from (1.3) is

$$(1.6) \qquad T(s) = -\gamma\kappa(s) - \nu/q;$$

i.e., it is undercooled by the dynamical quantity ν/q.

When $q = \infty$, the shape stability of solidifying fronts has been carried out in various geometries [**8**]. In the planar case independent linear analyses [**5,10**] established the stability of evolving planar fronts to cos my bumps if

$$(1.7) \qquad \frac{\gamma m^2}{\nu L} > 1.$$

In [**5**] the stability calculation was based on a planar solution with $t^{1/2}$ growth; i.e.,

$$(1.8) \qquad R(t) = \alpha t^{1/2}$$

where α is the unique solution of

$$(1.9) \qquad \alpha e^{\alpha^2/4} \int_{\alpha/2}^{\infty} e^{-y^2}\, dy = -\frac{T_\infty}{L}$$

when the ambient temperature is such that $-T_\infty/L < 1$ or $L + T_\infty > 0$ (no $t^{1/2}$ solution otherwise) [**2**]. Similar results are obtained for $q < \infty$ [**4**] with only a change in the temperature profile. For small ambient temperatures of this sort we also established [**3**] when $q = \infty$ that all planar solutions lead asymptotically to this $t^{1/2}$ solution. Moreover, for larger ambient temperatures, $-T_\infty/L > 1$, we showed [**3**] that planar solutions of problem (1.1–1.5) can blow up in finite time. It is interesting then, in this context, that the first result in section 2 is to show that when $q < \infty$ and $-T_\infty/L > 1$ (large ambient temperatures) we obtain an advancing travelling front solution; i.e.,

$$R(t) = at \tag{1.10}$$

and the velocity is selected by the data through

$$a = -q(T_\infty + L). \tag{1.11}$$

This suggests that the equilibrium model ($q = \infty$) breaks down for large ambient temperatures and must be replaced by the nonequilibrium version ($q < \infty$). From another viewpoint, when $q < \infty$ the $t^{1/2}$ growth changes to t growth when $-T_\infty/L$ crosses 1. We point out that this transition from $t^{1/2}$ to t growth when the ambient temperature $-T_\infty/L$ crossed 1 was also observed from numerical simulation of a spherical nonequilibrium model [**11**]. In sections 3, 4, 5 we study the linearized stability of this $q < \infty$, $-T_\infty/L > 1$ planar solution with respect to shape perturbations and find that precisely the same modes are unstable as in the $q = \infty$ model but that the magnitude of the exponential growth and decay is strongly influenced by the presence of the dynamical undercooling. Similar results are mentioned in [**12**] and worked out [**6**] in the case $-T_\infty/L \approx 1$. We remark that in a more general model [**7**] it was observed that it is possible to have the onset of oscillatory instabilities before those which are described here.

We thank the referee for informing us that Strain [**13**] recently also treated oscillatory instabilities.

2. The planar traveling wave solution. We propose a solution to (1.2)–(1.4) of the type

$$T(x, y, t) \equiv \tau(x - at), \qquad R(y, t) \equiv at, \qquad a > 0. \tag{2.1}$$

Defining the new variable $\xi = x - at$, one obtains the following transformation of (1.2)–(1.4):

$$\tau'' + a\tau' = 0 \tag{2.2}$$

$$a = -q\tau \qquad at \quad \xi = 0 \tag{2.3}$$

$$\tau' = -La \tag{2.4}$$

$$\tau \to \tau_\infty \quad \text{as } x \to \infty \tag{2.5}$$

with primes denoting differentiation with respect to ξ.

One may readily verify that

$$T(x, y, t) = T_\infty + (T_b - T_\infty)e^{-a(x-at)} \tag{2.6}$$

is the solution to (2.2)–(2.5) with $T_b = T_\infty + L$ which is clearly the value of the temperature at the interface. Furthermore, the consistency relation

$$a = -qT_b \tag{2.7}$$

is forced on the velocity a.

We note that the positivity of the velocity a, along with substitution of (2.6) into (2.4) implies

$$T_\infty + L < 0. \tag{2.8}$$

That is, the temperature at infinity must be sufficiently low compared with the latent heat.

