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GENERALIZATIONS OF THE GLEASON–KAHANE–ŻELAZKO THEOREM

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Let A be a commutative, complex Banach algebra with a unit and let M be a one codimensional subspace of A. A.M. Gleason [**5**] and, independently, J.P. Kahane and W. Żelazko [**8**] proved that

$$(*) \qquad M \text{ is an ideal if and only if } M \text{ consists only of noninvertible elements.}$$

Equivalently, if each element f of M belongs to a proper ideal I_f, which may depend on f, then M is actually an ideal. There is one-to-one correspondence between one codimensional subspaces (ideals) of any unital Banach algebra A and one dimensional subspaces of $A^{\#}$, the space of all linear functionals on A (linear-multiplicative functionals), hence the Gleason–Kahane–Żelazko theorem can be formulated:

$$\binom{*}{*} \qquad \text{Let } F \in A^{\#}. \text{ Then } F \text{ is multiplicative if and only if for any } f \in A \text{ we have } F(f) \in \sigma(f),$$

where $\sigma(f)$ denotes the spectrum of f.

The aim of this note is to give the history of various extensions and generalizations of the above result and to present some open problems.

In 1968 W. Żelazko [**15**] proved that the statement $\binom{*}{*}$ holds for any complex Banach algebra not necessarily unital and commutative. The proof is strictly algebraic, showing that $F \in A^*$ is multiplicative if and only if F restricted to any commutative subalgebra of A is multiplicative.

Statements $(*)$ and $\binom{*}{*}$ are equivalent for any complex, unital Banach algebras, not necessarily commutative, but they are not equivalent for nonunital Banach algebras. Here are two simple examples.

Example 1. Let S be a locally compact, not σ-compact, Hausdorff space. Put $A_0 = C_0(S)$, the algebra of all continuous functions defined

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on S which vanish at infinity. By the result of W. Żelazko A_0 has the $\binom{*}{*}$ property. To see that A does not satisfy $(*)$, let M be any one codimensional subspace of A. For any f in A the set $f^{-1}(\mathbf{C}\backslash\{0\})$ is σ-compact, the set $f^{-1}(0)$ is nonvoid, and thus any function from M is contained in some, and in fact in infinitely many, distinct maximal ideals. Evidently, M need not be an ideal.

Example 2. Put $B = \{(z,w) \in \mathbf{C}^2 : |z|^2 + |w|^2 \leq 1\}$. Let A_0 be the algebra of all complex, continuous functions defined on B, which are analytic functions of two complex variables on int B and which vanish at the point $(0,0)$. Any function from A_0 has infinitely many zeros in B but evidently not all one-codimensional subspaces are ideals.

The above examples are taken from the paper [**13**] of C.R. Warner and R. Whitley. In this paper they also prove that, although the statement $(*)$ is not satisfied for all nonunital algebras, it holds, for example, for algebras generated by a single element or for $L^1(G)$-algebras, where G is a locally compact, metrizable, abelian group. In 1983 C.P. Chen [**2**] proved that $(*)$ holds for any commutative Banach algebra with a countable maximal ideal space.

In all the above mentioned results only complex Banach algebras are considered. The reason is that even the Gleason–Kahane–Żelazko theorem does not hold for real Banach algebras. To get a simple example put $A = C_R[0,1]$ and define $F \in A^*$ by

$$F(f) = \int_0^1 f(t)\,dt, \quad \text{for } f \in A.$$

We have $F(f) \in \sigma(f)$ for any $f \in A$, but F is not multiplicative. In 1978 N. Farnum and R. Whitley [**4**] showed that an algebra $C_R(S)$ has the $\binom{*}{*}$ property if and only if S does not contain any nontrivial connected subset; so $\binom{*}{*}$ is rather an unusual property for this type of algebra. On the other hand, S.H. Kulkarni [**10**] recently noticed that the Gleason–Kahane–Żelazko theorem can be reformulated in a way which holds for real Banach algebras.

Theorem. *Let F be a linear map from a real Banach algebra A with unit $\mathbf{1}$, to the complex plane such that $F(\mathbf{1}) = 1$ and $F(a)^2 + F(b)^2 \neq 0$*

for all a, b in A, such that $ab = ba$ and $a^2 + b^2$ is invertible. Then F is multiplicative.

During the last ten years B. Aupetit, S. Kowalski and Z. Słodkowski, C.R. Warner and R. Whitley as well as M. Roitman and Y. Sternfeld extended the classical result is four distinct ways.

In 1979 B. Aupetit [**1**] showed that the statement $\binom{*}{*}$ can be extended to some operators between two Banach algebras as follows.

Theorem. *Let A and B be complex Banach algebras with identity and suppose that B has a separating family of finite dimensional irreducible representations. If T is a linear map from A onto B such that Tx is invertible in B for all invertible x in A, then we have $Tx = (T\mathbf{1}) \cdot Sx$ for every x in A, where S is a Jordan morphism.*

In 1980 S. Kowalski and Z. Słodkowski [**9**] obtained a surprising result that the assumption of linearity can be much weakened.

