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ORTHOMODULAR LATTICES AND QUADRATIC SPACES: A SURVEY

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Dedicated to the Memory of Charles H. Randall,
mathematician and friend

Introduction. The theory of quadratic and Hermitian forms has a long and fruitful history in mathematics. From the time of Fermat to the time of Minkowski, quadratic forms belonged to number theory and an impressive arithmetic theory of forms was developed. In 1937, Witt broke new ground bringing the theory of forms into a more modern algebraic setting emphasizing classification and general structure. In 1967, Pfister demonstrated the power of this approach when he published his celebrated structure theorems. To this day, mathematicians are uncovering the beauties of this algebraic theory (see the recent book of Scharlau [**B20**]).

There is an area of mathematics, not in the mainstream described in the paragraph above, where quadratic and sesquilinear forms have also made a significant, though perhaps unexpected, contribution in recent years. This is in the theory of orthocomplemented lattices, especially those known as "orthomodular." The story goes back (at least) to the seminal 1936 paper of Birkhoff and von Neumann [**10**] which has led to much research in what is commonly called the "logic of quantum mechanics" (Math Reviews classifies this area under 81B10). This historic paper argued against the classical Boolean algebra structure of logic identifying the distributive law as being untenable in the logic of empirically verifiable propositions concerning a quantum mechanical system. These authors argued for the structure of a projective geometry (essentially an orthocomplemented modular lattice). Subsequent researchers have generalized further to an orthomodular partially ordered set saying even a lattice structure is too much. In his monograph, Mackey [**B15**] presented some physically plausible axioms concerning states and observables and derived a logic which was a σ-orthocomplete orthomodular poset. He then postulated that this logic was orthoisomorphic to the lattice of closed subspaces of a separable

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infinite dimensional complex Hilbert space. This leap was expedient but not satisfying. Many researchers sought to fill this lacuna.

In an attempt to find orthomodular structures that resemble the lattice of closed subspaces of a Hilbert space, some researchers considered vector spaces over arbitrary division rings. These vector spaces are equipped with sesquilinear forms so that certain geometric concepts can be introduced. At first glance, it may seem hopeless to consider only an algebraic theory since the topology of a Hilbert space plays such a central role. In fact, when topology is introduced into algebra, it is usually employed in an infinite dimensional situation to limit the number of objects under study. So it is in Hilbert space where we consider *closed* linear subspaces instead of all linear subspaces, *continuous* linear operators instead of all linear operators, etc. There are, however, many "happy accidents" which occur in Hilbert space that correlate algebraic properties with topological ones. For example: a subspace of Hilbert space is topologically closed if and only if it is its own biorthogonal; a linear operator is continuous precisely when it has an everywhere defined adjoint; and a sub-$*$-algebra of the algebra of bounded operators on a Hilbert space is closed in the weak operator topology if and only if it is its own double centralizer. The Araki-Amemiya-Piron Theorem even gives an algebraic equivalent to metric completeness and so it goes. Since Hilbert space geometry is in so large part a consequence of the nature of the inner product and the nature of the underlying field of scalars, it is perhaps not so astounding that the algebraic object of interest, the "quadratic space" has been such a fruitful object of study.

It is the purpose of this survey to expose how, over the past several years, this theory of quadratic spaces has shed light on questions that have arisen in the theory of orthomodular lattices (see also the survey of Gross and Künzi [**40**]). Some of the questions below are sweeping in scope while others are somewhat technical, but all are motivated from either Hilbert space or the logic of quantum mechanics. With terminology to be defined later, the questions addressed are the following:

(1) Is there a set of "purely lattice axioms" that determines the lattice of closed subspaces of a complex infinite dimensional Hilbert space?

(2) Can a given lattice carry two orthocomplementations making it into nonisomorphic orthomodular lattices?

(3) Are there any infinite dimensional Hilbert lattices (Morash [**97**]) other than the classical ones based on real, complex or quaternionic Hilbert spaces?

(4) Are there infinite dimensional Hilbert lattices that contain orthogonal elements a and b of equal dimension such that there is no orthoautomorphism that maps a onto b?

(5) Are there infinite dimensional $\mathcal{O}$-symmetric Hilbert lattices that contain complementary a and b with no orthoautomorphism that maps a onto $b^{\perp}$?

(6) Is there an orthocomplemented AC lattice that is not $\mathcal{O}$-symmetric?

The structure of the paper is as follows. The first two sections deal with definitions and concepts central to the exposition in the paper. Question 1 is answered in the third section, Question 2 in the fourth, and Question 3 in the fifth. The next two questions are settled in Section 6 while angle bisecting quadratic spaces are discussed in the next section. Section 8 discusses the Algebraic Closed Graph Theorem and the last question. The final section is devoted to some very illustrative examples. Though the focus of this paper is on rather nice answers to some interesting questions, a number of challenging open questions are also presented.

1. Preliminaries. Though from time to time we shall mention order structures that are not lattices, the principle focus of this paper is on a class of lattices known as *orthocomplemented lattices* (or ortholattices for short). A lattice L with smallest element 0 and largest element 1 is called *orthocomplemented* if there is a mapping $a \to a'$ of L into itself such that

(1.1) $a = a''$ for all a in L,

(1.2) $a \leq b$ implies $b' \leq a'$,

(1.3) $a \wedge a' = 0$ for all a in L, and

(1.4) $a \vee a' = 1$ for all a in L.

Lattices have always been natural models for various logical structures. Interpreting the mapping $a \to a'$ as the passage from a proposition to its negation, we see the axioms above reflect familiar laws of logic. In particular, the DeMorgan Laws hold in any orthocomplemented lattice:

(1.5) $(a \vee b)' = a' \wedge b'$,

(1.6) $(a \wedge b)' = a' \vee b'$.

The concept of modularity and its various symmetries plays an important role in this paper. The lattice of closed subspaces of an infinite dimensional Hilbert space is not "globally" modular. However, there are many "local" modularities present. An ordered pair of elements (a, b) from a lattice L is called a *modular pair*, in symbols $\mathcal{M}(a, b)$, when $c \leq b$ implies $(c \vee a) \wedge b = c \vee (a \wedge b)$. The pair is called a *dual modular pair* if $c \geq b$ implies $(c \wedge a) \vee b = c \wedge (a \vee b)$. We denote this relation by $\mathcal{M}^*(a, b)$. The lattice L is called $\mathcal{M}$-*symmetric* (respectively, $\mathcal{M}^*$-*symmetric*) when the relation $\mathcal{M}$ (respectively, $\mathcal{M}^*$) is a symmetric relation on L. A lattice L is called *cross-symmetric* if $\mathcal{M}(a, b)$ implies $\mathcal{M}^*(b, a)$ and *dual cross-symmetric* if $\mathcal{M}^*(a, b)$ implies $\mathcal{M}(b, a)$. Note that in an orthocomplemented lattice $\mathcal{M}^*(a, b)$ is equivalent to $\mathcal{M}(a', b')$, hence $\mathcal{M}$-symmetry is equivalent to $\mathcal{M}^*$-symmetry. An orthocomplemented lattice L is called $\mathcal{O}$-*symmetric* when $\mathcal{M}(a, b)$ implies $\mathcal{M}(b', a')$. In an orthocomplemented lattice, $\mathcal{O}$-symmetry, cross-symmetry and dual cross-symmetry all coincide. The lattice of closed subspaces of an infinite dimensional Hilbert space is $\mathcal{O}$-symmetric. This fact entails many others. In fact, we have

(1.7) Theorem. *If an orthocomplemented lattice L is $\mathcal{O}$-symmetric, then it is $\mathcal{M}$- and $\mathcal{M}^*$-symmetric. Moreover, in L the four conditions $\mathcal{M}(a, b)$, $\mathcal{M}(b, a)$, $\mathcal{M}^*(a, b)$ and $\mathcal{M}^*(b, a)$ are all equivalent.*

Proof. See Maeda [**B16**, p. 131]. □

A relation of *orthogonality* can be defined on any orthocomplemented lattice as follows: $a \perp b$ if and only if $a \leq b'$. Our next purpose is to define the class of orthomodular lattices. To this purpose we cite

(1.8) Theorem. *Let L be an orthocomplemented lattice. The following are equivalent*:

(1.8.1) $a \perp b$ *implies* $\mathcal{M}(a,b)$;

(1.8.2) $\mathcal{M}(a,a')$ *for all* a *in* L;

(1.8.3) $\mathcal{M}^*(a',a)$ *for all* a *in* L;

(1.8.4) *if* $a \leq b$, *then* $b = a \vee (b \wedge a')$;

(1.8.5) *if* $a \leq b$, *then there exists* c *in* L *such that* $a \perp c$ *and* $a \vee c = b$.

Proof. See Maeda [**B16**, p. 132]. □

An orthocomplemented lattice is called *orthomodular* if any one (and hence all five) of the conditions of Theorem (1.8) is satisfied. Condition (1.8.4) is sometimes referred to as the *orthomodular identity.*

We need the concept of the dimension of a lattice element since dimension restrictions are important to certain coordinatization theorems. The *dimension* of a lattice element a, $\dim(a)$, is the least cardinal which is an upper bound for all the cardinals of subsets which do not contain zero, are chains, and are bounded by a. An element of dimension 1 is called an *atom.* If $a < b$ but $a < x < b$ is not satisfied for any x in a lattice, b is said to *cover* a. Clearly, an atom covers the order zero. The lattice L is *atomistic* if each nonzero element of L is the join of the atoms under it. The lattice L with 0 has the *covering property* if and only if p an atom and $a \wedge p = 0$ imply $a \vee p$ covers a. It is well known that the covering property is equivalent to $\mathcal{M}(p,a)$ for p an atom and every a in L. A lattice is said to satisfy the *atomic exchange property* if and only if whenever p and q are atoms and $q \not\leq b$, then $q \leq b \vee p$ implies $p \leq b \vee q$. In an atomistic lattice, the covering property is equivalent to the atomic exchange property (Maeda [**B16**, p. 32]). An *AC lattice* is an atomistic lattice with the covering property. For the definitions of further lattice concepts in this paper, refer to Birkhoff [**B4**], Grätzer [**B8**] or Maeda [**B16**].

Next we develop the concepts necessary to discuss quadratic spaces. A *quadratic space* is a triple (k, E, Φ) where

(1.9) (k, E) is a left vector space over the division ring k.