The case $q = \infty$ is the more traditional modification of the Stefan model in which (1.3) is replaced by

$$T = -\gamma\kappa \tag{2.9}$$

while (2.3) is replaced by $\tau = 0$ (since curvature is zero for the plane wave). In this case the traveling wave $\tau(\xi) = T_\infty + Be^{-a\xi}$ must satisfy $T_\infty + B = 0$ and $aB = -La$ so that a is arbitrary. For finite q, the velocity a is selected by (2.7).

3. Stability of a planar solution. Changing to moving coordinates

$$x' = x - at, \qquad y' = y, \qquad t' = t \tag{3.1}$$

we may now rewrite the temperature profile τ as

$$T^p(x') = T_\infty + (T_b - T_\infty)e^{-ax'} \tag{3.2}$$

with $T_b \equiv T_\infty + L$ and $a = -qT_b$.

Transforming equations (1.2)–(1.4) via the moving coordinates (3.1), one obtains (upon dropping the primes)

$$T_t = \Delta T + aT_x \qquad x > r(y,t) \tag{3.3}$$

$$\frac{a + r_t}{(1 + r_y^2)^{1/2}} = -q\left[T - \frac{\gamma r_{yy}}{(1 + r_y^2)^{3/2}}\right] \qquad x = r(y,t) \tag{3.4}$$

$$T_x - r_y T_y = -L(a + r_t) \tag{3.5}$$

$$T \to T_\infty \quad \text{as } x \to \infty \qquad T(x,y,0) = T_0(x,y) \tag{3.6}$$

where $r(y,t) \equiv R(y,t) - at$ has been defined.

We now assume a perturbed plane wave of the form

$$T(x,y,t) = T^p(x) + \delta\tilde{T}(x,y,t) \tag{3.7}$$

$$r(y,t) = 0 + \delta\tilde{r}(y,t) \tag{3.8}$$

where x, y, t are the moving coordinates and 0 is the planar front in this coordinate system. Substituting (3.7), (3.8) into (3.3)–(3.6), one obtains (upon dropping the tildes) the linearized equations

$$T_t = \Delta T + aT_x, \qquad x > 0 \tag{3.9}$$

$$r_t(y,t) = -q[r(y,t)T_x^p(0) + T(0,y,t) - \gamma r_{yy}(y,t)], \qquad x = 0 \tag{3.10}$$

$$T_{xx}^p(0)r + T_x(0,y,t) = -Lr_t(y,t), \qquad x = 0. \tag{3.11}$$

We now assume the following sinusoidal form for the perturbation (3.7), (3.8)

$$T(x,y,t) = \rho\hat{T}(x)e^{\sigma(m)t}\cos my, \tag{3.12}$$

$$r(y,t) = \rho e^{\sigma(m)t}\cos my, \tag{3.13}$$

i.e., we consider flow in a channel with zero flux boundary conditions. Substituting (3.12) and (3.13) into (3.9)–(3.11), one obtains (upon dropping the hats)

$$\sigma T = T'' - m^2 T + aT', \qquad x > 0 \tag{3.14}$$

$$\sigma = -q[-aL + T(0) + \gamma m^2] \tag{3.15}$$

$$a^2 L + T'(0) = -L\sigma, \qquad x = 0, \quad T \to 0 \quad \text{as } x \to \infty. \tag{3.16}$$

The solution to the differential equation (3.14) such that it satisfies the asymptotic condition (3.17) is

$$T(x) = Ae^{-|n|x} \; |n| \equiv \frac{a + (a^2 + 4(\sigma + m^2))^{1/2}}{2}. \tag{3.18}$$

The interfacial conditions (3.16), (3.17) imply the identity

$$\frac{1}{2}\left[a - \sqrt{a^2 + 4(\sigma + m^2)}\right]\left[aL - \frac{\sigma}{q} - \gamma m^2\right] = L\sigma + a^2 L. \tag{3.19}$$

The existence of a positive σ solution to (3.19) for a particular wave number m implies the instability of the mode. From (3.19) we can see immediately that a *necessary* condition for instability is:

$$aL - \frac{\sigma}{q} - \gamma m^2 > 0. \tag{3.20}$$

We note, first of all, that as $q \to \infty$ (neglecting the dynamical undercooling) this approaches the known instability criterion [**9,10**]

$$p \equiv \frac{\gamma m^2}{aL} < 1. \tag{3.21}$$

Secondly, the presence of a finite q in (3.20) makes it more difficult to satisfy this criterion. However, this necessary condition is not sufficient and hence one cannot yet infer that dynamical undercooling is a stabilizing factor.

