ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 4, Fall 1991

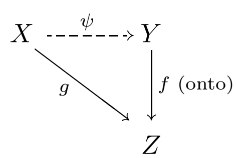
ℵ-PROJECTIVE SPACES IN NONCOMPACT CATEGORIES

LOUIS M. FRIEDLER AND STEPHEN WILLARD

ABSTRACT. Neville and Lloyd have defined ℵ-projective topological spaces and characterized them in the category of compact Hausdorff spaces and continuous maps. The present paper characterizes the spaces ℵ-projective in various noncompact categories of topological spaces and maps.

1. Introduction. A topological space X is *projective* in a category provided whenever $g : X \to Z$ and $f : Y \to Z$ are admissible maps with f onto, a map $\psi : X \to Y$ can be found with $f \circ \psi = g$. Thus, the requirement is precisely that a solution ψ can be found making diagram (1) below commutative.

(1)



Let ℵ be an infinite cardinal. In [**16**] Neville and Lloyd defined a space to be ℵ-*projective* (in the category of compact Hausdorff spaces and continuous maps) provided diagram (1) has a solution ψ whenever all spaces are compact Hausdorff and the weight of Y is less than ℵ. They then showed that a compact Hausdorff space X is ℵ-projective if and only if it is a totally disconnected $F_{\aleph}$-space. (A space is an $F_{\aleph}$-space if and only if disjoint ℵ-open sets have disjoint closures; a set is ℵ-open if it is the union of fewer than ℵ cozero sets.)

Our purpose here is to study ℵ-projectivity in various categories in which the objects are not necessarily compact. For this purpose, we will modify the definition of an ℵ-projective space by requiring that the weight of Z in diagram (1) also be less than ℵ. The resulting

Received by the editors on June 22, 1987, and in revised form on January 28, 1988.

notion is clearly equivalent to the Neville-Lloyd definition (hereafter, "$\aleph$-projective in the sense of Neville-Lloyd") in the category of compact T_2 spaces and continuous maps, and also in categories with perfect maps. We will later (Corollary 2.11) show that the two definitions are also equivalent in the category of Tychonoff spaces and continuous maps. In Section 2 below, we consider noncompact categories with continuous maps, deriving a general result from which it follows that *a Tychonoff space is $\aleph$-projective in the category $\tau_{3\frac{1}{2}}$ of Tychonoff spaces and continuous maps if and only if it is $\aleph$-discrete.* Other results in that section bear on the category $\mathcal{HC}$ of H-closed spaces and continuous maps. In Section 3 we study noncompact categories with perfect maps. The problem in this setting is complicated by the fact that the space Z in diagram (1) must have weight less than $\aleph$, and there may exist spaces X admitting no perfect map φ onto any space of weight less than $\aleph$. Such spaces X will be vacuously $\aleph$-projective. Our results in this section tend to characterize nonvacuously $\aleph$-projective spaces. For example, *in the category of Tychonoff spaces and perfect maps, every $F_{\aleph^*}$-space is $\aleph$-projective, and every nonvacuously $\aleph$-projective space is an $F_{\aleph^*}$-space.* (See Section 3 for the definition of an $F_{\aleph^*}$-space.)

In what follows, the weight of a topological space X will be denoted $w(X)$. All maps are assumed continuous.

2. Continuous maps. Let $\mathcal{C}$ be a fixed category of topological spaces and suppose $X \in \mathcal{C}$. We say $U \subseteq X$ is $\aleph$-*open*$(\mathcal{C})$ if and only if there exist a space $Z \in \mathcal{C}$ with weight less than $\aleph$, a continuous $g : X \to Z$ and an open set $W \subseteq Z$ such that $U = g^{-1}(W)$. The following theorem shows that this definition agrees with the definition of an $\aleph$-open set in [**16**] if the objects of $\mathcal{C}$ are Tychonoff and include the compact Hausdorff spaces.

Theorem 2.1. *Let $\mathcal{C}$ be a category such that every compact Hausdorff space is an object of $\mathcal{C}$ and every object of $\mathcal{C}$ is a Tychonoff space. Let $X \in \mathcal{C}$ and suppose $\aleph > \aleph_0$. Then $U \subseteq X$ is* $\aleph$-open$(\mathcal{C})$ *if and only if U is the union of fewer than $\aleph$ cozero sets.*

Proof. $(\Rightarrow)$. Suppose U is $\aleph$-open$(\mathcal{C})$ in X. Then there exists a $Z \in \mathcal{C}$ of weight $< \aleph$, a continuous $g : X \to Z$ and an open $W \subseteq Z$ with

$U = g^{-1}(W)$. Since Z is Tychonoff and $w(Z) < \aleph$, W can be written as the union of fewer than $\aleph$ cozero sets. Since the inverse image of a cozero set is a cozero set, U is the union of fewer than $\aleph$ cozero sets.

($\Leftarrow$). Assume $U = \cup_{\alpha \in A} U_\alpha$, where $|A| < \aleph$ and each U_α is a cozero set. For each $\alpha \in A$, there exists a continuous map g_α from X to a copy I_α of the closed unit interval such that $U_\alpha = \{x \in X : g_\alpha(x) \neq 0\}$. Let $g : X \to \prod_{\alpha \in A} I_\alpha$ be defined by $[g(x)]_\alpha = g_\alpha(x)$. Then g is continuous, $\prod_{\alpha \in A} I_\alpha$ is an object of $\mathcal{C}$ of weight $< \aleph$ and

$$\begin{aligned} U &= \bigcup_{\alpha \in A} g^{-1}[(I_\alpha - \{0\}) \times \prod_{\beta \neq \alpha} I_\beta] \\ &= g^{-1}\Big[\bigcup_{\alpha \in A}((I_\alpha - \{0\}) \times \prod_{\beta \neq \alpha} I_\beta)\Big] \end{aligned}$$

so that U is $\aleph$-open($\mathcal{C}$). □

A space X is $\aleph$-*discrete* if and only if the intersection of fewer than $\aleph$ open sets is open. (These spaces are called $<\aleph$-*discrete* in [**17**].) In a fixed category $\mathcal{C}$, we will refer to a space X as *weakly* $\aleph$-*discrete*($\mathcal{C}$) if and only if the intersection of fewer than $\aleph$ sets which are $\aleph$-open($\mathcal{C}$) is open.

Clearly, every $\aleph$-discrete space is weakly $\aleph$-discrete($\mathcal{C}$) and it is straightforward to establish the converse if the objects of $\mathcal{C}$ are Tychonoff and include the compact Hausdorff spaces.