(1.10) $\Phi : E \times E \to k$ is a $*$-sesquilinear form, that is,

$$\begin{aligned}\Phi(x+y,z) &= \Phi(x,z)+\Phi(y,z)\\ \Phi(x,y+z) &= \Phi(x,y)+\Phi(x,z)\\ \Phi(\lambda x,y) &= \lambda\Phi(x,y)\\ \Phi(x,\lambda y) &= \Phi(x,y)\lambda^*.\end{aligned}$$

Here $\lambda \to \lambda^*$ is an antiautomorphism of k.

(1.11) Φ is *nondegenerate*: $\Phi(x,y)=0$ for all x implies $y=\vec{0}$, and

(1.12) Φ is *orthosymmetric*: $\Phi(x,y)=0$ implies $\Phi(y,x)=0$.

The next theorem says that a quadratic space is almost Hermitian.

(1.13) Theorem. *Let (k,E,Φ) be a quadratic space and assume $\dim(E)\geq 2$. Then there exists an element ε in k such that*

(1.13.1) $\lambda^{**}=\varepsilon^{-1}\lambda\varepsilon$ *for all λ in k,*

(1.13.2) $\varepsilon^*\varepsilon=\varepsilon\varepsilon^*=1$, *and*

(1.13.3) $\Phi(y,x)=\varepsilon\Phi(x,y)^*$ *for all x,y in E.*

Proof. See Gross [**B9**, p. 7]. □

If ε above equals 1, then ** is the identity map and (k,E,Φ) is called a *Hermitian space*. From now on, we make the tacit assumption that all vector spaces E have dimension at least two.

Each quadratic space carries with it a relation of orthogonality: for x,y in E we say x is *orthogonal* to y and write $x\perp y$ when $\Phi(x,y)=0$. A vector x is called *isotropic* if $x\perp x$; otherwise, x is *anisotropic*. A form Φ is called *anisotropic* if it allows no nonzero isotropic vectors. For the sake of the geometry of subspaces considered in the next section, forms which yield the same orthogonality relation are considered the same. More precisely, we say that $(\Phi,*)$ is *equivalent* to $(\Psi,\#)$ when $\Phi(x,y)=0$ is equivalent to $\Psi(x,y)=0$. This equivalence is nicely characterized by

(1.14) Theorem. *The forms* $(\Phi, *)$ *and* $(\Psi, \#)$ *are equivalent if and only if there exists a constant* $\gamma \neq 0$ *such that* $\Psi(x, y) = \Phi(x, y)\gamma$ *and* $\lambda^{\#} = \gamma^{-1}\lambda^{*}\gamma$ *for all* x, y *in* E *and all* λ *in* k.

Proof. See Piziak [**110**]. □

We end this section by noting that if a form Φ admits an anisotropic vector, then Φ is equivalent to a Hermitian form which has a unit vector.

2. The geometry of subspaces of a quadratic space. Given any quadratic space (k, E, Φ), the orthogonality relation induced by the form Φ yields a map $M \to M^{\perp}$ on the lattice of all subspaces of (k, E), denoted $\mathrm{Lat}(k, E)$, by the prescription $M^{\perp} = \{x \text{ in } E | \Phi(x, y) = 0 \text{ for all } y \text{ in } M\}$. This yields the structure $(\mathrm{Lat}(k, E), \cap, +, ^{\perp}, (\vec{0}), E)$ which is ripe for generalization into pure lattice theory. This lattice is well known to be complete, complemented and modular and has a lattice theoretic characterization via the Fundamental Theorem of Projective Geometry. However, the mapping $M \to M^{\perp}$ falls short, in general, of providing an orthocomplementation. Even so, it is easy to verify that the following properties hold generally.

(2.1) Proposition. *Let* (k, E, Φ) *be a quadratic space. Let* M, N, M_{α} *denote subspaces of* E. *Then*

(2.1.1) $M \subseteq M^{\perp\perp}$;

(2.1.2) $M \subseteq N$ *implies* $N^{\perp} \subseteq M^{\perp}$;

(2.1.3) $E^{\perp} = (\vec{0})$;

(2.1.4) *if* $M_{\alpha} = M_{\alpha}^{\perp\perp}$ *for all* α, *then* $(\cap M_{\alpha})^{\perp\perp} = \cap M_{\alpha}$;

(2.1.5) *if* $M = M^{\perp\perp}$, *then* $(M + kx)^{\perp\perp} = M + kx$ *for any vector* x *in* E; *and*

(2.1.6) $M^{\perp} = M^{\perp\perp\perp}$.

The knowledgeable reader may have already noted that, in general, the mapping $M \to M^{\perp}$ establishes a Galois autoconnection on $\mathrm{Lat}(k, E)$. Since this map does not in general yield an orthocomplementation, we are forced to consider various kinds of subspaces of a quadratic space. These are listed below.

(2.2) Definition. Let (k, E, Φ) be a quadratic space. The *semisimple* subspaces are defined by

$$L_{ss}(k, E, \Phi) = \{M \in \mathrm{Lat}(k, E) \mid M \cap M^{\perp} = (\vec{0})\},$$

the *splitting* subspaces by

$$L_{s}(k, E, \Phi) = \{M \in \mathrm{Lat}(k, E) \mid M + M^{\perp} = E\},$$

and the $^{\perp}$-*closed* subspaces by

$$L_{c}(k, E, \Phi) = \{M \in \mathrm{Lat}(k, E) \mid M = M^{\perp\perp}\}.$$

In a general study, each of these partially ordered sets and several others would play a role. However, for the purposes of this paper, only selected highlights will be given. The first question is where does the orthomodularity reside?

(2.3) Theorem. *Let (k, E, Φ) be a quadratic space. Then*

(2.3.1) *Every splitting subspace is $^{\perp}$-closed and semisimple.*

(2.3.2) *$L_s(k, E, \Phi)$ is always an orthomodular poset but need not be a lattice.*

(2.3.3) *If $L_s(k, E, \Phi)$ contains a subspace of countably infinite dimension, $L_s(k, E, \Phi)$ is not a lattice. In particular, if E is an infinite dimensional subspace of a space spanned by an orthogonal basis, then $L_s(k, E, \Phi)$ is not a lattice.*

(2.3.4) *$L_s(k, E, \Phi)$ can be the orthocomplemented modular lattice consisting of subspaces M of E where either M or $M^{\perp}$ is of finite dimension.*

Proof. For (2.3.1) see Piziak [**102**]. For (2.3.2) see Piziak [**105**]. For (2.3.3) see Gross and Keller [**38**, Theorem 4, p. 73], and for (2.3.4) see [**38**, Theorem 3, page 71]. □

Though orthomodularity resides in the poset of splitting subspaces, the structure most closely associated with the lattice of closed subspaces of a Hilbert space is $L_c(k,E,\Phi)$. This will be considered next.

(2.4) Theorem. *Let* (k,E,Φ) *be a quadratic space. Then* $(L_c(k,E,\Phi), \bigsqcap, \bigsqcup, ^\perp, (\vec{0}), E)$ *is a lattice where* $M \bigsqcap N = M \bigcap N$ *but* $M \bigsqcup N = (M+N)^{\perp\perp}$. *As a lattice it is complete, atomistic and enjoys the covering property making it an* AC *lattice. The local modularities are beautifully characterized as follows*:

(2.4.1) $\mathcal{M}^*(A,B)$ *in* $L_c(k,E,\Phi)$ *if and only if* $A+B$ *is* $^\perp$*-closed.*

(2.4.2) $\mathcal{M}(A,B)$ *in* $L_c(k,E,\Phi)$ *if and only if* $A^\perp + B^\perp$ *is* $^\perp$*-closed.*

In particular, then, $L_c(k,E,\Phi)$ *has both* $\mathcal{M}$ *and* $\mathcal{M}^*$*-symmetry.*

Moreover, $L_c(k,E,\Phi)$ *contains all finite dimensional subspaces of* E. *Indeed, we have*

(2.4.3) $L_c(k,E,\Phi)$ *is modular if and only if* E *is finite dimensional.*

Proof. See Maeda [**B16**]. For (2.4.3) see Keller [**71**] and Frapolli [**32**]. □

We remark that (2.4.3) may look innocent in the list of properties above. However, it is actually an impressive result due to H.A. Keller. Frapolli extended this result to characteristic two.

Though $L_c(k,E,\Phi)$ carries the full-fledged involution $M \to M^\perp$ (i.e., $M = M^{\perp\perp}$ and $M \subseteq N$ implies $N^\perp \subseteq M^\perp$ for M,N in $L_c(k,E,\Phi)$) we still cannot be assured of an orthocomplementation. This issue is finally laid to rest by

(2.5) Proposition. *Let* (k,E,Φ) *be a quadratic space. Then the following are equivalent*:

(2.5.1) $L_c(k,E,\Phi)$ *is orthocomplemented by* $M \to M^\perp$;

(2.5.2) $L_{ss}(k,E,\Phi) = \mathrm{Lat}(k,E)$;

(2.5.3) $(M+M^\perp)^\perp = (\vec{0})$ *for all* M *in* $\mathrm{Lat}(k,E)$; *and*

(2.5.4) *Φ is anisotropic.*

Proof. The proof is routine and will be omitted. □

(2.6) Corollary. *Let (k, E, Φ) be an anisotropic quadratic space. Then the following are equivalent in $L_c(k, E, \Phi)$:*

(2.6.1) *$\mathcal{O}$-symmetry; $\mathcal{M}(A, B)$ implies $\mathcal{M}(B^\perp, A^\perp)$;*

(2.6.2) *Cross symmetry: $\mathcal{M}(A, B)$ implies $\mathcal{M}^*(B, A)$; and*

(2.6.3) *Dual cross symmetry: $\mathcal{M}^*(A, B)$ implies $\mathcal{M}(B, A)$.*

Proof. We need only note that anisotropy implies $L_c(k, E, \Phi)$ is orthocomplemented and these equivalences hold in any orthocomplemented lattice. □

The real issue for this paper is the following: When is $L_c(k, E, \Phi)$ an orthomodular lattice? Clearly, it suffices to consider only anisotropic quadratic spaces since an orthomodular lattice is in particular orthocomplemented. Also, since only the orthogonality relation determines the structure of $L_c(k, E, \Phi)$, there is no loss in generality in assuming Φ is a Hermitian form which admits a unit vector.