We now give the generalization of the analysis in [**5,10**] of instabilities associated with real σ's. A separate, independent analysis for complex σ's is given in [**13**].

4. Analysis of spectrum (unstable regime). Our basic goal is to find the general properties of the solution σ to (3.19) and to examine its dependence on m and q. Writing identity (3.19) as

(4.1)
$$g(m,\sigma) \equiv -\sqrt{a^2+4(\sigma+m^2)} = \frac{(2Lq+a)\sigma+a^2qL+a\gamma m^2q}{-\sigma+aLq-\gamma m^2q} \equiv f(m,\sigma),$$

provided $\sigma \neq aLq - \gamma m^2 q$, we must determine the value σ_i at which the curves f and g intersect. An instability will occur if this value of σ_i is positive.

The minimum value of σ in the domain of g will be denoted by

$$\sigma_c = \frac{-a^2}{4} - m^2. \tag{4.2}$$

The function f has vertical and horizontal asymptotes of

$$\sigma_\nu = q(aL - \gamma m^2), \tag{4.3}$$
$$f_h = -(2qL + a), \tag{4.4}$$

respectively (see Figure 1). We note that (4.3) places an upper bound on the value of σ_i. It is natural to consider separately the cases in which

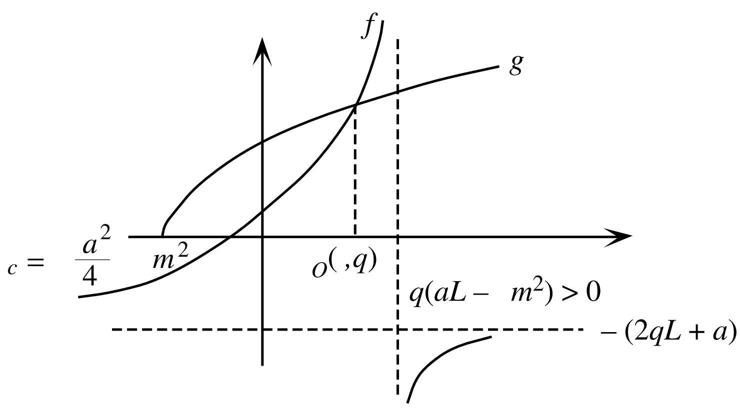

FIGURE 1. Intersection of curves for $p < 1$.

the vertical asymptote is on the left or right half plane. We define the parameters

$$p = \frac{\gamma m^2}{La}, \qquad r \equiv \frac{m^2}{a^2}, \qquad s \equiv \frac{\gamma a}{L} = p/r \tag{4.5}$$

and consider first the case $p < 1$.

With f and g defined by (4.1) we let

$$h(m,\sigma) \equiv g(m,\sigma) - f(m,\sigma). \tag{4.6}$$

An instability will occur for a particular value of m and the physical parameters if and only if σ_i, the point of intersection of f and g, is positive. By looking at the left portion of Figure 1, one sees that this will occur if $h(m,0) > 0$. Similarly, stable growth will occur if σ_i is negative, i.e., $h(m,0) < 0$.

Using (4.5) we may rewrite $h(m,0)$ as

$$h(m,0) = a\left\{(1+4r)^{1/2} - \frac{1+p}{1-p}\right\} \tag{4.7}$$

so that the curve $h(m,0) = 0$ which separates stability and instability is just the parabola

$$s = (1-p)^2, \qquad 0 < p < 1 \tag{4.8}$$

with the large s side describing the stable part of the spectrum, as shown in Figure 2.

An interesting aspect of (4.8) is the absence of a role by q, the dynamical undercooling. That is, in the region $0 < p < 1$, the stability or instability of each mode described by m is unaffected by the magnitude of q.