More generally, let $\mathcal{C}$ be a category, X a space in $\mathcal{C}$ which has an open base of $\aleph$-open($\mathcal{C}$) sets. Then X is weakly $\aleph$-discrete($\mathcal{C}$) if and only if X is $\aleph$-discrete. This includes the above-mentioned case since if compact $T_2 \subset \mathcal{C} \subset$ Tychonoff, and $X \in \mathcal{C}$, then X has a base of cozero-sets which are $\aleph$-open ($\mathcal{C}$) by 2.1. Other cases are included as well: e.g., if $\mathcal{C}$ consists of 0-dimensional T_2-spaces and $\{0, 1\} \in \mathcal{C}$, then any $X \in \mathcal{C}$ has a base of open-closed sets which are $\aleph$-open($\mathcal{C}$) for any $\aleph$.

The following example shows that these two classes do not coincide in general.

Example 2.2. Let τ_3 be the category of T_3-spaces with continuous maps. Then for $X \in \tau_3$, $U \subset X$ is $\aleph_1$-open(τ_3) if and only if U is a cozero-set (since second countable T_3-spaces are metrizable). But there exist T_3-spaces X on which every real valued continuous function is constant. For such a space X, U is a cozero-set if and only if $U = \varnothing$ or $U = X$. Hence, such a space is weakly $\aleph_1$-discrete (τ_3), but cannot be $\aleph_1$-discrete unless $|X| = 1$. Note that such a space is easily $\aleph_1$-projective($\mathcal{T}_3$).

In the example above, not every open subset is $\aleph$-open($\mathcal{C}$). Note that in a space of weight $< \aleph$ and in a discrete space, every open set is $\aleph$-open($\mathcal{C}$). The next example shows these are not the only spaces with this property.

Example 2.3. Let τ_2 be the category of T_2-spaces with continuous maps. We provide a connected H-closed space of weight $\geq \aleph$ in τ_2 which has every open set $\aleph$-open(τ_2).

Let $\mathcal{U}$ be the usual topology on $[0, 1]$, and let $\mathcal{S}$ be the collection of all countably infinite sequences in $[0, 1]$ which converge in $\mathcal{U}$ to 0. Then $\{U - S \mid U \in \mathcal{U}, S \in \mathcal{S}\}$ is a base for a topology τ on $[0, 1]$ and every open set in τ is of the form $U - S^1$ where $S^1 \subseteq S$ for some $S \in \mathcal{S}$. Let $X = ([0, 1], \tau)$. Then X is Hausdorff, H-closed and not first countable at 0, so $w(X) > \aleph_0$ [**8**, Example 4.3]. Clearly, X is connected.

We claim every open set in X is $\aleph_1$-open(τ_2). Let V be open in X. Then $V = U_0 - S_0$ for some $U_0 \in \mathcal{U}$ and $S_0 \subset S \in \mathcal{S}$. We define a new topology τ_0 on $[0, 1]$. Neighborhoods of 0 are of the form $U - S_0$ for $U \in \mathcal{U}$ and other points have τ-neighborhoods. Let $Y = ([0, 1], \tau_0)$. Then Y is H-closed, Hausdorff and has weight $\aleph_0$. Let $f : X \to Y$ be the identity map. Then $f(V)$ is open, $V = f^{-1}f(V)$, f is continuous and $w(Y) < \aleph_1$, so that V is $\aleph_1$-open(τ_2).

The same example works if τ_2 is replaced by the category $\mathcal{HC}$ of H-closed spaces with continuous maps.

The weakly $\aleph$-discrete ($\mathcal{C}$) spaces play a central role in the characterization of $\aleph$-projective spaces in categories whose maps are the continuous maps.

Theorem 2.4. *Let $\mathcal{C}$ be a category in which*:

(a) *the admissible maps are precisely the continuous functions*;

(b) *given an object X in $\mathcal{C}$ and $p \in X$, the space formed from X by making $\{p\}$ open is an object in $\mathcal{C}$*;

(c) *objects are productive.*

Then an ℵ-projective space in $\mathcal{C}$ is weakly ℵ-discrete($\mathcal{C}$).

Proof. Let X be ℵ-projective in $\mathcal{C}$, and suppose $U = \cap_{\alpha \in A} U_\alpha$ in X, where $|A| < \aleph$ and each U_α is ℵ-open($\mathcal{C}$). We must show U is open in X.

Let $p \in U$. For each $\alpha \in A$, there are a space Z_α in $\mathcal{C}$ of weight $< \aleph$, a continuous $g_\alpha : X \to Z_\alpha$ and an open $W_\alpha \subseteq Z_\alpha$ such that $U_\alpha = g_\alpha^{-1}(W_a)$. Let $Z = \prod_{\alpha \in A} Z_\alpha$. Then $w(Z) < \aleph$, and, by (c), Z is an object of $\mathcal{C}$. Define $g : X \to Z$ by $[g(x)]_\alpha = g_\alpha(x)$. Then g is continuous, and $g(p) \in \cap_{\alpha \in A}(W_\alpha \times \prod_{\beta \neq \alpha} Z_\alpha)$.

Let Y be the space Z with $\{g(p)\}$ made open. By (b), Y is an object of $\mathcal{C}$, and clearly $w(Y) < \aleph$. Let $f : Y \to Z$ be the identity. Then f is continuous and onto so, since X is ℵ-projective, there is a (continuous) lifting $\psi : X \to Y$ with $f \circ \psi = g$.

Now $\psi^{-1}(f^{-1}g(p))$ is open and contains p. To show U open, it suffices to show $\psi^{-1}(f^{-1}g(p)) \subseteq U$. But if $x \notin U$, then $x \notin U_\alpha$ for some $\alpha \in A$, whence $g_\alpha(x) \notin W_\alpha$. Thus, $g(x) \notin \cap_{\alpha \in A}(W_\alpha \times \prod_{\beta \neq \alpha} g_\beta)$, and so $g(x) \neq g(p)$. It follows that $x \notin \psi^{-1}(f^{-1}g(p))$. Thus, $\psi^{-1}(f^{-1}g(p)) \subseteq U$, completing the proof. □

We wish now to establish the converse, that each weakly ℵ-discrete($\mathcal{C}$) space is ℵ-projective. In the development below, we assume $\mathcal{C}$ is a fixed category whose objects are Hausdorff and whose admissible maps are precisely the continuous maps, X is weakly ℵ-discrete($\mathcal{C}$), Y has weight $< \aleph$, and we have the following diagram:

$$\begin{array}{ccc} & & Y \\ & & \big\downarrow f \text{ (onto)} \\ X & \xrightarrow[g]{} & Z \end{array} \tag{2}$$

We further assume that either Z has weight $< \aleph$ or X is $\aleph$-discrete.