To address the issue of the orthomodularity of $L_c(k, E, \Phi)$ we first need

(2.7) Lemma. *Let (k, E, Φ) be any quadratic space. Then M is a splitting subspace of E if and only if there is a linear operator $P : E \to E$ with $P^2 = P$, $\operatorname{im}(P) = M$ and $\Phi(Px, y) = \Phi(x, Py)$ for all x, y in E.*

Proof. See Piziak [**102**]. □

This lemma says that the splitting subspaces of a quadratic space are exactly the images of projection operators.

(2.8) Theorem. *Let (k, E, Φ) be an anisotropic quadratic space. Then the following are equivalent:*

(2.8.1) $L_c(k,E,\Phi)$ *is an orthomodular lattice*;

(2.8.2) $M^\perp + M^{\perp\perp} = E$ for all M in $\mathrm{Lat}(k,E)$;

(2.8.3) $L_c(k,E,\Phi) = L_s(k,E,\Phi)$;

(2.8.4) $M = M^{\perp\perp}$ *implies there is a linear operator* $P : E \to E$ *with* $P^2 = P$, $\Phi(Px,y) = \Phi(x,Py)$ *for all* x,y *in* E *and* $\mathrm{im}(P) = M$; *and*

(2.8.5) $M = M^{\perp\perp}$ *implies* $M + M^\perp = E$.

Proof. (2.8.1) implies (2.8.2). Suppose $L_c(k,E,\Phi)$ is orthomodular and suppose there exists M a subspace of E with $M^\perp + M^{\perp\perp} \neq E$. Then there is a nonzero vector x in E with x not in $M^\perp + M^{\perp\perp}$. Now $M^\perp \subseteq M^\perp + kx$ (the latter being $^\perp$-closed by (2.1.5)) in $L_c(k,E,\Phi)$, so by orthomodularity $M^\perp + kx = M^\perp \bigsqcup ((M^\perp + kx) \cap M^{\perp\perp})$. If $(M^\perp + kx) \cap M^{\perp\perp} = (\vec{0})$, then $M^\perp + kx = M^\perp$ putting x in $M^\perp \subseteq M^\perp + M^{\perp\perp}$, a contradiction. Hence, there must exist a nonzero vector y in $(M^\perp + kx) \cap M^{\perp\perp}$. This means $y \in M^{\perp\perp}$ and $y = z + \alpha x$ where $z \in M^\perp$ and $\alpha \in k$. But then $\alpha x = y - z \in M^{\perp\perp} + M^\perp$ putting x again in $M^\perp + M^{\perp\perp}$, also contradictory. Hence, no such subspace M can exist.

The only other interesting argument is (2.8.5) implies (2.8.1): Suppose M,N belong to $L_c(k,E,\Phi)$ with $M \subseteq N$. Then $N = N \cap E = N \cap (M + M^\perp) = M + (N \cap M^\perp)$. This illustrates the power of having the modular law available in the background lattice $\mathrm{Lat}(k,E)$. Now $N = N^{\perp\perp} = (M + (N \cap M^\perp))^{\perp\perp} = M \bigsqcup (N \cap M^\perp) = M \bigsqcup (N \bigsqcap M^\perp)$ which is the orthomodular identity in $L_c(k,E,\Phi)$. □

The above theorem can be viewed as an analogue of the Projection Theorem in Hilbert space theory. It appears to give the essence behind the orthomodularity of $L_c(k,E,\Phi)$.

Varadarajan [**B21**] calls quadratic spaces satisfying (2.8.5) "Hilbertian." Others, following Kaplansky, call it an "orthomodular space." Note that in finite dimensions, being Hilbertian is equivalent to the anisotropy of the form, because in this case $L_c(k,E,\Phi) = \mathrm{Lat}(k,E)$ and the latter is then an orthocomplemented modular lattice. Clearly, then the interesting results (as well as the enormous challenges) lie in infinite dimensional Hilbertian spaces.

Now it is time to address the questions related in the introduction.

3.

Question 1. Is there a set of "purely lattice axioms" that determines the lattice of closed subspaces of a complex infinite dimensional Hilbert space?

Though this question is of purely mathematical interest, the impetus for its solution comes from the logical foundations of quantum mechanics. As described in the introduction, it would be nice to have a physically plausible set of axioms that would lead naturally to the conventional Hilbert space model of quantum mechanics. In fact, invoking some reasonably well known coordinatization theorems, it is not all that difficult to get a quadratic space from familiar lattice properties. We begin then with this basic representation theorem. Recall that a lattice is called irreducible if it cannot be written (nontrivially) as a product of other lattices.

(3.1) Theorem. *Let L be a lattice which is irreducible, complete, atomistic and which has the covering property. Suppose L has a mapping $\prime : L \to L$ such that $x = x''$ and $x \leq y$ implies $y' \leq x'$ for all x, y in L. Finally suppose there is sufficient height in the lattice to allow all constructions to work, say height at least 4. Then there exists a quadratic space (k, E, Φ) such that L is orthoisomorphic to $L_c(k, E, \Phi)$. Moreover, L is orthocomplemented if and only if Φ is Hermitian.*

Proof. See Gross [**B9**] and Maeda [**B16**]. □

So we see, it is not all that difficult to write a list of lattice axioms to get $L_c(k, E, \Phi)$ as an orthocomplemented lattice which coordinatizes our logic. The real difficulty comes in determining the nature of the coordinatizing division ring. In order to get a connection with the field **R** of real numbers, Zierler [**130**] assumed the topological compactness of certain sublattices of the logic and the existence of a continuous nonconstant function from the unit interval to some finite dimensional sublattice of the logic. In [**100**] Piron simply postulated a direct relation between the coordinatizing division ring k and the field of real numbers. By making assumptions about observables, Gudder and Piron [**46**] and Maczynski [**86**] were able to derive that k was an extension of the real field. It is well known that if this extension is finite dimensional, k is either the reals $\mathbf{R}$, the complex field $\mathbf{C}$ or the division ring of real quaternions $\mathbf{H}$. All these results were quite nice, but the crowning achievement comes in the ingenious paper of Wilbur [**128**] on which this section is based.

Suppose then that we can produce a Hilbertian quadratic space (k, E, Φ) with our lattice L orthoisomorphic to $L_c(k, E, \Phi) = L_s(k, E, \Phi)$. Then Φ is Hermitian for some $^* : k \to k$ where $(\alpha+\beta)^* = \alpha^* + \beta^*$, $(\alpha\beta)^* = \beta^*\alpha^*$ and $\alpha^{**} = \alpha$. In Hilbert space, physical states correspond to unit vectors so we ought to be able to "normalize" any nonzero vector in E. Thus given $x \neq \vec{0}$ in E, there should exist α in k with $\Phi(\alpha x, \alpha x) = \pm 1$. But this essentially says that $\Phi(x, x) = \pm\beta\beta^*$ where $\beta = \alpha^{-1}$ for all $x \neq \vec{0}$ in E. Unfortunately, this condition is not invariant under equivalence of forms and so is not an invariant of the lattice $L_c(k, E, \Phi)$. Let $\mathrm{sym}(k) = \{\alpha \in k \mid \alpha = \alpha^*\}$ and $Z(k)$ denote the center of the division ring k. Wilbur makes two assumptions:

(P1) for each α in $\mathrm{sym}(k)$, there exists β in $\mathrm{sym}(k)$ with $\alpha = \pm\beta\beta^*$, and

(P2) $\mathrm{sym}(k) \subseteq Z(k)$.

With these two properties, he quickly gets the field $\mathrm{sym}(k)$ to be ordered and classifies k to have one of three forms (for convenience, let $F = \mathrm{sym}(k)$):

(I) $k = F$,

(II) $k = F(i)$ where i^2 is negative and $i^* = -i$,

(III) $k = F(i, j)$ where $ij = -ji$ and i^2 and j^2 are negative while $i^* = -i$, $j^* = -j$.

Then using the infinite dimension assumption and a sequence of ingenious arguments, he proves:

(3.2) Theorem. *Let* (k, E, Φ) *be an infinite dimensional Hilbertian space satisfying* (P1) *and* (P2). *Then one of the following three cases must hold*:

(3.2.1) $k = \mathbf{R}$ *the reals and* $*$ *is the identity*, (3.2.2) $k = \mathbf{C}$ *the complex field and* $*$ *is the usual conjugation.*

(3.2.3) $k = \mathbf{H}$ *the quaternions with* $*$ *the usual conjugation.*

Then as a consequence of the theorem of Amemiya-Araki-Piron [**1**], (k, E, Φ) *must be one of the classical Hilbert spaces.*

Now it is a relatively straightforward matter to answer the main question of this section. The axioms we shall list are not those given by Wilbur since we have not developed the language of Varadarajan [**B21**]. However, they are equivalent.

(3.3) Theorem (Wilbur's Axioms). *Let* L *be a lattice which is*

(3.3.1) *irreducible*,

(3.3.2) *complete*,

(3.3.3) *atomistic*,

(3.3.4) *of height at least* 4,

(3.3.5) *orthomodular*,

(3.3.6) *separable* (*in the sense that every orthogonal set of atoms is countable*), *and*

(3.3.7) *infinite dimensional.*

Suppose also that L *has*

(3.3.8) *the covering property.*

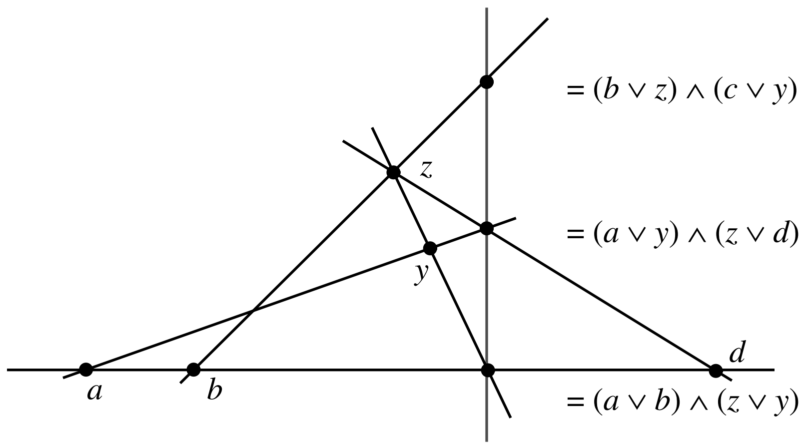
The axioms that determine the division ring are

(3.3.9) (Pappus' Theorem). *Let* $a_0, a_1, a_2, b_0, b_1, b_2$ *be atoms with* $\dim(a_0 \vee a_1 \vee a_2 \vee b_0 \vee b_1 \vee b_2) = 2$ *(that is, all six atoms are in the same plane). Then*

$$(a_1 \vee b_0) \wedge (a_0 \vee b_1) \\ \leq [(a_2 \vee b_0) \wedge (b_2 \vee a_0)] \vee [(a_2 \vee b_1) \wedge [(a_1 \vee b_2)]$$

and, finally,

(3.3.10) (The Square Root Axiom). *Given four distinct atoms* a, b, c, d *with* $a \vee b = c \vee d$ *there exist atoms* y *and* z *with* $y \not\leq b \vee z$, $y \not\leq c \vee d$, $z \not\leq c \vee d$, $z \not\leq a \vee y$ *such that* $\dim(c \vee d \vee y \vee z) = 2$ *and* $(a \vee y) \wedge (z \vee d) \leq [(b \vee z) \wedge (c \vee y)] \vee [(a \vee b) \wedge (z \vee y)]$.