Theorem. *Let A be a complex, not necessarily unital nor commutative Banach algebra, and let F be a complex valued function defined on A such that*

$$F(0) = 0$$

and

$$F(f) - F(g) \in \sigma(f - g), \quad \textit{for all } f, g \textit{ in } A.$$

Then F is linear, continuous, and multiplicative.

In 1981 M. Roitman and Y. Sternfeld [**12**] proved that the classical result can be extended to a wide class of topological algebras, which satisfy some spectral conditions. For example, if each element of a topological algebra A has a bounded spectrum, then $\binom{*}{*}$ holds. The following example shows that $\binom{*}{*}$ does not hold in general.

Example 3. Let $H(\mathbf{C})$ be the algebra of all entire functions and put

$$A = \{f \in H(\mathbf{C}) : ||f||_k = \sup_{z \in \mathbf{C}} |f(z) \exp(-|z|/k)| < \infty \text{ for } k \in \mathbf{N}\}.$$

A is a topological algebra and for any nonconstant function f from A we have $\sigma(f) = \mathbf{C}$. Hence, for any linear functional F defined on A such that $F(\mathbf{1}) = 1$ we have $F(f) \in \sigma(f)$ but F need not be multiplicative.

Finally, C.R. Warner and R. Whitley [**14**] considered the following problem.

Let A be a Banach algebra and let M be an n-codimensional closed subspace of A. Assume that each element f of M belongs to at least n distinct regular, maximal ideals of A. Does this imply that M is an ideal and hence the intersection of n distinct maximal ideals?

They proved that the answer is positive if $A = C(S)$, where S is a compact subset of the real line or if $A = L^1(R)$. They also pointed out that the answer is negative in general even for complex, commutative Banach algebras with unit; one example being $A = A_0 \oplus \mathbf{C}e$, where A_0 is the algebra of Example 1 or of Example 2.

With the above n-codimensional problem in mind, we define a more general $P(k,n)$ property.

Definition. Let A be a Banach algebra. We say that A has $P(k,n)$ property, n and k positive integers, if the following holds: Let M be any closed k codimensional subspace of A with the property that for any f in M there are at least n distinct closed maximal, regular ideals $I_1^f, \dots, I_n^f$ with $f \in I_j^f$ for $j = 1, \dots, n$. Then there are n distinct regular maximal ideals $I_1, \dots, I_n$ of A such that $M \subset I_1 \cap \dots \cap I_n$.

The problem of characterizing the Banach algebras with the $P(k,n)$ property is far from being solved in general but the answer is known for some Banach algebras. In [**6**] it has been proved that a $C(S)$ algebra has $P(k,1)$ property for any positive integer k and for any compact Hausdorff space S; and [**6**] also shows that this algebra has the $P(k,n)$ property, with $n \geq 2$ if and only if any one point subset of S is G_δ. This result has been generalized by C.P. Chen and P.J. Cohen [**3**], N.V. Rao [**11**], and the author [**7**], for various other self-adjoint, complex Banach algebras. The following theorem is now known.

Theorem. *Let A be a commutative, self-adjoint Banach algebra. Assume that*

A is regular, the maximal ideal space of A is σ-compact and each point of this space is a G_δ,

or

A is point spectral.

Then A has the $P(k,n)$ property for any positive integers k and n.

Here A is said to be a point spectral if for any finite subset K of $\mathfrak{M}(A)$, the maximal ideal space of A, there is exactly one closed ideal J such that $\{I \in \mathfrak{M}(A) : J \subset I\} = K$.

For the non self-adjoint Banach algebras, the situation is much less clear. The only nontrivial result here is due to Pomorski [manuscript]. He proved that the disc algebra has the $P(2,2)$ property. The disc algebra is the algebra of all complex, continuous functions defined on the closed unit disc, which are analytic in the interior of the disc, taken with the supremum norm.

The following problems are open.

Problem 1. Does any commutative, complex, unital Banach algebra A have the $P(k,1)$ property, for $k \geq 2$?

Remark . Note that if M is a finite codimensional subspace of A, then $\mathrm{cl}(\hat{M})$, the sup norm closure of the space of all Gelfand transforms of the elements of M, is a finite codimensional subspace of a function algebra $\mathrm{cl}(\hat{A}) \subset C(\mathfrak{M}(A))$. Moreover, if M consists only of noninvertible elements, then the same holds for $\mathrm{cl}(\hat{M})$. Hence, for this problem, we can assume without loss of generality that A is a function algebra.

Problem 2. Does the disc algebra have the $P(k,1)$ property, with $k \geq 2$?

The above problem is a very special case of the first one but this case seems to be the basic one. If this problem can be solved for the disc algebra, then, by some very general induction arguments, it can be shown that the result follows for a large class of other Banach algebras.

Problem 3. Let A be a commutative complex Banach algebra containing an f which is not an element of any maximal, regular ideal of A. Does A have the $P(k,1)$ property with $k \geq 1$?

Problem 4. Let A be as in the previous problem and assume that all the points of $\mathfrak{M}(A)$ are G_δ. Then, does A have the $P(k,n)$ property for all positive integers k and n?

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