We will call a space $\aleph$-*Lindelöf* if every open cover has a subcover of cardinality less than $\aleph$ and *hereditarily* $\aleph$-*Lindelöf* if every subset is $\aleph$-*Lindelöf*.

Lemma 2.5. *Z (in diagram 2) is hereditarily $\aleph$-Lindelöf.*

Proof. Let $A \subseteq Z$. Then $f^{-1}(A)$ has weight $< \aleph$ and is thus $\aleph$-Lindelöf. Now $A = f[f^{-1}(A)]$ is the continuous image of an $\aleph$-Lindelöf space and is therefore $\aleph$-Lindelöf. □

Lemma 2.6. *Each $q \in Z$ is the intersection of fewer than $\aleph$ open sets.*

Proof. Let $\{V_\alpha\}$ be the collection of all open sets containing q. Since Z is Hausdorff, $\{q\} = \cap \bar{V}_\alpha$. Then $\{Z - \bar{V}_\alpha\}$ is an open cover of the $\aleph$-Lindelöf space $Z - \{q\}$, so $Z - \{q\}$ is covered by fewer than $\aleph$ of the sets $Z - \bar{V}_\alpha$. Thus, $\{q\}$ is the intersection of fewer than $\aleph$ of the V_α.

Lemma 2.7. *For each $p \in X$, g is constant on a neighborhood of p.*

Proof. Say $g(p) = q$. Then $\{q\} = \cap_{\alpha \in A} V_\alpha$ where each V_α is open and $|A| < \aleph$. For each α, let $U_\alpha = g^{-1}(V_\alpha)$. Then if $U = \cap U_\alpha$, U is the intersection of fewer than $\aleph$ open sets. If $w(Z) < \aleph$, then each U_α is $\aleph$-open$(\mathcal{C})$ and since X is weakly $\aleph$-discrete$(\mathcal{C})$, U is open. On the other hand, if X is $\aleph$-discrete, U is the intersection of fewer than $\aleph$ open sets, so again U is open. But $g(U) \subseteq \cap\, g(U_\alpha) \subseteq \cap\, V_\alpha = \{q\}$. Thus, g is constant on a neighborhood of p. □

Lemma 2.8. *For each $p \in X$, $g^{-1}[g(p)]$ is open.*

Proof. Say $g(p) = q$. Then $g(U) = q$ where U is an open set containing p. By Zorn's Lemma, there exists a maximal open V containing p such that $g(V) = q$. We claim $V = g^{-1}(q)$. Since $g(V) = q$, $V \subseteq g^{-1}(q)$. But if $x \notin V$ and $g(x) = q$, then $g(W) = q$ for some open W containing x by Lemma 2.7, and then $V \cup W$ contradicts the maximality of V. Thus, $V = g^{-1}(q) = g^{-1}[g(p)]$ and the latter is open. □

Theorem 2.9. *Let $\mathcal{C}$ be a category whose objects are Hausdorff and whose maps are precisely the continuous maps. Then*

(a) *every weakly* ℵ-discrete*($\mathcal{C}$) space is* ℵ-projective *in $\mathcal{C}$;*

(b) *every* ℵ-discrete *space is* ℵ-projective *in the sense of Neville–Lloyd in $\mathcal{C}$.*

Proof. We do (a) and (b) together. Assume the situation of diagram (2) where either (i) X is weakly ℵ-discrete($\mathcal{C}$) and $w(Z) < \aleph$, or (ii) X is ℵ-discrete. In either case, by Lemma 2.8, those $g^{-1}(q)$ which are nonempty partition X into disjoint open sets. For each such q, choose any point $y_q \in f^{-1}(q)$ and define ψ on $g^{-1}(q)$ to be constant and equal to y_q. If $p \in g^{-1}(q)$, then $(f \circ \psi)(p) = f(y_q) = q = g(p)$ so $f \circ \psi = g$. Since ψ is continuous on each element of a partition of X into open sets, it is continuous. □

Corollary 2.10. *In the following categories, the ℵ-projective spaces are exactly the weakly* ℵ-discrete($\mathcal{C}$) *spaces*:

(a) *the category τ_3 of T_3 spaces with continuous maps,*

(b) *the category τ_2 of T_2 spaces with continuous maps.*

In the category $\tau_{3\frac{1}{2}}$ of Tychonoff spaces with continuous maps, a space is weakly ℵ-discrete($\tau_{3\frac{1}{2}}$) if and only if it is ℵ-discrete. Consequently,

Corollary 2.11. *The following are equivalent for a Tychonoff space X*:

(a) *X is* ℵ-discrete:

(b) *X is weakly* ℵ-discrete($\tau_{3\frac{1}{2}}$);

(c) *X is* ℵ-projective($\tau_{3\frac{1}{2}}$);

(d) *X is* ℵ-projective($\tau_{3\frac{1}{2}}$) *in the sense of Neville–Lloyd.*

Note that the category $\mathcal{HC}$ of H-closed spaces and continuous maps does not satisfy condition (b) of Theorem 2.4. In the theory of projective spaces, it is known (e.g., Theorem 5.3 in [**12**]) that an H-closed space is projective in $\mathcal{HC}$ if and only if it is finite. We conjecture, but have not proved, that this result holds also for $\aleph$-projective spaces. However, it is easily verified that every finite space is $\aleph$-projective in $\mathcal{HC}$, and we show that every $\aleph$-projective space is weakly $\aleph$-discrete ($\mathcal{HC}$) and also that every $\aleph$-discrete H-closed space is finite.

In the following theorem, K will represent the second countable, H-closed, noncompact Urysohn space described in [**2**, Example 3.13]. Thus, $K = ([0,1] \times N) \cup \{a\}$, where $a \notin [0,1] \times N$, and basic neighborhoods of a take the form $V_k = \cup_{j \geq k}((0,1] \times \{j\}) \cup \{a\}$.

Theorem 2.12. *Let X be H-closed and $\aleph$-projective in $\mathcal{HC}$. Then X is weakly* $\aleph$-discrete($\mathcal{HC}$).