Perhaps a figure would help on this last axiom.

Proof. The first eight axioms give us a Hilbertian quadratic space (k, E, Φ) with L orthoisomorphic to $L_c(k, E, \Phi)$. The square root axiom yields that each element of k is a square and by the Pappus Theorem, k is commutative. The latter gives us (P2) trivially. Also, if $\alpha = \alpha^*$ and $\alpha = \beta^2$ then $\alpha = \alpha^* = (\beta^2)^* = (\beta^*)^2$ so $\beta^2 = (\beta^*)^2$. This means $\beta = \pm\beta^*$. Thus, $\alpha = \beta^2 = \pm\beta\beta^*$ and (P1) follows. Since the lattice is infinite dimensional, Wilbur's Theorem (3.2) applies and we must be looking at one of the three classical Hilbert spaces. The commutativity rules out the quaternions and the universal existence of square roots rules out the reals.

We close this section with the remark that Holland has generalized Wilbur's theorem (3.2) in [**58**].

4.

Question 2. Does there exist a lattice with 0 and 1 which admits two orthocomplementations in such a way that the resulting orthocomplemented lattices are orthomodular but not orthoisomorphic?

This is Problem 27 in the book by G. Kalmbach [**B11**]. She raised this question at a conference in Banff in 1981 although it has been around at least since the 1960's. G. Birkhoff presented the solution given below at the same conference. The construction is briefly sketched in his paper *Ordered Sets in Geometry* which is published in the proceedings of the conference [**9**]. First we lay out the theory underlying this construction.

(4.1) Theorem. *Let (k_1, E_1) and (k_2, E_2) be vector spaces of finite dimension n where $n \geq 3$. Suppose $\eta : \mathrm{Lat}(k_1, E_1) \to \mathrm{Lat}(k_2, E_2)$ is a lattice isomorphism. Then there exists an isomorphism $\tau : k_1 \to k_2$ and a τ-semilinear bijection $T : E_1 \to E_2$ such that $\eta(M) = T(M)$.*

Proof. See Varadarajan [**B21**, p. 35]. □

(4.2) Theorem. *Let (k_1, E_1, Φ) be a quadratic space of dimension at least two, and let (k_2, E_2, Ψ) be any quadratic space. Let $T : E_1 \to E_2$ be a τ-semilinear transformation such that $\Phi(x, y) = 0$ implies $\Psi(Tx, Ty) = 0$. Then there exists a unique γ in k_2 such that $\Psi(Tx, Ty) = (\Phi(x, y))^\tau \cdot \gamma$ for all x, y in E.*

Proof. See Piziak [**110**]. □

We are now ready to present the Birkhoff example. It is remarkably simple considering the length of time this question was open. Let $\mathbf{Q}$ denote the field of rationals and consider the lattice $L = \mathrm{Lat}(\mathbf{Q}, \mathbf{Q}^4)$ the lattice of all subspaces of the four dimensional 4-tuple space over $\mathbf{Q}$. Define $\Phi : \mathbf{Q}^4 \times \mathbf{Q}^4 \to \mathbf{Q}$ by $\Phi((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$ and $\Psi((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)) = x_1y_1 + x_2y_2 + x_3y_3 + 2x_4y_4$. These are positive definite symmetric bilinear forms over $\mathbf{Q}$. Thus $(\mathrm{Lat}(\mathbf{Q}, \mathbf{Q}^4), ^{\perp_1}) = L_c(\mathbf{Q}, \mathbf{Q}^4, \Phi)$ and $(\mathrm{Lat}(\mathbf{Q}, \mathbf{Q}^4), ^{\perp_2}) = L_c(\mathbf{Q}, \mathbf{Q}^4, \Psi)$ are orthocomplemented modular (hence orthomodular) lattices based on the same lattice $\mathrm{Lat}(\mathbf{Q}, \mathbf{Q}^4)$. We claim that as orthocomplemented structures, $(\mathrm{Lat}(\mathbf{Q}, \mathbf{Q}^4), ^{\perp_1})$ and $(\mathrm{Lat}(\mathbf{Q}, \mathbf{Q}^4, ^{\perp_2})$ could not be orthoisomorphic. Suppose there is an orthoisomorphism $\eta : L_c(\mathbf{Q}, \mathbf{Q}^4, \Phi) \to L_c(\mathbf{Q}, \mathbf{Q}^4, \Psi)$. Then, in particular, η is a lattice isomorphism. Thus, by (4.1) there is a (necessarily) linear bijection $T : \mathbf{Q}^4 \to \mathbf{Q}^4$ such that $\eta(M) = T(M)$ and, moreover, $T(M^{\perp_1}) =$

$\eta(M^{\perp_1}) = (\eta(M))^{\perp_2} = T(M)^{\perp_2}$. But $x \perp_1 y$ if and only if $kx \subseteq (ky)^{\perp_1}$ if and only if $T(kx) \subseteq T((ky)^{\perp_1}) = T(ky)^{\perp_2}$ if and only if $k(Tx) \subseteq (k(Ty))^{\perp_2}$ if and only if $Tx \perp_2 Ty$. Hence, (4.2) applies and there exists γ in $\mathbf{Q}$ with $\Psi(Tx, Ty) = \Phi(x, y)\gamma$. Consider the basis $\mathbf{b} = \{e_1 = (1,0,0,0), e_2 = (0,1,0,0), e_3 = (0,0,1,0), e_4 = (0,0,0,1)\}$. Since T is a vector space isomorphism, $\mathbf{c} = \{Te_1, Te_2, Te_3, Te_4\}$ is also a basis. The matrix of Ψ relative to $\mathbf{b}$ is

$$\mathrm{Mat}(\Psi; \mathbf{b}) = (\Psi(e_i, e_j)) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

while, with respect to $\mathbf{c}$,

$$\mathrm{Mat}(\Psi; \mathbf{c}) = (\Psi(Te_i, Te_j)) = (\Phi(e_i, e_j)\gamma) = \begin{bmatrix} \gamma & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma \end{bmatrix}.$$

Now by O'Meara [**B17**, p. 85] there exists an invertible matrix P over $\mathbf{Q}$ such that $P^t \mathrm{Mat}(\Psi; \mathbf{c})P = \mathrm{Mat}(\Psi; \mathbf{b})$. Taking determinants produces the equation

$$\det(P)^2\gamma^4 = 2$$

or

$$(\det(P)\gamma^2)^2 = 2.$$

This says there is a rational number whose square is 2. Thus, there can be no such orthoisomorphism η.

More recently, Gross [**42**] has used rather sophisticated constructions (alluded to in future sections of this paper) to construct a nonmodular bounded lattice L which admits infinitely many orthocomplementations $^{\perp_i}$ such that $(L, ^{\perp_i})$ are pairwise nonisomorphic orthomodular lattices. However, it should be noted that nonmodular orthomodular lattices with distinct orthocomplementations can be easily constructed using the Birkhoff examples above. Simply take a direct product (or horizontal sum) of the two examples above with a nonmodular orthomodular lattice. Gross's constructions are of interest beyond the fact that they are nonmodular.

5.

Question 3. Are there any infinite dimensional Hilbert lattices other than the classical ones based on real, complex or quaternionic Hilbert spaces?

In his thesis [**97**] and subsequent papers, Morash defined a *Hilbert lattice* to be a lattice which is

(5.1) irreducible,

(5.2) complete,

(5.3) atomistic,

(5.4) orthocomplemented,

(5.5) $\mathcal{M}$-symmetric,

(5.6) separable, and

(5.7) orthomodular.

Of course, (5.4) is redundant with (5.7) but Morash was interested in knowing what could be proved without (5.7). The classical example of a Hilbert lattice is the lattice of all closed subspaces of a separable real, complex or quaternionic Hilbert space. Indeed, let $F = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$, * the usual conjugation on F and let $\mathcal{H} = \ell_2(F) = \{(\alpha_i) \mid \sum_{i=1}^{\infty} \alpha_i \alpha_i^* < \infty\}$ taken with the usual inner product $\langle(\alpha_i), (\beta_i)\rangle = \sum_{i=1}^{\infty} \alpha_i \beta_i^*$. Recall that two elements of a lattice are called perspective if they share a common complement. This relation is quite important in the theory of orthomodular lattices. For the classical examples we have the following well-known facts.

(5.8) Theorem. *$L_c(F, \mathcal{H}, <,>)$ is an infinite dimensional $\mathcal{O}$-symmetric Hilbert lattice in which perspectivity is a transitive relation.*

Morash and others lament the fact that certain lattice properties known to hold in the classical examples have not yet been given purely lattice theoretic proofs. For example, Fillmore [**25**] proved the transitivity of perspectivity but his proof makes heavy use of analytical arguments and, in particular, uses the properties of the underlying scalars. Another such example is in establishing $\mathcal{O}$-symmetry. Also, evidence was accumulated by Morash and many others (see Gross and Keller [**38**]) to the effect that there could be no other Hilbert lattices other than the classical ones. Many types of scalar fields were ruled out if orthomodularity was demanded. Moreover, Morash gave lattice theoretic proofs of the following properties known to hold in the classical examples.