The next question we address within the $p < 1$ regime is the dependence of σ_i on q. That is, how does the actual growth rate of the instability depend on q? From Figure 1 we see that as q decreases (i.e., increased dynamical undercooling) the horizontal asymptote moves upward while the vertical asymptote moves to the left. In order to

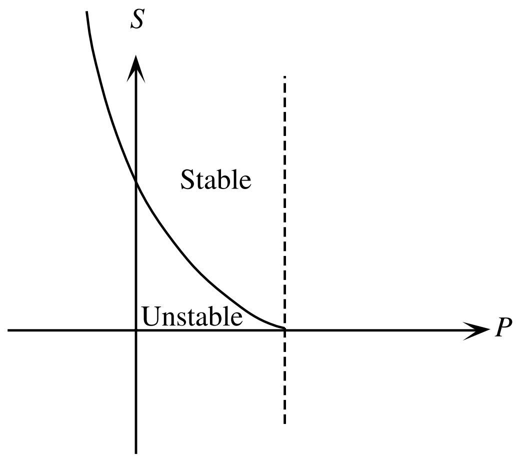

FIGURE 2.

see that the intersection σ_i moves to the left as g decreases, we may rewrite (4.1) as

$$-\sqrt{a^2 + 4(\sigma + m^2)} = \frac{(2L\sigma + a^2 L + a\gamma m^2) + a\sigma/q}{(La - \gamma m^2) - \sigma/q}. \tag{4.9}$$

The left-hand side (i.e., g) is independent of q, while the right-hand side (i.e., f) is a decreasing function of q for $0 < \sigma < \sigma_\nu$. Hence the intersection, σ_i, occurs at a smaller value (see Figure 3). Hence, in the unstable regime, the dynamical undercooling has a stabilizing effect in that the exponential growth rate σ_i becomes smaller while still remaining positive. The results are summarized below.

Theorem. *If p satisfies $0 < p < 1$, then the plane wave solution* (2.6) *to* (1.2)–(1.4) *is stable if $s > (1-p)^2$ and unstable if $s < (1-p)^2$ independently of q. Furthermore, σ_i varies monotonically with q and is bounded above by $\sigma_i(\infty)$. If $\gamma = 0$ ($p = s = 0$), then the solution is unstable with σ_i varying monotonically with q.*

5. Analysis of spectrum (stable regime). We consider now the regime $p > 1$. Once again, we seek a solution σ_i to (4.1), which will necessarily be negative if it exists at all (see Figures 4a, b, c).

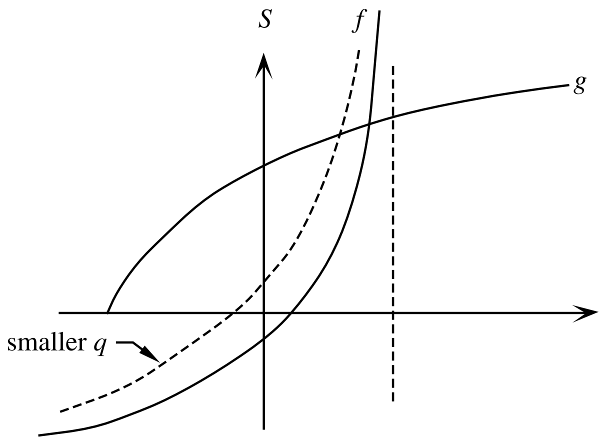

FIGURE 3.

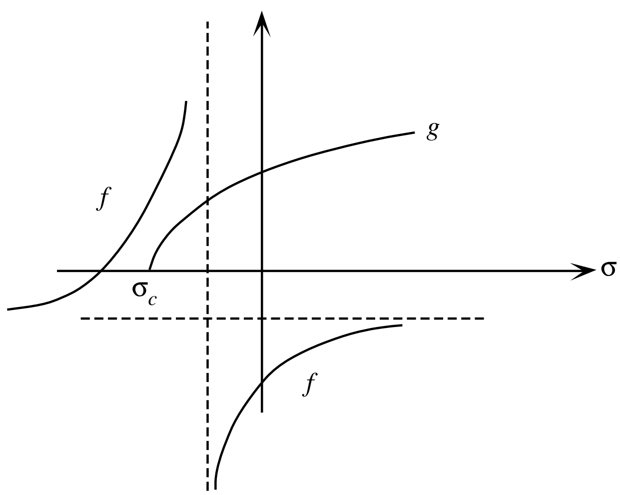

FIGURE 4a. No intersection and consequently no solution.