Proof. Let X be $\aleph$-projective in $\mathcal{HC}$ and let $U = \cap_{\alpha \in A} U_\alpha$, where $|A| < \aleph$ and each U_α is $\aleph$-open ($\mathcal{HC}$). To show U is open, let $p \in U$.

For each $\alpha \in A$, there are an H-closed space Z_α of weight less than $\aleph$, a continuous $g_\alpha : X \to Z_\alpha$ and an open $W_\alpha \subseteq Z_\alpha$ with $U_\alpha = g_\alpha^{-1}(W_\alpha)$. Let $h : X \to \prod_{\alpha \in A} Z_\alpha$ be the continuous function defined by $[h(x)]_\alpha = g_\alpha(x)$. Since the product of H-closed spaces is H-closed [**2**, Theorem 3.3], $\prod Z_\alpha$ is H-closed, and clearly $w(\prod Z_\alpha) < \aleph$. Let $Z = K \times \prod Z_\alpha$, embed $\prod Z_\alpha$ in Z by $i(y) = (a, y)$, and let $g = i \circ h$.

Now $g : X \to Z$ is continuous, Z is H-closed and $w(Z) < \aleph$. Let Y be the space formed from Z by making the dense subspace $A = (K - \{a\}) \times \prod Z_\alpha \cup \{(a, h(p))\}$ open in Y. By a result of Vermeer [**21**, Theorem 1.1.11(i)], Y is H-closed. Let $f : Y \to Z$ be the natural continuous projection. Since $w(Y) < \aleph$ and X is $\aleph$-projective, there is a continuous $\psi : X \to Y$ such that $f \circ \psi = g$.

Now $\psi^{-1} f^{-1}(A)$ is open and contains p. To show U open, it thus suffices to show that $\psi^{-1} f^{-1}(A) \subseteq U$. But if $q \notin U$, then $q \notin U_\alpha$ for some $\alpha \in A$. Then $g_\alpha(q) \notin W_\alpha$ so that $h(q) \notin \cap_{\alpha \in A}(W_\alpha \times \prod_{\beta \neq \alpha} Z_\beta)$. Since $h(p)$ is in the latter set, $h(q) \neq h(p)$. Thus, $g(q) \notin A$ and $q \notin \psi^{-1} f^{-1}(A)$. The result follows. $\square$

Corollary 2.13. *In the category $\mathcal{HC}$ of H-closed spaces and continuous maps, the $\aleph$-projective spaces are precisely the weakly $\aleph$-discrete $(\mathcal{HC})$ spaces.*

Lemma 2.14. *Suppose $\aleph > \aleph_0$ and let X be an $\aleph$-discrete Hausdorff space. Let $S = \{x_1, x_2, \dots\}$ be a countable, closed, discrete subset of X. If $p \notin S$, then p and S can be separated by disjoint open sets.*

Proof. Let $S_n = \{x_1, x_2, \dots, x_n\}$ for each $n = 1, 2, \dots$. For each n, choose disjoint open sets U_n and V_n so that $S_n \subseteq U_n$ and $p \in V_n$. Let $U = \cup U_n$ and $V = \cap V_n$. Then V is open, since X is $\aleph$-discrete, and, thus, U and V are disjoint open sets containing S and p, respectively. □

Theorem 2.15. *Let $\aleph > \aleph_0$. An* $\aleph$-discrete *H-closed space is necessarily finite.*

Proof. If X is an infinite $\aleph$-discrete H-closed space, let $S = \{x_1, x_2, \dots\}$ be a denumerable subset of X. Then [**9**, 4K.1] S is closed and discrete. Let $\mathcal{U}_m$ be the collection of all open sets in X containing $\{x_m, x_{m+1}, \dots\}$, and let $\mathcal{U} = \cup_{m=1}^{\infty} \mathcal{U}_m$. Then $\mathcal{U}$ is an open filter base on the H-closed space X, so there is a point p in $\cap\{\bar{U} : U \in \mathcal{U}\}$. If $p = x_m$ for some m, then by Lemma 2.14, p and $\{x_{m+1}, x_{m+2}, \dots\}$ can be separated by disjoint open sets; but this easily contradicts $p \in \cap\{\bar{U} : U \in \mathcal{U}\}$. Thus, $p \notin S$. But now Lemma 2.14 can be applied to p and S, again contradicting $p \in \cap\{\bar{U} : U \in \mathcal{U}\}$.

Thus X must be finite. □

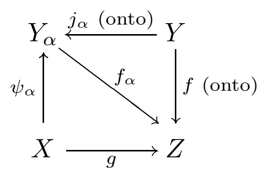
3. Perfect maps. In this section we will employ the partial lifting technique used by Neville and Lloyd [**16**]. We review their terminology here. Suppose the situation of diagram (3) below:

$$\begin{array}{ccc} & & Y \\ & & \Big\downarrow {\scriptstyle f\ (\text{onto})} \\ X & \xrightarrow[g]{} & Z \end{array} \tag{3}$$

Then a *partial lifting* of g over f will be a quadruple $(\psi_\alpha, j_\alpha, f_\alpha, Y_\alpha)$

making diagram (4) below commutative:

(4)



The partial lifting $(\psi_\alpha, j_\alpha, f_\alpha, Y_\alpha)$ is said to be *subordinate* to the partial lifting $(\psi_\beta, j_\beta, f_\beta, Y_\beta)$ if there exists a map $j = j_{\beta\alpha} : Y_\beta \to Y_\alpha$ such that $j \circ j_\beta = j_\alpha$, $j \circ \psi_\beta = \psi_\alpha$ and $f_\alpha \circ j = f_\beta$. Note that in this case, ψ_β is a lifting of ψ_α over j.

For notation and terminology concerning inverse limit systems $(X_\alpha; f_{\alpha\beta})$ we refer the reader to [**5**].

A closed surjection $f : Y \to Z$ is called *irreducible* if and only if for no proper closed $A \subseteq Y$ is $f(A) = Z$.