(5.9) Theorem. *Every Hilbert lattice*

(5.9.1) *is finite-modular* (*i.e., every finite element* (*a finite join of atoms*) *forms a modular pair with every other element*),

(5.9.2) *has the relative center property,*

(5.9.3) *has any section* $[0, x]$ *again a Hilbert lattice, and*

(5.9.4) *has perspectivity transitive on atoms.*

Proof. See Morash [**97**]. □

Then the dramatic event occurred for the theory of orthomodular lattices and quadratic spaces. In 1979, H.A. Keller discovered the first example of a nonclassical Hilbert space. It was published in [**72**] in 1980. To get Keller's example, begin with $\Gamma = Z^{(N)}$ the additive group of all finitely nonzero sequences of integers. Ordered antilexicographically ($(\gamma_i) \leq (\mu_i)$ if and only if there is an n with $\gamma_i \leq \mu_i$ and for all $m > n$, $\gamma_m = \mu_m$), Γ becomes an ordered abelian group. Let K_0 be the rational field $\mathbf{Q}$ (or the reals or complexes for that matter) with a countable number of indeterminates adjoined, $K_0 = \mathbf{Q}(X_1, X_2, X_3, \dots)$. There is a unique nonarchimedean valuation $w : K_0 \to \Gamma \cup \{\infty\}$ such that $w(\alpha) = 0$ for nonzero α in $\mathbf{Q}$ and $w(X_n) = (0, \dots, 0, -1, 0, \dots)$ where -1 is in the n-th position. Let K be the completion of (K_0, w) by Cauchy sequences. Extend w to K. The vector space E is a "sequence space" analogous to ℓ_2. Let $E = \left\{(\beta_i) \mid \beta_i \in K, \sum_{i=1}^{\infty} \beta_i^2 X_i < \infty\right\}$. Here convergence is taken in the valuation topology. The form Φ is defined by $\Phi(x, y) = \sum_{i=1}^{\infty} \beta_i \gamma_i X_i$ where $x = (\beta_i)$ and $y = (\gamma_i)$. Then (K, E, Φ) is an orthomodular (i.e., Hilbertian) quadratic space and $L_c(K, E, \Phi)$ is a Hilbert lattice. This Hilbertian space is quite different from the classical ones. For example, it is not isometric to any proper subspace and no pair of orthogonal vectors have the same length!

Holland [**58**] has given a general construction which encompasses Keller's example. For this we need his concept of a *-valuation. A *-*valuation* is a mapping $w : k \to \Gamma$ from the nonzero elements of a division ring k with involution * to an ordered abelian group Γ such that (1) $w(\alpha\beta) = w(\alpha) + w(\beta)$; (2) $w(\alpha + \beta) \geq \min(w(\alpha), w(\beta))$ if $\alpha + \beta \neq 0$; (3) $w(\alpha^*) = w(\alpha)$ and (4) w is onto. For more on valuations see Ribenboim [**B19**], Endler [**B7**] and Holland [**58**].

(5.10) Theorem. *Suppose K is a division ring with involution * and suppose K is complete with respect to a nontrivial *-valuation $w : K \to \Gamma$. Select from K a sequence X_n, $n = 1, 2, \ldots,$ of nonzero symmetric elements and set $p_n = w(X_n)$.*

(5.10.1) Let

$$E = \left\{ (\beta_i) \mid \sum_{i=1}^{\infty} \beta_i X_i \beta_i^* \text{ converges in } K \text{ where all } \beta_i \in K \right\}.$$

Define $\Phi(x, y) = \sum_{i=1}^{\infty} \beta_i X_i \gamma_i^*$. Then (K, E, Φ) is a quadratic space.

(5.10.2) *If the p_n satisfy $m \neq n$ implies $p_m \not\equiv p_n \bmod 2\Gamma$, then Φ is anisotropic and, if $x \perp y$ in E, $w(\Phi(x, x)) \not\equiv w(\Phi(y, y)) \bmod 2\Gamma$. Moreover, E is complete in the valuation topology and in E any orthogonal family of nonzero vectors is countable.*

(5.10.3) *Suppose the sequence p_n also has the property that $p_n \to \infty$ and for any bounded below sequence q_n for which $q_n \equiv p_n \bmod 2\Gamma$ also $q_n \to \infty$. Then, given any maximal orthogonal set $\{f_i\}$ of nonzero vectors, $x = \sum_{i=1}^{\infty} \Phi(x, f_i)\Phi(f_i, f_i)^{-1} f_i$ for any x in E.*

Every topologically closed subspace M of E is $^\perp$-closed and every $^\perp$-closed subspace is splitting (i.e., (K, E, Φ) is Hilbertian).

Proof. See Holland [**58**, p. 237]. □

Other generalizations of Keller's original construction have been obtained by Gross and his students. In particular, Fässler-Ullman has constructed explicit orthomodular spaces over generalized power series over prescribed fields [**24**]. She has also proved that for certain orthomodular quadratic spaces, the associated Clifford algebra is a division ring. With its canonical form, it is a definite quadratic space and the completion is orthomodular.

6. The classes $\mathcal{E}$ and $\mathcal{D}$. Gross and Künzi have studied a class of anisotropic quadratic spaces they call "norm-topological" and designate by the symbol $\mathcal{E}$ [**40, 76**]. Let (k, E, Φ) be an anisotropic quadratic space where k carries a *-valuation $w : k \to \Gamma \cup \{\infty\}$. For $\varepsilon \in \Gamma$, let $U_\varepsilon = \{x \text{ in } E \mid w(\Phi(x,x)) > \varepsilon\}$. A quadratic space is in the class $\mathcal{E}$ if the U_ε's are the zero-neighborhood filter of a vector space topology on E that makes Φ continuous (separately). Let $\mathcal{D}$ be the "definite" spaces in $\mathcal{E}$; that is, those spaces in $\mathcal{E}$ that satisfy Schwarz's inequality: $2w(\Phi(x,y)) \geq w(\Phi(x,x)) + w(\Phi(y,y))$. This may look a little strange but remember that the value group Γ is abelian and customarily written additively. Though it would be nice to have a purely algebraic theory for orthomodular quadratic spaces, the nonclassical examples initiated by Keller all seem to use topological considerations in a crucial way. Thus, we let $L(k, E, w) = \{M \in \mathrm{Lat}(k, E) \mid M = \overline{M}\}$. By the continuity of Φ we see that $L_s(k, E, \Phi) \subseteq L_c(k, E, \Phi) \subseteq L(k, E, w)$.

The concept of "types" invented by Keller have so far been crucial in dealing with the nonclassical orthomodular spaces. These are types defined for elements of Γ, scalars in k and vectors in E. In fact, at least two kinds of types have proved useful. There are topological types and algebraic types. The topological types are proper convex subgroups of the ordered abelian group Γ. These are well known to be totally ordered by inclusion. The algebraic types are a bit easier. They are just the cosets of $\Gamma/2\Gamma$. Here, if $\gamma \in \Gamma$, define the algebraic type of γ, by $T(\gamma) = \gamma + 2\Gamma$. Then, if α is a scalar, the type of α, $T(\alpha) = w(\alpha) + 2\Gamma$, and if x is a nonzero vector, its type is $T(x) = w(\Phi(x,x)) + 2\Gamma$. To get topological types, begin with $\delta \in \Gamma$ and let $\Delta(\delta)$ be the largest convex subgroup of Γ contained in the interval $[-|\delta|, |\delta|]$. The topological type of δ is $\overline{T}(\delta) = \cap\Delta(\delta + 2\lambda)$ taken over all λ in Γ. The types of scalars and vectors are then defined as above. Since $\overline{T}(\delta + 2\lambda) = \overline{T}(\delta)$ it is clear that a type is assigned to each one dimensional subspace of E. Also, since topological types are made up of full cosets mod 2Γ, the equality of algebraic types implies the equality of the corresponding topological types. It is important that orthogonal vectors have distinct types. Even more important is that to each maximal orthogonal family of vectors, the associated collection of types for certain spaces in $\mathcal{E}$ is an invariant of the space and does not depend on the orthogonal family. Let us now describe these spaces.

The space (k, E, Φ) of class $\mathcal{E}$ is called a *K-space* provided

(6.1) k is complete under the *-valuation w;

(6.2) Γ contains a countable cofinal subset;

(6.3) E is topologically complete; and

(6.4) E admits a maximal orthogonal family $(e_i)_{i\in I}$ that satisfies the "type condition": for all $(\alpha_j) \in k^I$, $(w(\Phi(\alpha_j \cdot e_j, \alpha_j \cdot e_j))$ for j in I bounded below implies $(\alpha_j e_j)$ converges to $\vec{0}$ in E.

(6.5) Theorem. *Let (k, E, Φ) be of class $\mathcal{E}$. Then the following are equivalent*:

(6.5.1) $L(k, E, w) = L_c(k, E, \Phi)$,

(6.5.2) $L_c(k, E, \Phi) = L_s(k, E, \Phi)$,

(6.5.3) (k, E, Φ) *is a K-space.*

In particular, if (k, E, Φ) is a K-space, then $L_c(k, E, \Phi)$ is an $\mathcal{O}$-symmetric Hilbert lattice.

Proof. See Künzi [**76**]. □

It turns out that the Hilbert lattice $L_c(k, E, \Phi)$ for a K-space is quite different from the classical Hilbert lattices.

(6.6) Theorem. *Let (k, E, Φ) be a K-space. Then $L_c(k, E, \Phi)$ cannot be orthoisomorphic to any classical Hilbert lattice. Indeed, unlike classical infinite dimensional Hilbert spaces where any infinite dimensional closed subspace is isometric to the entire space, a K-space is not isometric to any of its proper subspaces.*

Proof. See Gross [**41**]. □

Two other questions due to Morash (Questions (4) and (5) of the introduction) have been answered by Gross.

(6.7) Theorem. *There exist infinite dimensional Hilbert lattices L that contain orthogonal elements A, B of equal dimension such that there is no orthoautomorphism of L that maps A onto B.*

Proof. See Gross [**41**, Theorem 6]. □

We also have

(6.8) Theorem. *There exist infinite dimensional $\mathcal{O}$-symmetric Hilbert lattices L that contain elements A, B with $A \wedge B = 0$, $A \vee B = 1$ and such that no orthoautomorphism of L maps A onto $B^\perp$.*

Proof. See Gross [**41**, Theorem 7]. □

This latter result is quite relevant in the face of

(6.9) Theorem. *An orthomodular lattice L is $\mathcal{O}$-symmetric if it satisfies*

(6.9.1) *If $a \wedge b = 0$ and $a \vee b = 1$, then there exists an orthoautomorphism $\theta(a) = b^\perp$ and $\theta(b) = a^\perp$.*

Proof. See Maeda [**B16**, p. 170]. □

7. Angle bisecting quadratic spaces. An anisotropic quadratic space (k, E, Φ) is called *angle bisecting* if and only if x, y in E, $x \perp y$ implies there exists α in k with $\alpha y + x \perp \alpha y - x$. This definition is given in terms of the orthogonality relation and does not seem to agree with that given by Morash [**97**]. However, there is

(7.1) Lemma. (k, E, Φ) *is angle bisecting if and only if, given $x \perp y$, there exists α in k with*

(7.1.1) $\alpha\Phi(y,y)\alpha^* = \Phi(x,x)$.