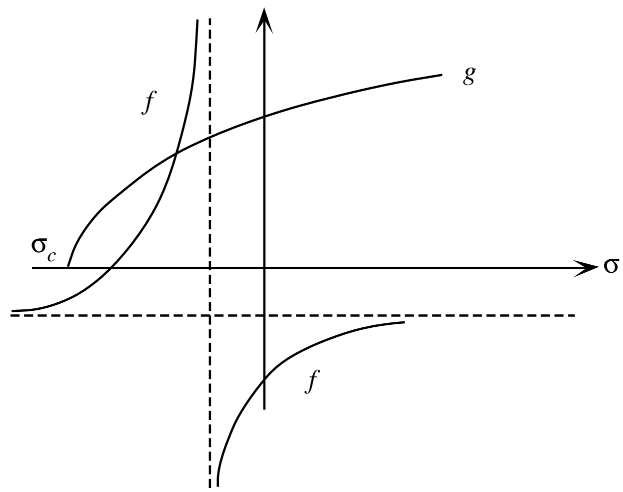

FIGURE 4b. Intersection at negative value; stable growth.

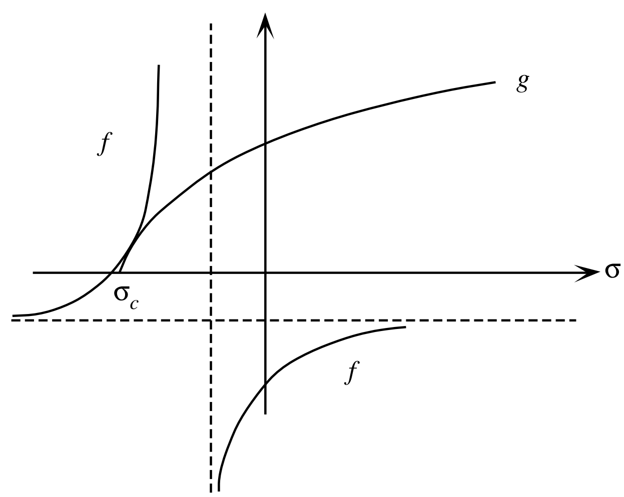

FIGURE 4c. The borderline case $f(\sigma_c) = 0$.

The function g is unchanged while f differs mainly in that the vertical asymptote σ_ν is now in the left half plane. A real intersection of the two curves will occur, resulting in a stable solution if $f(\sigma_c) \leq 0$. Otherwise, an intersection will not necessarily occur.

Substituting $\sigma_c \equiv -a^2/4 - m^2$ into $f(m,\sigma)$, we write

$$f(\sigma_c) = \frac{\left(-\frac{a^2}{4} - m^2\right)(2Lq + a) + (La + \gamma m^2)aq}{\left(-\frac{a^2}{4} - m^2\right) + (La - \gamma m^2)q}. \tag{5.1}$$

We note that if either γ or q becomes large while a and m are fixed, then $f(\sigma_c)$ will be negative, resulting in stable growth of the interface. Alternatively, if q becomes very small (i.e., strong dynamical undercooling term) then $f(\sigma_c) \approx a > 0$ which means that there is no perturbed solution for a which is sufficiently large.

In the case where $f(\sigma_c) < 0$, we may examine the dependence of σ_i on q. For an intersection in the second quadrant, equation (4.1) may be written as

$$-\sqrt{a^2 + 4(\sigma + m^2)} = \frac{(-2L + La^2 + a\gamma m^2)/|\sigma| + a/q}{|La - \gamma m^2|/|\sigma| + 1/q}. \tag{5.2}$$

Hence, if a is sufficiently large, the right-hand side will increase as q becomes smaller. This means that the solution will become even more stable. If a is sufficiently small, the right-hand side will decrease, resulting in a less negative rate of growth σ_i.

The conclusion for the $p > 1$ case then is that the role of finite q is more ambiguous, with the dynamical undercooling exerting a stabilizing influence for large velocities but destabilizing for small ones. In either case, however, the influence is not capable of changing the sign of the exponential growth, though it may change a stable perturbed interface to one which does not correspond to a solution.

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