Given a category $\mathcal{C}$ of topological spaces, a space X is said to be $F_\aleph(\mathcal{C})$ if and only if disjoint $\aleph$-open($\mathcal{C}$) sets in X have disjoint closures. We say X is $F_{\aleph^*}(\mathcal{C})$ if and only if for any two disjoint $\aleph$-open($\mathcal{C}$) sets U and V in X, there is a partition of X into open-closed sets X_1 and X_2 such that $\mathrm{Cl}U \subseteq X_1$ and $\mathrm{Cl}V \subseteq X_2$. If the objects of $\mathcal{C}$ are Tychonoff and include the compact Hausdorff spaces, then in view of Theorem 2.1, X is $F_\aleph(\mathcal{C})$ precisely when X is an $F_\aleph$-space in the sense of Neville and Lloyd [**12**]. It is also clear that a zero dimensional compact Hausdorff space is $F_\aleph(\mathcal{C})$ if and only if it is $F_{\aleph^*}(\mathcal{C})$.

A category $\mathcal{C}$ of topological spaces will be called *nice* provided:

(a) the maps are exactly the perfect (continuous) maps;

(b) the objects are regular closed hereditary;

(c) the objects are Hausdorff;

(d) the disjoint union of two objects in $\mathcal{C}$ is an object in $\mathcal{C}$;

(e) whenever $(X_\alpha; f_{\alpha\beta})$ is an inverse system of spaces which are objects in $\mathcal{C}$ and maps which are perfect, onto and irreducible, then $\varprojlim X_\alpha$ is an object of $\mathcal{C}$; and

(f) either the objects of $\mathcal{C}$ are closed hereditary or else irreducible, perfect preimages of objects of $\mathcal{C}$ are also objects in $\mathcal{C}$.

Lemma 3.1. *Let* $(X_\alpha; f_{\alpha\beta})$ *be an inverse system with each bonding map* $f_{\alpha\beta}$ *perfect, onto and irreducible and each space* X_α *Hausdorff. Then*

(i) *each projection* $f_\alpha : \varprojlim X_\alpha \to X_\alpha$ *is perfect and onto*;

(ii) *each projection is irreducible*;

(iii) *if* $\psi_\alpha : X \to Y_\alpha$ *is perfect for each* α*, then the natural map* $\psi : X \to \varprojlim X_\alpha$ *is perfect*;

(iv) *if* $\psi_\alpha : X \to X_\alpha$ *is onto for each* α*, then* $\psi : X \to \varprojlim X_\alpha$ *is onto*;

(v) *if* $\psi_\alpha : X \to X_\alpha$ *is perfect and irreducible for each* α*, then* $\psi : X \to \varprojlim X_\alpha$ *is irreducible.*

Proof. (i). See [**5**, Theorem 3.7.12]

(ii). See [**7**, Lemma 4].

(iii). See [**5**, Theorem 3.7.11].

(iv). See [**5**, Theorem 2.5.9]

(v). The proof is straightforward. □

The following categories are nice (in each case, the maps are the perfect maps):

(i) T_2 (or T_3, or Tychonoff) spaces;

(ii) Compact T_2 spaces;

(iii) H-closed spaces (the required conditions except for (e) and (f) are easily seen to be true. Condition (f) follows since the preimage of an H-closed space under a perfect, irreducible onto map is H-closed [**23**] and condition (e) follows since the projections are perfect, irreducible and onto by 3.1(i) and (ii), and so the inverse limit is H-closed by condition (f)).

(iv) Paracompact T_2 spaces (again, all but (e) are easily true and (e) follows since the projections are perfect and onto by 3.1(i), and the preimage of a paracompact space under a perfect, onto map is paracompact).

Theorem 3.2. *Every Hausdorff space X is the continuous perfect irreducible image of a Hausdorff space $\tilde{X}$ which has a basis with open closures. Further, $w(\tilde{X}) = w(X)$ and if X is H-closed, then so is $\tilde{X}$.*

Proof. See [**8**]. The proof depends on the fact that Hausdorff and H-closed are regular closed hereditary properties which are inherited by inverse limits when the bonding maps are perfect and irreducible. In short, we need the categories of Hausdorff spaces and of H-closed spaces to be nice. □

Since the inverse limit of T_3 (Tychonoff, compact T_2) spaces is T_3 (Tychonoff, compact T_2, respectively) and since a T_3 space with a base of sets with open closures is zero dimensional, we have a number of corollaries.

Corollary 3.3. (i) *Let $\mathcal{C}$ be a nice category and X an object of $\mathcal{C}$. Then X is the perfect, irreducible image of a space $\tilde{X}$ in $\mathcal{C}$ which has a base with open closures and weight equal to the weight of X.*

(ii) *Every T_3 (Tychonoff) space is the perfect, irreducible image of a zero dimensional T_3 (Tychonoff) space of the same weight.*

(iii) [**4**, Corollary 2.3.8] *Every compact Hausdorff space is the perfect irreducible image of a zero dimensional compact Hausdorff space of the same weight.*

Lemma 3.4. *Let $f : X \to Y$ be perfect, irreducible and onto, and let $U \subseteq X$ be open. Then*

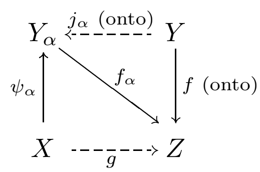
$$f(\mathrm{Cl}U) = \mathrm{Cl}(Y - f(X - U)).$$

Proof. See [**11**]. □

Corollary 3.5. *Let $f : X \to Y$ be perfect, irreducible, and onto. Let V_1 and V_2 be regular closed sets in X with disjoint interiors. Then $f(V_1)$ and $f(V_2)$ are regular closed sets with disjoint interiors.*

Lemma 3.6. *In a nice category $\mathcal{C}$, assume the situation in diagram* (5)*. Further*

(5)



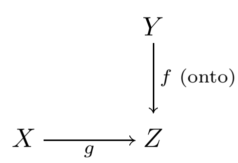
assume that f is irreducible, Y has weight $< \aleph$ and X is $F_{\aleph^*}(\mathcal{C})$. Let E be an open-closed set in Y. Then the partial lifting $(\psi_\alpha, j_\alpha, f_\alpha, Y_\alpha)$ is subordinate to a partial lifting $(\psi_\beta, j_\beta, f_\beta, Y_\beta)$ where Y_β is the disjoint union of $j_\alpha(E)$ and $j_\alpha(Y - E)$ (and thus has weight $< \aleph$).