Then we easily have that

(7.2) Corollary. *Let (k, E, Φ) be angle bisecting. Then*:

(7.2.1) *If there exists x_0 in E with $\Phi(x_0, x_0) = \gamma$ and $y \perp x_0$, then there exists α in k such that $\Phi(\alpha y, \alpha y) = \gamma$.*

(7.2.2) *If there exists x_0 in E with $\Phi(x_0, x_0) = 1$ and $y \perp x_0$, then there exists α in k such that $\Phi(\alpha y, \alpha y) = 1$.*

Clearly, the classical Hilbert spaces are the angle bisecting. However, note that the quadratic space $(\mathbf{Q}, \mathbf{Q}^3, \langle , \rangle)$ with the usual inner product from $\mathbf{R}^3$ is not angle bisecting. The vectors $(1,1,1)$ and $(-2,1,1)$ are orthogonal yet if

$$\alpha\langle(1,1,1),(1,1,1)\rangle\alpha = \langle(-2,1,1),(-2,1,1)\rangle.$$

Then $3\alpha^2 = 6$ or $\alpha^2 = 2$ for some α in $\mathbf{Q}$. This example is due to Morash. Even more interesting is

(7.3) Proposition. *Suppose the quadratic space (k, E, Φ) carries a type function T as described in the previous section. Then it cannot be angle bisecting.*

Proof (Gross). Suppose $x \perp y$. Then $T(x) \neq T(y)$ and yet there exists α in k with $\Phi(\alpha y, \alpha y) = \Phi(x,x)$ so $w(\Phi(\alpha y, \alpha y)) = w(\Phi(x,x))$. Thus, $T(w(\Phi(\alpha y, \alpha y)) = T(w(\Phi(x,x))$ whence $T(\alpha y) = T(x)$. But $T(\alpha y) = T(y)$, yielding a contradiction. □

An angle bisecting quadratic space forces much structure on the division ring including that under certain conditions it be ordered. However, the ordering is in the sense given by Baer [**B2**] and exploited by Holland [**58**]. A *-division ring is *Baer ordered* if it contains a subset Π with the following properties: (1) $\Pi + \Pi \subseteq \Pi$, (2) $0 \notin \Pi$ but $1 \in \Pi$, (3) $\alpha \in \Pi$ implies $\alpha = \alpha^*$, (4) for $\rho \neq 0$ define the map $\hat{\rho}$ by $\hat{\rho}(\lambda) = \rho\lambda\rho^*$. Then $\hat{\rho}(\Pi) \subseteq \Pi$ for all $\rho \neq 0$, and (5) for each $\lambda = \lambda^* \neq 0$, either λ or $-\lambda$ is in Π.

(7.4) Lemma. *Every finite dimensional angle bisecting quadratic space which admits a unit vector has an orthonormal basis.*

The proof is easy and will be omitted.

(7.5) Theorem. *Let* (k, E, Φ) *be an infinite dimensional angle bisecting quadratic space which admits a unit vector. Then*

(7.5.1) *k is "formally real" in the sense that if* $\lambda_1\lambda_1^* + \cdots + \lambda_n\lambda_n^* = 0$ *where* $\lambda_i \in k$ *then* $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

(7.5.2) *the characteristic of k is zero so k contains the rational field.*

(7.5.3) *k is "Pythagorean" in the sense that, given* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *in k, there exists γ in k with* $\lambda_1\lambda_1^* + \cdots + \lambda_n\lambda_n^* = \gamma\gamma^*$.

(7.5.4) *Every* $\Phi(x, x)$ *is of the form* $\lambda\lambda^*$ *for some* λ *in* k.

(7.5.5) $\Pi = \{\Phi(x, x) \mid x \neq \vec{0}\}$ *constitutes a Baer ordering for k if and only if, given any nonzero symmetric α in k (i.e.,* $\alpha = \alpha^*$*), there is an x in E with* $\Phi(x, x) = \pm\alpha$.

Proof. See Morash [**94**]; and for (7.5.5) see Piziak [**111**, Theorem 1]. □

Morash has given a purely lattice theoretic definition of angle bisecting atoms in a Hilbert lattice ([**93**] or [**97**]). The definition is a little complicated. Let p, q, x, y, r be distinct atoms such that $p \perp q$, $r < p \vee q$, $x \perp r$, x is not orthogonal to $p \vee q$, $y \perp x$ and $y < p \vee q \vee x$. We write $(p, q)H(x, y)$ via r if and only if $y \perp r$. Now let p, q, r be distinct atoms with $p \perp q$ and $r < p \vee q$. We say *r bisects the angle between p and q*, symbolized $rB(p, q)$, if and only if for any pair (x, y) of orthogonal atoms with $(p, q)H(x, y)$ via r, we have

$$r \leq [(p \vee x) \wedge (q \vee y)] \vee [(p \vee y) \wedge (q \vee x)].$$

(7.6) Theorem. $L_c(k, E, \Phi)$ *is an angle bisecting Hilbert lattice if and only if* (k, E, Φ) *is a Hilbertian angle bisecting quadratic space.*

Proof. See Morash [**97**]. □

(7.7) Open Question. Are there any nonclassical angle bisecting Hilbertian lattices?

In [**15**], D.E. Catlin defines the *hyperoctant property*: for any orthogonal family of atoms $\{p_\alpha\}$ with cardinality at least 2, there exists an atom q under the join of the p_α and q fails to be orthogonal to any of the p_α. This condition holds in the classical Hilbert lattices. It is known that any complete atomic orthomodular lattice with the hyperoctant property also has the property that every interval in it is irreducible.

(7.8) Theorem. *Any infinite dimensional Hilbert lattice that is angle bisecting has the hyperoctant property.*

Proof. See Morash [**96**]. □

Apparently, all currently known examples of orthomodular spaces have the hyperoctant property.

8. The algebraic closed graph theorem and $\mathcal{O}$-symmetry. In 1972, Piziak published a paper [**105**] dealing with certain "happy accidents" in Hilbert space. These are instances where algebraic statements are equivalent to topological statements in the Hilbert space context. For example, a subspace is orthoclosed (or $^\perp$-closed, i.e., $M = M^{\perp\perp}$) if and only if it is closed in the norm topology. This naturally leads to the asking of topological-like questions in the purely algebraic setting of a quadratic space. For example, the notion of orthocontinuity of a linear operator can be formulated in the natural way (see below). The Algebraic Closed Graph Theorem then becomes the following: If T is an everywhere defined orthoclosed linear operator, then T has an everywhere defined adjoint, i.e., T is orthocontinuous. In [**105**], Piziak conjectured that this theorem is not true in general but is true for an orthomodular quadratic space. In 1978, Obi [**98**] claimed to have proved that a $^\perp$-closed linear operator with semisimple graph is $^\perp$-continuous and in 1980 [**99**] that the Algebraic Closed Graph Theorem holds without restriction. However, in 1981, Saarimäki provided counterexamples to Obi's conclusions. In the course of this work, he also answered part of Maeda's Problem 7 [**B16**, p. 135] which is Question (6) of the introduction of this paper: Is there an orthocomplemented AC lattice which is not $\mathcal{O}$-symmetric?

The von Neumann [**126**] formulation of the notion of adjoint translates nicely into the context of quadratic spaces (k, E, Φ). Let T be any relation on E with graph $G(T) \subseteq E \times E$. Call T a *linear relation* if and only if $G(T)$ is a vector subspace of $E \times E$. Call T a $^{\perp}$*-closed relation* provided $G(T) = G(T)^{\perp\perp}$ where $^{\perp}$ is taken with respect to the form $\Phi \oplus \Phi$ on $E \times E$. Note that any $^{\perp}$-closed relation is necessarily linear and $\ker(T)$ is a $^{\perp}$-closed linear subspace of E. Now define $U : E \times E \to E \times E$ by $U(x, y) = (-y, x)$. Then U is an everywhere defined linear bijection with $U^{-1}(y, x) = (x, -y)$. Also note that $\Phi \oplus \Phi(Uz, w) = \Phi \oplus \Phi(z, U^{-1}w)$ and for $M \subseteq E \times E$, $U(M^{\perp}) = U(M)^{\perp}$. For T any relation on E, define T^*, the *adjoint linear relation*, by $G(T^*) = U(G(T))^{\perp}$. Since under this definition, every linear operator has an adjoint, the question of interest is whether it is single valued.

(8.1) Theorem. *Let T be a relation on E. Then T^* is single valued if and only if* $(\mathrm{dom}(T))^{\perp\perp} = E$.

Proof. See Piziak [**105**] or Arens [**3**]. □

(8.2) Corollary. *The following are equivalent for a linear operator* $T : E \to E$:

(8.2.1) T *is* $^{\perp}$*-closed*;

(8.2.2) $T = T^{**}$;

(8.2.3) $\mathrm{dom}(T^*)$ *is* $^{\perp}$*-dense, i.e.,* $(\mathrm{dom}(T^*))^{\perp\perp} = E$;

(8.2.4) T^{**} *is single valued.*

Let (k, E, Φ) be a quadratic space and let $T : E \to E$ be a linear operator. Define T to be *orthocontinuous* ($^{\perp}$-continuous for short) when $T(M^{\perp\perp}) \subseteq T(M)^{\perp\perp}$ for all subspaces M of E. Note that this purely algebraic concept of orthocontinuity of T implies that T is $^{\perp}$-closed and this in turn implies T has a $^{\perp}$-closed kernel. Note also that there is a topology lurking in the background that matches these algebraic definitions. One can introduce the weak linear topology which has as zero neighborhood basis the orthogonals of all finite dimensional subspaces of E. The weak closure of a subspace F coincides with $F^{\perp\perp}$ and T is $^{\perp}$-continuous if and only if it is continuous with respect to the weak linear topology.