Proof. Let $V_1 = E$, $V_2 = Y - E$ and, for $i = 1, 2$, let $W_i = j_\alpha(V_i)$. Let Y_β be the disjoint union of W_1 and W_2. Since f is irreducible, it is clear that both j_α and f_α are irreducible. Since j_α is perfect, if follows from Corollary 3.5 that W_1 and W_2 are regular closed sets with disjoint interiors. Since $\mathcal{C}$ is nice, Y_β is an object of $\mathcal{C}$. Now $\psi_\alpha^{-1}(\operatorname{Int} W_1)$ and $\psi_\alpha^{-1}(\operatorname{Int} W_2)$ are disjoint $\aleph$-open($\mathcal{C}$) sets since $w(Y_\alpha) < \aleph$. Since X is $F_{\aleph^*}(\mathcal{C})$, X can be partitioned into open-closed sets X_1 and X_2 such that $\operatorname{Cl}[\psi_\alpha^{-1}(\operatorname{Int} W_i)] \subseteq X_i$ for $i = 1, 2$. Note that if $\psi_\alpha(x) \notin W_1$, then $\psi_\alpha(x) \in \operatorname{Int} W_2$ hence $x \in X_2$. Thus $\psi_\alpha(X_i) \subseteq W_i$ for $i = 1, 2$.

Define $\psi_\beta : X \to Y_\beta$ to be a copy of $\psi_\alpha \mid X_1$ on X_1 and a copy of $\psi_\alpha \mid X_2$ on X_2. Since X_1 and X_2 are open-closed, ψ is continuous, and since ψ_α is perfect, ψ is perfect. Define j_β to be $j_\alpha \mid V_1$ on V_1 and $j_\alpha \mid V_2$ on V_2. Again, j_β is continuous and perfect. Let j be the projection map from Y_β to Y_α and set $f_\beta = f_\alpha \circ j$. Then $j_\alpha = j \circ j_\beta$, $\psi_\alpha = j \circ \psi_\beta$ and $f = f_\beta \circ j$. Clearly, j is irreducible so that f_β and j_β are irreducible, and f_β is perfect. □

Lemma 3.7. *In a nice category $\mathcal{C}$, assume the situation in diagram* (6)*. Further suppose f is irreducible, Y has a basis with open closures and $w(Y) < \aleph$ and X is $F_{\aleph^*}(\mathcal{C})$. Then a solution $\psi : X \to Y$ exists, i.e., there exists a perfect map $\psi : X \to Y$ such that $f \circ \psi = g$.*

(6)



Proof. Let $\mathcal{U}$ be a basis for Y such that each $u \in \mathcal{U}$ has open closure and $|\mathcal{U}| < \aleph$. Let $\{E_\alpha : \alpha \in A\}$ be a well ordering of the closures of the elements of $\mathcal{B}$, where $|A| < \aleph$. If β is an ordinal of cardinality $w(Z)$, we determine for each $\alpha \leq \beta$ a partial lifting $(\psi_\alpha, j_\alpha, f_\alpha, Y_\alpha)$ as follows.

Let $(\psi_0, j_0, f_0, Y_0) = (g, f, i, Z)$ where i is the identity on Z, and note that since f is perfect, $w(Y_0) = w(Z) \leq w(Y) < \aleph$. Assume $(\psi_\alpha, j_\alpha, f_\alpha, Y_\alpha)$ has been defined for each $\alpha < \gamma$, $w(Y_\alpha) = w(Z) < \aleph$, and that for all $\alpha, \kappa < \gamma$, the $f_{\alpha\kappa} : Y_\alpha \to Y_\kappa$ are perfect, irreducible maps (with appropriate commutative digrams). Let $Z_\gamma = \varprojlim(Y_\alpha; f_{\alpha_\kappa})$ with projections $f'_{\gamma\alpha} : Z_\gamma \to Y_\alpha$. By [**18**, Lemma 2.8] $w(Z_\gamma) = w(Z) < \aleph$. By Lemma 3.1(i) and (ii), each $f'_{\gamma\alpha}$ is perfect and irreducible and thus so is $f'_\gamma = f_\alpha \circ f'_{\gamma\alpha}$, and by 3.1(iii), (iv) and (v), the natural maps $\psi'_\gamma : X \to Z_\gamma$ and $j'_\gamma : Y \to Z_\gamma$ are perfect and j'_γ is onto and irreducible. (Note that if $\gamma = \alpha + 1$ for some α, then $Z_\gamma \simeq Y_\alpha$.) Now by Lemma 3.6, $(\psi'_\gamma, j'_\gamma, f'_\gamma, Z_\gamma)$ is subordinate to a partial lifting $(\psi_\gamma, j_\gamma, f_\gamma, Y_\gamma)$, where $E = E_\gamma$. For each $\alpha \leq \kappa$, let $f_{\gamma\alpha} : Y_\gamma \to Y_\alpha$ be $f'_{\gamma\alpha} \circ j$. Note that $f_\gamma = f'_{\gamma 0} \circ j$ and f_γ is irreducible. Also, $f_\gamma \circ j_\gamma = f$, $f_\gamma \circ \psi_\gamma = g$, and $w(Y_\gamma) < \aleph$.

When $\gamma = \beta$, each open-closed set E_γ which is the closure of a basis element for Y has the property that $E_\gamma = j_\beta^{-1}[j_\beta(E_\gamma)]$. But then j_β is one-to-one, continuous and closed and, hence, a homeomorphism.

Let $\psi = j_\beta^{-1}\psi_\beta : X \to Y$. Then ψ is perfect since j_β and ψ_β are, and $f \circ \psi = g$. $\square$

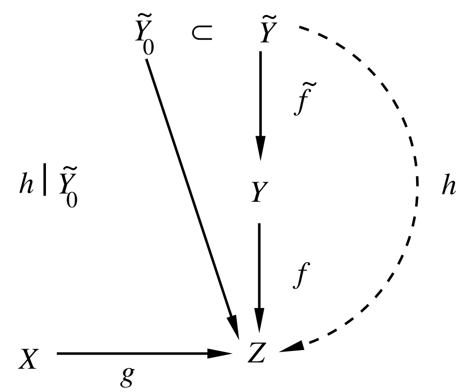
Theorem 3.8. *If $\mathcal{C}$ is a nice category and if X is an object of $\mathcal{C}$ which is $F_{\aleph^*}(\mathcal{C})$, then X is $\aleph$-projective in $\mathcal{C}$.*

Proof. Assume we have our original diagram in $\mathcal{C}$:

$$\begin{array}{ccc} & & Y \\ & & \Big\downarrow{\scriptstyle f\ (\text{onto})} \\ X & \xrightarrow[g]{} & Z \end{array} \tag{1}$$

where f is onto and $w(Y) < \aleph$, but where we now assume X is $F_{\aleph^*}(\mathcal{C})$. Since $\mathcal{C}$ is a nice category, Corollary 3.3 provides us with a space $\tilde{Y}$

which is an object of $\mathcal{C}$ of weight equal to the weight of Y and a perfect irreducible map $\tilde{f}$ of $\tilde{Y}$ onto Y, where $\tilde{Y}$ has a base of sets with open closures. Let $h = f \circ \tilde{f}$. Then h is perfect and onto. We use the standard technique (see [**23**]) to choose $\tilde{Y}_0$, a closed subset of $\tilde{Y}$ such that $h \mid \tilde{Y}_0$ is perfect, irreducible and onto. By condition (f) of the definition of a nice category, $\tilde{Y}_0$ is an object of $\mathcal{C}$. Also, $\tilde{Y}_0$ still has a base of sets with open closures and $w(\tilde{Y}_0) \leq w(\tilde{Y}) = w(Y) < \aleph$. We now have the following situation:



It follows from Lemma 3.7 that there is a perfect map $\psi : X \to \tilde{Y}_0$ such that $(h|\tilde{Y}_0) \circ \psi = g$. But then $(\tilde{f}|\tilde{Y}_0) \circ \psi : X \to Y$ is perfect and $f \circ [(\tilde{f}|\tilde{Y}_0) \circ \psi] = g$, so that (1) has a solution in $\mathcal{C}$. Thus X is $\aleph$-projective in $\mathcal{C}$. □

Corollary 3.9. *In each of the following categories $\mathcal{C}$, every $F_{\aleph^*}(\mathcal{C})$-space is $\aleph$-projective:*

(i) T_2 (or T_3 or Tychonoff) spaces and perfect maps,

(ii) H-closed spaces and perfect maps,

(iii) Paracompact T_2 spaces and perfect maps.

We center attention now on the question: can a converse to Theorem 3.8 be developed? Our first observation: the full converse fails in a rather frustrating way. Call a space X from a nice category $\mathcal{C}$ $\aleph$-*bad*($\mathcal{C}$)

if there is no perfect map of X into a space from $\mathcal{C}$ of weight $< \aleph$. If X is $\aleph$-bad$(\mathcal{C})$, then X is (vacuously) $\aleph$-projective in $\mathcal{C}$ (because no diagram 6 exists, since $w(Z) < \aleph$). That $\aleph$-bad$(\mathcal{C})$ spaces exist which are not $F_{\aleph^*}(\mathcal{C})$ can be seen by observing that whenever X is not $\aleph$-Lindelöf (every open cover admits a subcover of cardinality $< \aleph$) then X is $\aleph$-bad$(\tau_{3\frac{1}{2}})$. [Otherwise $f : X \to Z$ exists where $w(Z) < \aleph$, hence Z is $\aleph$-Lindelöf, hence X is $\aleph$-Lindelöf]. Thus, for example, every non-Lindelof space is $\aleph_1$-bad$(\tau_{3\frac{1}{2}})$. In particular, the Moore plane is a *connected* $\aleph_1$-bad$(\tau_{3\frac{1}{2}})$ space.

A partial converse to Theorem 3.8 *is* available if it is first restated to take account of $\aleph$-bad$(\mathcal{C})$ spaces.

Theorem 3.10. *If $\mathcal{C}$ is a nice category, an object X of $\mathcal{C}$ which is either $F_{\aleph^*}(\mathcal{C})$ or* $\aleph$-bad$(\mathcal{C})$ *is* $\aleph$-projective *in $\mathcal{C}$.*

Theorem 3.11. *Let $\mathcal{C}$ be a category whose maps are the perfect maps and whose objects are*

(i) *Hausdorff,*

(ii) *regular closed hereditary,*

(iii) *preserved by disjoint unions*

and either

(iv) *preserved by perfect maps into Hausdorff spaces or*

(v) *productive and closed hereditary.*

Then an object X of $\mathcal{C}$ which is $\aleph$-projective is either $\aleph$-bad$(\mathcal{C})$ *or* $F_{\aleph^*}(\mathcal{C})$.

Proof. Suppose X is not $\aleph$-bad$(\mathcal{C})$. Then there exists a space Z_0 which is an object of $\mathcal{C}$ of weight $< \aleph$ and a perfect map g_0 of X into Z_0.

Let U and V be disjoint $\aleph$-open$(\mathcal{C})$ sets in X. To show X is $F_{\aleph^*}(\mathcal{C})$, we construct an open-closed partition $\{X_1, X_2\}$ of X with $\mathrm{Cl}U \subseteq X_1$ and $\mathrm{Cl}V \subseteq X_2$. Since U and V are $\aleph$-open$(\mathcal{C})$, there are Hausdorff spaces Z_1 and Z_2 in $\mathcal{C}$ of weight $< \aleph$ and, for $i = 1, 2$, continuous maps $g_i : X \to Z_i$ and open sets $W_i \subseteq Z_i$ such that $U = g_1^{-1}(W_1)$ and

$V = g_2^{-1}(W_2)$.

Let $g : X \to Z_0 \times Z_1 \times Z_2$ be defined by $g(x) = (g_0(x), g_1(x), g_2(x))$. Since g_0 is perfect and $Z_0 \times Z_1 \times Z_2$ is Hausdorff, it follows immediately from [**5**, Theorem 3.7.9] that g is a perfect map. Note that $w(Z_0 \times Z_1 \times Z_2) < \aleph$.

We claim:

(1) If $g^{-1}(y) \cap U \neq \varnothing$, then $g^{-1}(y) \subseteq U$. (As a result, $g(U) \cap g(\mathrm{Cl}V) = \varnothing$.)

(2) $g(U) \subseteq g(X) - \mathrm{Cl}\ g(V)$ and $\mathrm{Int}g(U) \cap \mathrm{IntCl}g(V) = \varnothing$.

(3) $g(X) \subseteq \mathrm{Cl}[g(X) - \mathrm{Cl}g(U)] \cup \mathrm{Cl}[g(X) - \mathrm{Cl}g(V)]$.

Note that since $g(X)$ is closed in $Z_0 \times Z_1 \times Z_2$, $\mathrm{Cl}_{Z_0 \times Z_1 \times Z_2} E = \mathrm{Cl}_{g(X)} E$ when $E \subseteq g(X)$. By $\mathrm{Int}E$ we mean $\mathrm{Int}_{g(X)} E$.