(8.3) Theorem. *Let $T : E \to E$ be a linear operator. Then the following are equivalent*:

(8.3.1) *T is orthocontinuous*,

(8.3.2) *$M = M^{\perp\perp}$ implies $T^{-1}(M) = (T^{-1}(M))^{\perp\perp}$*,

(8.3.3) *$M^{\perp}$-closed implies $T^{-1}(M)$ is $^{\perp}$-closed*,

(8.3.4) *$T^{-1}(N^{\perp\perp}) \subseteq (T^{-1}(N))^{\perp\perp}$ for all $N \in \mathrm{Lat}(k, E)$*,

(8.3.5) *T^* is everywhere defined, i.e., $\mathrm{dom}(T^*) = E$*,

(8.3.6) *T is continuous with respect to the weak linear topology on E*,

(8.3.7) *$G(T)^{\perp} + ((\vec{0}) \times E)^{\perp} = E \times E$, and*

(8.3.8) *$\mathcal{M}(G(T), (\vec{0}) \times E)$ in $L_c(k, E \times E, \Phi \oplus \Phi)$*.

Proof. For (8.3.1) through (8.3.5) see Piziak [**105**]. For (8.3.6) through (8.3.8) see Saarimäki [**120**]. ◻

(8.4) Corollary. *Suppose $L_c(k, E \times E, \Phi \oplus \Phi)$ is dual cross symmetric. Then the Algebraic Closed Graph Theorem is valid for E. That is, every $^{\perp}$-closed linear operator $T : E \to E$ is $^{\perp}$-continuous.*

Proof. See Saarimäki [**120**]. ◻

Also in [**120**], Saarimäki has constructed examples of anisotropic quadratic spaces (k, E, Φ) which have $^{\perp}$-closed linear operators that are not $^{\perp}$-continuous. This proves

(8.5) Theorem. *There exist orthocomplemented* AC *lattices which are not $\mathcal{O}$-symmetric. In fact, if (k, E, Φ) is anisotropic and of countably infinite dimension there always exist $^{\perp}$-closed subspaces F and G with $F \cap G = (\vec{0})$, $F + G = E$ and $E \neq F^{\perp} + G^{\perp}$. Thus $L_c(k, E, \Phi)$ is not dual cross symmetric and hence not $\mathcal{O}$-symmetric.*

Proof. See Saarimäki [**120**]. □

We also have

(8.6) Theorem. *If the Algebraic Closed Graph Theorem holds in the Hilbertian space* (k, E, Φ)*, then* $L_c(k, E, \Phi)$ *is* $\mathcal{O}$*-symmetric, cross symmetric and dual cross symmetric.*

In the class $\mathcal{D}$ of definite spaces described previously, a linear map T is called *bounded* if and only if there exists $\gamma \in \Gamma$ such that for all x we have $w(\Phi(Tx, Tx)) \geq \gamma + w(\Phi(x, x))$. Fässler-Ullman [**24**] has given an explicit example of a continuous linear operator that is not bounded. If the definite spaces are taken over a field whose valuation topology is first countable, then the topological version of the closed graph theorem can be proved by the classical Baire category argument (see Künzi [**76**]).

We end this section with a few important remaining open questions.

(8.8) Open Question. Is there an orthomodular AC lattice that is not $\mathcal{O}$-symmetric? More generally, does an $\mathcal{M}$-symmetric orthomodular lattice have to be $\mathcal{O}$-symmetric?

(8.9) Open Question. Does the Algebraic Closed Graph Theorem hold in all Hilbertian spaces?

9. Some very illustrative examples. The first example goes back to ideas of Kaplansky [**67**]. Let k be the field of real numbers. Define a k-vector space E by taking all finitely nonzero sequences of scalars from k under componentwise operations:

$$E = \{(\alpha_1, \alpha_2, \alpha_3, \dots) \mid \alpha_i = 0 \text{ except for finitely many } i\}$$
$$(\alpha_i) + (\beta_i) = (\alpha_i + \beta_i)$$
$$\lambda(\alpha_i) = (\lambda\alpha_i).$$

Define a form $\Phi((\alpha_i), (\beta_i)) = \sum_{i=1}^{\infty} \alpha_i\beta_i$. Then Φ is a nondegenerate symmetric bilinear form on E. Note that the summation above is finite. Also note Φ admits no isotropic vectors, so (k, E, Φ) is an anisotropic quadratic space of countably infinite dimension. In particular, $L_c(k, E, \Phi)$ is a nonmodular orthocomplemented lattice. The sequence $e_1 = (1, 0, 0, \dots)$, $e_2 = (0, 1, 0, \dots)$, $e_3 = (0, 0, 1, 0, \dots)$, $\dots$ is an orthonormal sequence for Φ and is also a Hamel basis for E. Note if $x = (\alpha_i) = \sum_{i=1}^{\infty} \alpha_i e_i$ then $\Phi(x, e_i) = \alpha_i$ so $x = \sum_{i=1}^{\infty} \Phi(x, e_i) e_i$.

Now consider another sequence of vectors:

$$\begin{aligned}
f_0 &= e_1 = (1, 0, 0, \dots) \\
f_1 &= e_1 - e_2 = (1, -1, 0, 0, \dots) \\
f_2 &= e_1 + e_2 - 2e_3 = (1, 1, -2, 0, 0, \dots) \\
f_3 &= e_1 + e_2 + e_3 - 3e_4 = (1, 1, 1, -3, 0, 0, \dots) \\
&\vdots \\
f_k &= e_1 + e_2 + \dots + e_k - ke_{k+1}.
\end{aligned}$$

The following are easily established.

(9.1) Lemma.

(9.1.1) $\Phi(f_0, f_j) = 1$ *for all* $j \geq 0$,

(9.1.2) $\Phi(f_i, f_j) = 0$ *for all* $i, j \geq 1$, $i \neq j$,

(9.1.3) $\Phi(f_j, f_j) = j + j^2$ *for all* $j \geq 1$,

(9.1.4) $\{f_0, f_1, f_2, \dots\}$ *is a Hamel basis of* E.

Thus we see f_0 is not orthogonal to any of the $f_1, f_2, f_3, \dots$ and $f_1, f_2, f_3, \dots$ is an orthogonal sequence in E.

Now let $H = \operatorname{span}\{f_1, f_2, f_3, \dots\}$. Note $f_0 \notin H$ so $H \neq E$ even though H and E have the same Hamel dimension. Let

$$M = \operatorname{span}\{f_1, f_3, f_5, \dots\},$$
$$N = \operatorname{span}\{f_2, f_4, f_6, \dots\}.$$

Note $M + N = H$ and $M \cap N = (\vec{0})$. Even better, we have

(9.2) Lemma. $M = N^{\perp}$ *and* $N = M^{\perp}$.

Proof. Let x be in $N^\perp$. Then $x = \sum_{i=0}^{\infty} \alpha_i f_i$ by (9.1.4). For $n = 2, 4, 6, 8, \ldots$, $0 = \Phi(f_n, x) = \Phi(f_n, \sum_{i=0}^{\infty} \alpha_i f_i) = \alpha_0 \Phi(f_n, f_0) + \alpha_n \Phi(f_n, f_n) = \alpha_0 + \alpha_n \Phi(f_n, f_n)$. For n large enough, $\alpha_n = 0$ so it must be that $\alpha_0 = 0$. Also $0 = \Phi(f_{2m}, x) = \alpha_{2m} \Phi(f_{2m}, f_{2m})$ for all $m = 1, 2, 3, \ldots$. Thus x is in M, and so $N^\perp \subseteq M$. But clearly $M \subseteq N^\perp$, hence $M = N^\perp$. A similar argument yields $M^\perp = N$. □

(9.3) Corollary. *M and N are closed but not splitting, $H^\perp = (\vec{0})$, $H^{\perp\perp} = E$ and $L_c(k, E, \Phi)$ is not an orthomodular lattice.*

Proof. First, $M^{\perp\perp} = N^{\perp\perp\perp} = N^\perp = M$ and $N^{\perp\perp} = N$ similarly. Also $H = M + N = M + M^\perp$, so $H^\perp = (M + M^\perp)^\perp = M^\perp \cap M^{\perp\perp} = M^\perp \cap M = (\vec{0})$. Thus, $H^{\perp\perp} = (\vec{0})^\perp = E$. Note that H is a nonclosed hyperplane and $\mathcal{M}^*(M, N)$ fails in $L_c(k, E, \Phi)$. □

Note we also have here an explicit failure of a familiar law from finite dimensions. Namely, $(M \cap N)^\perp \neq M^\perp + N^\perp$, since $(M \cap N)^\perp = (\vec{0})^\perp = E$ while $M^\perp + N^\perp = H \neq E$.

In [**105**], Piziak proved that a linear functional is orthocontinuous if and only if it has an orthoclosed kernel. It is now possible to give an example of a nonorthocontinuous functional.

Define $\varphi : E \to k$ by $\varphi(x) = \sum_{i=1}^{\infty} \Phi(x, e_i)$.

(9.4) Theorem. *φ is a linear functional and* $\ker(\varphi) = H$*, so φ is not orthocontinuous.*

Proof. Clearly $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(\lambda x) = \lambda \varphi(x)$. Also $\varphi(e_1) = 1$, so $\varphi \neq 0$. Thus, $\ker(\varphi) = \{x \mid \sum_{i=1}^{\infty} \Phi(x, e_i) = 0\}$ is a hyperplane. Clearly, all of $f_1, f_2, f_3, \ldots$, lie in $\ker(\varphi)$ so $H \subseteq \ker(\varphi) \subseteq E = H + \mathrm{span}(f_0)$. By the covering property, $H = \ker(\varphi)$. Since $\ker(\varphi) = H$ is not closed, φ is not orthocontinuous. □

We can also give an explicit example of a nonorthocontinuous linear operator.

Define a linear operator $T : E \to E$ by its action on the basis $\{e_i\}$. Define $T(e_n) = \sum_{i=1}^{n} e_i$.