To prove (1), assume $z \in g^{-1}(y) \cap U$. Then $g_1(z) = y_1 \in W_1$. If $x \in g^{-1}(y)$, then $g_1(x) = y_1 \in W_1$ so that $x \in U$. To prove (2), observe that since g is closed, $g(\mathrm{Cl}V) = \mathrm{Cl}g(V)$. Both statements now follow easily. To prove (3), let $x \in X$. If $g(x) \in \mathrm{IntCl}g(U)$, then $g(x) \notin \mathrm{Cl}g(V)$, so $g(x) \in g(X) - \mathrm{Cl}g(V)$. If $g(x) \in \mathrm{Cl}g(U) - \mathrm{IntCl}g(U)$, then every neighborhood of $g(x)$ in $g(X)$ intersects $g(X) - \mathrm{Cl}g(U)$, so $g(x) \in \mathrm{Cl}[g(X) - \mathrm{Cl}g(V)]$. Finally, if $g(x) \notin \mathrm{Cl}g(U)$, then $g(x) \in g(X) - \mathrm{Cl}g(U)$. (3) now follows.

Let $Y_1 = \mathrm{Cl}[g(X) - \mathrm{Cl}g(V)]$ and $Y_2 = \mathrm{Cl}[g(X) - \mathrm{Cl}g(U)]$, and let $Z = Y_1 \cup Y_2$. By (2), $g(X) = Z$. If objects in $\mathcal{C}$ are preserved by perfect maps onto Hausdorff spaces, Z is an object of $\mathcal{C}$, while if $\mathcal{C}$ is productive and closed hereditary, Z is again an object of $\mathcal{C}$. Clearly, $w(Z) < \aleph$. Let Y be the disjoint union of Y_1 and Y_2, and let $f : Y \to Z$ be the natural projection. Then Y is an object of $\mathcal{C}$ (since $\mathcal{C}$ is regular closed hereditary and the disjoint union of objects in $\mathcal{C}$ is an object of $\mathcal{C}$), f is perfect and onto, and $w(Y) < \aleph$.

Since X is ℵ-projective, there is a lifting $\psi : X \to Y$ such that $f \circ \psi = g$. Since $g(U) \subseteq Z - f(Y_2)$ by the first claim, and $g(U) = (f \circ \psi)(U)$, it follows that $\psi(U) \subseteq Y_1$. Similarly, $\psi(V) \subseteq Y_2$. Since $\{Y_1, Y_2\}$ is an open-closed partition of Y and ψ is continuous, if $X_i = \psi^{-1}(Y_i)$ for $i = 1, 2$, then $\{X_1, X_2\}$ is an open-closed partition of X with $\mathrm{Cl}U \subseteq X_1$ and $\mathrm{Cl}V \subseteq X_2$. Hence, X is $F_{\aleph^*}(\mathcal{C})$. ◻

Corollary 3.13. *In each of the following categories $\mathcal{C}$, if X is $\aleph$-projective, then X is either* $\aleph$-bad($\mathcal{C}$) *or* $F_{\aleph^*}(\mathcal{C})$:

(i) T_2 (*or* T_3 *or Tychonoff*) *spaces with perfect maps*,

(ii) H*-closed spaces with perfect maps*,

(iii) *Paracompact* T_2 *spaces with perfect maps.*

Many thanks to the referee, who pointed out one serious mistake and contributed Example 2.2.

REFERENCES

1. B. Banaschewski, *Projective covers in categories of topological spaces and topological algebras*, Proc. Kanpur Top. Conf., Academic Press, New York, 1970, 63–91.

2. M.P. Berri, J.R. Porter, R.M. Stephensen, Jr., *A survey of minimal topological spaces*, Proc. Kanpur Top Conf., Academic Press, New York, 1970, 93–114.

3. A. Blaszcyk, *On projectiveness in H-closed spaces*, Colloq. Math. **32** (1975), 185–193.

4. W.W. Comfort and S. Negrepontis, *The theory of ultrafilters*, Springer-Verlag, NY, 1974.

5. R. Engelking, *General topology*, PWN-Polish Scientific Publishers, Warsaw, 1977.

6. J. Flachsmeyer, *Topologische Projektivroume*, Math. Nachr. **26** (1963), 57–66.

7. L.M. Friedler, *Resolutions of H-closed spaces*, Proc. Amer. Math. Soc. **71** (1978), 309–312.

8. ——— and D.H. Pettey, *Mappings and inverse limits of minimal topological spaces*, Pacific J. Math. **71** (1977), 429–448.

9. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, 1960.

10. A. Gleason, *Projective topological spaces*, Illinois J. Math. **2** (1958), 482–489.

11. S. Iliadis and S. Fomin, *The method of centered systems in the theory of topological spaces*, Russian Math. Surveys **21** (1966), No. 4, 37–62.

12. C.T. Liu, *Absolutely closed spaces*, Trans. Amer. Math. Soc. **130** (1968), 86–104.

13. ——— and G.E. Stecker, *Concerning almost real compactifications*, Czech Math. J. **22** (1972), 181–190.

14. I. Loncar, *A note on inverse systems of H-closed spaces*, Glas. Mat. Ser. III **19** (1984), 169–175.

15. J. Mioduszewski and L. Rudolf, *H-closed and extremally disconnected Hausdorff spaces*, Diss. Math. **66** (1969), 1–55.

16. C.W. Neville and S.P. Lloyd, *ℵ-projective spaces*, Illinois J. Math. **25** (1981), 159–168.

17. N. Noble, *Products with closed projections*, Trans. Amer. Math. Soc. **140** (1969), 381–391.

18. D.H. Pettey, *Locally P-closed spaces and rim P-closed spaces*, Proc. Amer. Math. Soc. **87** (1983), 543–548.

19. P. Ponomarev, *The absolute of a topological space*, Dokl. Akad. Nauk SSSR **149** (1963), 26; Soviet Math. Dokl. **4** (1963), 299–302.

20. D.P. Strauss, *Extremally disconnected spaces*, Proc. Amer. Math. Soc. **18** (1967), 305–309.

21. J. Vermeer, *Expansions of topological spaces*, Ph.D. Thesis, Vrije Universitect te Amsterdam, 1983.

22. ——— and E. Wattel, *Projective elements in categories with perfect θ-continuous maps*, Canad. J. Math. **33** (1984), 872–884.

23. G. Woods, *A survey of absolutes of topological spaces*, in *Top structures* II, Math. Centrum, Amsterdam, 1979.

Department of Computer Science and Mathematics, Beaver College, Glenside, PA 19038

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1