(9.5) Theorem. *T is an everywhere defined linear operator on E, the graph of T is closed, the domain of T^* is dense but unequal to E and T is not orthocontinuous.*

Proof. Now $y \in \operatorname{dom}(T^*)$ if and only if there exists x in E with $(y, x) \in G(T^*) = U(G(T)^\perp)$ if and only if $(x, -y) \in G(T)^\perp$ if and only if $\Phi \oplus \Phi((x, y), (e_n, Te_n)) = 0$ for all n (since $G(T) = \operatorname{span}\{(e_n, Te_n)\}$) if and only if $0 = \Phi(x, e_n) + \Phi(-y, Te_n)$ if and only if $\Phi(x, e_n) = \Phi(y, Te_n) = \Phi(y, \sum_{i=1}^n e_i) = \sum_{i=1}^n \Phi(y, e_i)$. But for n large enough, $\Phi(x, e_n) = 0$ so $y \in \operatorname{dom}(T^*)$ if and only if $\sum_{i=1}^\infty \Phi(y, e_i) = 0$. Thus $\operatorname{dom}(T^*) = H$ which is dense but not closed. Clearly T is everywhere defined, $\operatorname{dom}(T^*)$ is dense so, by Piziak [**105**], T has a closed graph. However, since $\operatorname{dom}(T^*) \neq E$, T is not orthocontinuous. $\square$

(9.6) Corollary. *The Algebraic Closed Graph Theorem fails for (k, E, Φ) and $L_c(k, E \times E, \Phi \oplus \Phi)$ is an orthocomplemented* AC *lattice which is not dual cross symmetric and hence not $\mathcal{O}$-symmetric.*

Indeed, we can exhibit closed subspaces A and B such that $\mathcal{M}^*(A, B)$ holds in $L_c(k, E, \Phi)$ while $\mathcal{M}(A, B)$ fails (recall $L_c(k, E, \Phi)$ is $\mathcal{M}$ and $\mathcal{M}^*$-symmetric). Let $A = N + \operatorname{span}(f_0)$ and $B = M$. Then A and B are closed and $A + B = E$ is closed, so $\mathcal{M}^*(A, B)$ in $L_c(k, E, \Phi)$. Note A is not splitting however. Note $N \subseteq N + \operatorname{span}(f_0)$, so $A^\perp = (N + \operatorname{span}(f_0))^\perp \subseteq N^\perp = M = B$. That is, $A^\perp \subseteq B$. Could $A^\perp = B$? If so, $M = B = A^\perp = (N + \operatorname{span}(f_0))^\perp = N^\perp \cap \operatorname{span}(f_0)^\perp = M \cap \operatorname{span}(f_0)^\perp \subseteq \operatorname{span}(f_0)^\perp$, a contradiction since none of the f_i in M are orthogonal to f_0. Note $A \cap B = (\vec{0})$ and $A^\perp + B^\perp \subseteq B + B^\perp = M + M^\perp = H \neq E$, so $A^\perp + B^\perp \neq E$. Thus, $A^\perp + B^\perp$ cannot be closed since, if it were, $A^\perp + B^\perp = (A^\perp + B^\perp)^{\perp\perp} = (A^{\perp\perp} \cap B^{\perp\perp}) = (A \cap B)^\perp = (\vec{0})^\perp = E$. Thus $\mathcal{M}^*(A^\perp, B^\perp)$ fails in $L_c(k, E, \Phi)$, and so $\mathcal{M}(A, B)$ fails since $L_c(k, E, \Phi)$ is orthocomplemented. Thus, we see the explicit failure of $\mathcal{O}$-symmetry. By the way, the quadratic space in this example is angle bisecting simply because of the choice of field k.

Another illustrative example is due essentially to Künzi [**77**]. First we need an ordered abelian group Γ that admits no nontrivial order preserving automorphism and that is "big enough." This can be accomplished as follows. For $i \in \mathbf{N}$, let $G_i = \{p/q \in \mathbf{Q} \mid p, q \in \mathbf{Z}$, q is nonzero odd and not divisible by the $(i+1)$th power of any prime). So, for example, $G_0 = \mathbf{Z}$ and G_1 would consist of all fractions p/q with p, q integers, q odd and squarefree.

Let $\Gamma = \{(g_i) \in \mathbf{Q}^{\mathbf{N}} \mid g_i \in G_i \text{ and } g_i = 0 \text{ for almost all } i \text{ in } \mathbf{N}\}$. Γ is an ordered abelian group under component addition and antilexicographic ordering. Note for each i, $[G_i : 2G_i] = 2$. Define $d_i \in \Gamma$ for $i \in \mathbf{N}$ by $d_1 = (1, 0, 0, \dots)$, $d_2 = (0, 1, 0, 0, \dots)$, $d_3 = (0, 0, 1, 0, \dots)$, etc. Γ is generated by elements gd_i, $g \in G_i$ so Γ admits no nontrivial order preserving automorphism.

Now let K be a complete henselian valued field with valuation $w : K \to \Gamma$ satisfying $w(2) = 0$ and with residue class field K_w quadratically closed. That is, $\mathrm{char}(K_w) \neq 2$ and $K_w = K_w^2$. It can be argued that such fields exist. The valuation ring of w, $A_w = \{\alpha \in K \mid w(x) \geq 0\}$ has a purely algebraic characterization. Namely, $A_w = \{\alpha \in K \mid \text{ for all } n \in \mathbf{N},\ \alpha = \beta^{2^n} \text{ or } 1 + \alpha = \beta^{2^n} \text{ for some } \beta \in K\}$. Now if $\varphi : K \to K$ is any field automorphism, then $w(\alpha) \geq 0$ if and only if $w(\varphi(\alpha)) \geq 0$ so $w \circ \varphi$ is a valuation equivalent to φ. Thus there is an order preserving group automorphism on Γ, say $\psi : \Gamma \to \Gamma$ such that $w \circ \varphi = \psi \circ w$. But by the choice of our Γ, ψ is the identity map so we conclude $w(\varphi(\lambda)) = w(\lambda)$ for all λ in K. In other words, all field automorphisms of K preserve the valuation.

Next take a sequence α_n from K with $w(\alpha_n) = d_n$ for each $n \in \mathbf{N}$. Then put $E = \{(\lambda_i) \in K^{\mathbf{N}} \mid \sum_{i=1}^{\infty} \lambda_i^2 \alpha_i < \infty\}$. E is a K-vector space with componentwise operations. Define $\Phi((\lambda_i), (\eta_i)) = \sum_{i=1}^{\infty} \lambda_i \alpha_i \eta_i$. Then (K, E, Φ) is an orthomodular space. Moreover, if $\{e_i\}_{i \in I}$ is a maximal orthogonal family of vectors in E, then there exists a bijection $\tau : I \to \mathbf{N}$ such that for $i \in I$, $w(\Phi(e_i, e_i)) \equiv d_{\tau(i)} \bmod 2\Gamma$. From this we see that $\Phi(x, y) = 0$ implies $w(\Phi(x, x)) \not\equiv w(\Phi(y, y)) \bmod 2\Gamma$. Thus if $x \perp y$ in $E \backslash \{\vec{0}\}$, then $\Phi(x, x) \neq \Phi(y, y)$. This shows that (K, E, Φ) is *not* angle bisecting.

A similitude on (K, E, Φ) is a mapping $S : E \to E$ and a field automorphism $\sigma : K \to K$ such that S is semilinear for σ and for all x, y in E, $\Phi(Sy, Sy) = \sigma(\Phi(x, y)) \cdot \mu$ where μ is a nonzero scalar fixed for S. Any similitude induces an orthoautomorphism on $L_c(K, E, \Phi)$ and, conversely, using the Fundamental Theorem of Projective Geometry (for dimension at least 3), any orthoautomorphism is induced by a similitude. So let S be a similitude and say $w(\mu) =$

$(m_1, m_2, \ldots, m_r, 0, 0, \ldots)$. Choose x in E such that $w(\Phi(x,x)) = d_n$ for some $n \geq r$. Then $w(\Phi(Sx,Sx)) = w(\sigma(\Phi(x,x)) + w(\mu) = w(\Phi(x,x)) + w(\mu) = d_n + w(\mu)$. Thus $w(\mu) \in 2\Gamma$ so that μ is a square, and without loss of generality we can take $\mu = 1$. Thus $w(\Phi(Sy,Sy)) = w(\Phi(y,y))$ for all y in E. Thus $y \perp Sy$ only if $y = \vec{0}$. Thus we see that if A and B are closed subspaces of the same dimension with $A \perp B$, there can be no orthoautomorphism of the lattice mapping A to B. This is quite the opposite of the situation in a classical Hilbert space. There is more, however. Let

$$A = \{(\lambda_i) \in E \mid \lambda_{2i} = \lambda_{2i+1}\} \quad \text{and} \quad B = \{(\lambda_i) \in E \mid \lambda_{2i+1} = 0\}.$$

One computes the orthogonals of these spaces as

$$A^{\perp} = \{(\eta_i) \in E \mid \eta_{2i}\alpha_{2i} + \eta_{2i+1}\alpha_{2i+1} = 0\}$$

and

$$B^{\perp} = \{(\eta_i) \in E \mid \eta_{2i} = 0\}.$$

We see that $A = A^{\perp\perp}$, $B = B^{\perp\perp}$, $A \cap B = (\vec{0}) = A^{\perp} \cap B^{\perp}$. For any $(\eta_i) \in B^{\perp}\backslash\{\vec{0}\}$ there is an $n \in \mathbf{N}$ with $w(\Phi((\eta_i),(\eta_i))) = w(\sum \eta^{w}_{2i+1}, \alpha_{2i+1}) = \min_{i \in N} w(\eta^2_{2i+1}, \alpha_{2i+1}) \equiv d_{2n+1j} \bmod 2\Gamma$, and for any $(\lambda_i) \in A\backslash\{\vec{0}\}$ there is an $m \in \mathbf{N}$ with $w(\Phi((\lambda_i),(\lambda_i))) \equiv w(\sum_{i \in 2\mathbf{N}} \lambda_i^2(\alpha_i + \alpha_{i+1})) = \min_{i \in 2\mathbf{N}} w(\lambda_i^2 \alpha_i) \equiv d_{2m} \bmod 2\Gamma$. As above, there cannot be an orthoautomorphism of $L_c(K, E, \Phi)$ mapping A to $B^{\perp}$.

Finally, in classical Hilbert spaces, any closed infinite dimensional subspace is isomorphic to the whole space. Our example fails on this count also even though it is $\mathcal{O}$-symmetric. Let S be a similarity between E and a closed subspace U of E. Now let $\{e_i\}_{i \in \mathbf{N}}$ be a maximal orthogonal family in E. It follows that $\{Se_i\}_{i \in \mathbf{N}}$ is also a maximal orthogonal family in E. But each Se_i is in U so $U^{\perp\perp} = E$. That is, $U = E$.

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